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Homogeneous algebras

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Abstract

Various concepts associated with quadratic algebras admit natural generalizations when the quadratic algebras are replaced by graded algebras which are finitely generated in degree 1 with homogeneous relations of degree N. Such algebras are referred to as *homogeneous algebras of degree* N. In particular, it is shown that the Koszul complexes of quadratic algebras generalize as N-complexes for homogeneous algebras of degree N.

1. Introduction and preliminaries

Our aim is to generalize the various concepts associated with quadratic algebras as described in [27] when the quadratic algebras are replaced by the homogeneous algebras of degree N with $N \geqslant 2$ ($N = 2$ is the case of quadratic algebras). Since the generalization is natural and relatively straightforward, the treatment of [25–27] will be directly adapted to homogeneous algebras of degree N. In other words we dispense ourselves to give a review of the case of quadratic algebras (i.e., the case $N = 2$) by referring to the above quoted nice treatments.

Besides the fact that it is natural to generalize for other degrees what exists for quadratic algebras, this paper produces a very natural class of N-complexes which generalize the

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Koszul complexes of quadratic algebras [19,25–27,33] and which are not of simplicial type. By N-complexes of simplicial type we here mean N-complexes associated with simplicial modules and Nth roots of unity in a very general sense [12] which cover cases considered, e.g., in [11,16,20,21,28] the generalized homology of which has been shown to be equivalent to the ordinary homology of the corresponding simplicial modules [12]. This latter type of constructions and results has been recently generalized to the case of cyclic modules [35]. In spite of the fact that they compute the ordinary homology of the simplicial modules, the usefulness of these N-complexes of simplicial type comes from the fact that they can be combined with other N-complexes [17,18]. In fact, the BRS-like construction [4] of [18] shows that spectral sequences arguments (e.g., in the form of a generalization of the homological perturbation theory [31]) are still working for N-complexes. Other nontrivial classes of N-complexes which are not of simplicial type are the universal construction of [16] and the N-complexes of [14,15] (see also in [13] for a review). It is worth noticing here that elements of homological algebra for N-complexes have been developed in [21] and that several results for N-complexes and more generally N-differential modules like Lemma 1 of [12] have no nontrivial counterpart for ordinary complexes and differential modules. It is also worth noticing that besides the above mentioned examples, various problems connected with theoretical physics implicitly involve exotic N-complexes (see, e.g., [23,24]).

In the course of the paper we shall point out the possibility of generalizing the approach based on quadratic algebras of [27] to quantum spaces and quantum groups by replacing the quadratic algebras by N-homogeneous ones. Indeed one also has in this framework internal **end**, etc., with similar properties.

Finally we shall revisit in the present context the approach of [8,9] to Koszulity for N-homogeneous algebras. This is in order since as explained below, the generalization of the Koszul complexes introduced in this paper for N-homogeneous algebras is a canonical one. We shall explain why a definition based on the acyclicity of the N-complex generalizing the Koszul complex is inappropriate and we shall identify the ordinary complex introduced in [8] (the acyclicity of which is the definition of Koszulity of [8]) with a complex obtained by contraction from the above Koszul N-complex. Furthermore we shall show the uniqueness of this contracted complex among all other ones. Namely we shall show that the acyclicity of any other complex (distinct from the one of [8]) obtained by contraction of the Koszul N-complex leads for $N \geqslant 3$ to an uninteresting (trivial) class of algebras.

Some examples of Koszul homogeneous algebras of degree > 2 are given in [8], including a certain cubic Artin–Schelter regular algebra [1]. Recall that Koszul quadratic algebras arise in several topics as algebraic geometry [22], representation theory [5], quantum groups [26,27,33,34], Sklyanin algebras [30,32]. A classification of the Koszul quadratic algebras with two generators over the complex numbers is performed in [7]. Koszulity of nonquadratic algebras and each of the above items deserve further attention.

The plan of the paper is the following.

In Section 2 we define the duality and the two (tensor) products which are exchanged by the duality for homogeneous algebras of degree N (N-homogeneous algebras). These are the direct extension to arbitrary N of the concepts defined for quadratic algebras ($N = 2$)

[25–27], and our presentation here as well as in Section 3 follows closely the one of Ref. [27] for quadratic algebras.

In Section 3 we elaborate the categorical setting and we point out the conceptual reason for the occurrence of N-complexes in the framework of N-homogeneous algebras. We also sketch in this section a possible extension of the approach of [27] to quantum spaces and quantum groups in which relations of degree N replace the quadratic ones.

In Section 4 we define the N-complexes which are the generalizations for homogeneous algebras of degree N of the Koszul complexes of quadratic algebras [26,27]. The definition of the cochain N-complex $L(f)$ associated with a morphism f of N-homogeneous algebras follows immediately from the structure of the unit object $\bigwedge_N\{d\}$ of one of the (tensor) products of N-homogeneous algebras. We give three equivalent definitions of the chain N-complex $K(f)$: A first one by dualization of the definition of $L(f)$, a second one which is an adaptation of [25] by using Lemma 1, and a third one which is a componentwise approach. It is pointed out in this section that one cannot generalize naively the notion of Koszulity for N-homogeneous algebras with $N \geqslant 3$ by the acyclicity of the appropriate Koszul N-complexes.

In Section 5, we recall the definition of Koszul homogeneous algebras of [8] as well as some results of [8,9] which justify this definition. It is then shown that this definition of Koszulity for homogeneous N-algebras is optimal within the framework of the appropriate Koszul N-complex.

Let us give some indications on our notations. Throughout the paper all vector spaces, algebras, coalgebras are over a fixed field $\mathbb{K}$. Furthermore, unless otherwise specified, the algebras are unital associative and the coalgebras are counital coassociative. The symbol $\otimes$ denotes the tensor product over the basic field $\mathbb{K}$. Concerning the generalized homology of N-complexes we shall use the notation of [20] which is better adapted than other ones to the case of chain N-complexes, that is if $E = \bigoplus_n E_n$ is a chain N-complex with N-differential d, its generalized homology is denoted by ${}_pH(E) = \bigoplus_{n\in\mathbb{Z}} {}_pH_n(E)$ with

$$ {}_pH_n(E) = \mathrm{Ker}(d^p : E_n \to E_{n-p}) \big/ \mathrm{Im}\big(d^{N-p} : E_{n+N-p} \to E_n\big) $$

for $p \in \{1, \ldots, N-1\}$ ($n \in \mathbb{Z}$).

2. Homogeneous algebras of degree N

Let N be an integer with $N \geqslant 2$. A *homogeneous algebra of degree N* or *N-homogeneous algebra* is an algebra of the form

$$ \mathcal{A} = A(E, R) = T(E)/(R) \tag{1} $$

where E is a finite-dimensional vector space (over $\mathbb{K}$), $T(E)$ is the tensor algebra of E and (R) is the two-sided ideal of $T(E)$ generated by a linear subspace R of $E^{\otimes^N}$. The homogeneity of (R) implies that $\mathcal{A}$ is a graded algebra $\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ with $\mathcal{A}_n = E^{\otimes^n}$ for $n < N$ and $\mathcal{A}_n = E^{\otimes^n} / \sum_{r+s=n-N} E^{\otimes^r} \otimes R \otimes E^{\otimes^s}$ for $n \geqslant N$ where we have set $E^{\otimes^0} = \mathbb{K}$ as usual. Thus $\mathcal{A}$ is a graded algebra which is connected ($\mathcal{A}_0 = \mathbb{K}$), generated in degree 1 ($\mathcal{A}_1 = E$) with the ideal of relations among the elements of $\mathcal{A}_1 = E$ generated by $R \subset E^{\otimes^N} = (\mathcal{A}_1)^{\otimes^N}$.

A *morphism of N-homogeneous algebras* $f : A(E, R) \to A(E', R')$ is a linear mapping $f : E \to E'$ such that $f^{\otimes^N}(R) \subset R'$. Such a morphism is a homomorphism of unital graded algebras. Thus one has a category $\mathbf{H}_N\mathbf{Alg}$ of N-homogeneous algebras and the forgetful functor $\mathbf{H}_N\mathbf{Alg} \to \mathbf{Vect}$, $\mathcal{A} \mapsto E$, from $\mathbf{H}_N\mathbf{Alg}$ to the category **Vect** of finite-dimensional vector spaces (over $\mathbb{K}$).

Let $\mathcal{A} = A(E, R)$ be a N-homogeneous algebra. One defines *its dual* $\mathcal{A}^!$ to be the N-homogeneous algebra $\mathcal{A}^! = A(E^*, R^\perp)$ where E^* is the dual vector space of E and where $R^\perp \subset E^{*\otimes^N} = (E^{\otimes^N})^*$ is the annihilator of R, i.e., the subspace $\{\omega \in (E^{\otimes^N})^* \mid \omega(x) = 0, \forall x \in R\}$ of $(E^{\otimes^N})^*$ identified with $E^{*\otimes^N}$. One has canonically

$$\left(\mathcal{A}^!\right)^! = \mathcal{A} \tag{2}$$

and if $f : \mathcal{A} \to \mathcal{A}' = A(E', R')$, is a morphism of $\mathbf{H}_N\mathbf{Alg}$, the transposed of $f : E \to E'$ is a linear mapping of E'^* into E^* which induces the morphism $f^! : (\mathcal{A}')^! \to \mathcal{A}^!$ of $\mathbf{H}_N\mathbf{Alg}$ so ($\mathcal{A} \mapsto \mathcal{A}^!$, $f \mapsto f^!$) is a contravariant (involutive) functor.

Let $\mathcal{A} = A(E, R)$ and $\mathcal{A}' = A(E', R')$ be N-homogeneous algebras; one defines $\mathcal{A} \circ \mathcal{A}'$ and $\mathcal{A} \bullet \mathcal{A}'$ by setting

$$\mathcal{A} \circ \mathcal{A}' = A\left(E \otimes E',\ \pi_N\left(R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R'\right)\right),$$
$$\mathcal{A} \bullet \mathcal{A}' = A\left(E \otimes E',\ \pi_N(R \otimes R')\right),$$

where π_N is the permutation

$$(1, 2, \ldots, 2N) \mapsto (1, N+1, 2, N+2, \ldots, k, N+k, \ldots, N, 2N) \tag{3}$$

belonging to the symmetric group S_{2N} acting as usually on the factors of the tensor products. One has canonically

$$(\mathcal{A} \circ \mathcal{A}')^! = \mathcal{A}^! \bullet \mathcal{A}'^!, \qquad (\mathcal{A} \bullet \mathcal{A}')^! = \mathcal{A}^! \circ \mathcal{A}'^! \tag{4}$$

which follows from the identity $\{R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R'\}^\perp = R^\perp \otimes R'^\perp$. On the other hand, the inclusion $R \otimes R' \subset R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R'$ induces a surjective algebra homomorphism $p : \mathcal{A} \bullet \mathcal{A}' \to \mathcal{A} \circ \mathcal{A}'$ which is of course a morphism of $\mathbf{H}_N\mathbf{Alg}$.

It is worth noticing here that in contrast with what happens for quadratic algebras if $\mathcal{A}$ and $\mathcal{A}'$ are homogeneous algebras of degree N with $N \geqslant 3$ then the tensor product algebra $\mathcal{A} \otimes \mathcal{A}'$ is no more a N-homogeneous algebra. Indeed $\mathcal{A} \otimes \mathcal{A}'$ is generated in degree 1 by $E \oplus E'$ with the relation

$$R + R' \subset (E \oplus E')^{\otimes^N}$$

and the quadratic relation

$$[E, E'] = \{e \otimes e' - e' \otimes e \mid e \in E,\ e' \in E'\} \subset (E \oplus E')^{\otimes^2}$$

so that $\mathcal{A} \otimes \mathcal{A}'$ is homogeneous if and only if $N = 2$ in which case it is quadratic. Nevertheless there still exists (for $N \geqslant 2$) an injective homomorphism of unital algebra $i : \mathcal{A} \circ \mathcal{A}' \to \mathcal{A} \otimes \mathcal{A}'$ doubling the degree which we now describe. Let $\tilde{\imath} : T(E \otimes E') \to T(E) \otimes T(E')$ be the injective linear mapping which restricts as

$$\tilde{\imath} = \pi_n^{-1} : (E \otimes E')^{\otimes^n} \to E^{\otimes^n} \otimes E'^{\otimes^n}$$

on $T^n(E \otimes E') = (E \otimes E')^{\otimes^n}$ for any $n \in \mathbb{N}$. It is straightforward that $\tilde{\imath}$ is an algebra homomorphism which is an isomorphism onto the subalgebra $\bigoplus_n E^{\otimes^n} \otimes E'^{\otimes^n}$ of $T(E) \otimes T(E')$. The following proposition is not hard to verify.

Proposition 1. *Let $\mathcal{A} = A(E, R)$ and $\mathcal{A}' = A(E', R')$ be two N-homogeneous algebras. Then $\tilde{\imath}$ passes to the quotient and induces an injective homomorphism i of unital algebras of $\mathcal{A} \circ \mathcal{A}'$ into $\mathcal{A} \otimes \mathcal{A}'$. The image of i is the subalgebra $\bigoplus_n \mathcal{A}_n \otimes \mathcal{A}'_n$ of $\mathcal{A} \otimes \mathcal{A}'$.*

The proof is almost the same as for quadratic algebras [27].

Remark. As pointed out in [27], any finitely related and finitely generated in degree 1 graded algebra (so in particular any N-homogeneous algebra) gives rise to a quadratic algebra. Indeed if $\mathcal{A} = \bigoplus_{n \geqslant 0} \mathcal{A}_n$ is a graded algebra, define $\mathcal{A}^{(d)}$ by setting $\mathcal{A}^{(d)} = \bigoplus_{n \geqslant 0} \mathcal{A}_{nd}$. Then it was shown in [3] that if $\mathcal{A}$ is generated by the finite-dimensional subspace $\mathcal{A}_1$ of its elements of degree 1 with the ideal of relations generated by its components of degree $\leqslant r$, then the same is true for $\mathcal{A}^{(d)}$ with r replaced by $2 + (r - 2)/d$.

3. Categorical properties

Our aim in this section is to investigate the properties of the category $\mathbf{H}_N\mathbf{Alg}$. We follow again closely [27] replacing the quadratic algebras considered there by the N-homogeneous algebras.

Let $\mathcal{A} = A(E, R)$, $\mathcal{A}' = A(E', R')$ and $\mathcal{A}'' = A(E'', R'')$ be three homogeneous algebras of degree N. Then the isomorphisms $E \otimes E' \simeq E' \otimes E$ and $(E \otimes E') \otimes E'' \simeq E \otimes (E' \otimes E'')$ of **Vect** induce corresponding isomorphisms $\mathcal{A} \circ \mathcal{A}' \simeq \mathcal{A}' \circ \mathcal{A}$ and $(\mathcal{A} \circ \mathcal{A}') \circ \mathcal{A}'' \simeq \mathcal{A} \circ (\mathcal{A}' \circ \mathcal{A}'')$ of N-homogeneous algebras (i.e., of $\mathbf{H}_N\mathbf{Alg}$). Thus $\mathbf{H}_N\mathbf{Alg}$ endowed with $\circ$ is a tensor category [10] and furthermore to the one-dimensional vector space $\mathbb{K}t \in$ **Vect** which is a unit object of (**Vect**, $\otimes$) corresponds the polynomial algebra $\mathbb{K}[t] = A(\mathbb{K}t, 0) \simeq T(\mathbb{K})$ as unit object of $(\mathbf{H}_N\mathbf{Alg}, \circ)$. In fact the isomorphisms $\mathbb{K}[t] \circ \mathcal{A} \simeq \mathcal{A} \simeq \mathcal{A} \circ \mathbb{K}[t]$ are obvious in $\mathbf{H}_N\mathbf{Alg}$. Thus one has part (i) of the following theorem.

Theorem 1. *The category $\mathbf{H}_N\mathbf{Alg}$ of N-homogeneous algebras has the following properties*:

(i) $\mathbf{H}_N\mathbf{Alg}$ *endowed with $\circ$ is a tensor category with unit object $\mathbb{K}[t]$.*
(ii) $\mathbf{H}_N\mathbf{Alg}$ *endowed with $\bullet$ is a tensor category with unit object $\bigwedge_N\{d\} = \mathbb{K}[t]^!$.*

Part (ii) follows from (i) by the duality $\mathcal{A} \mapsto \mathcal{A}^!$. In fact (i) and (ii) are equivalent in view of (2) and (4).

The N-homogeneous algebra $\bigwedge_N\{d\} = \mathbb{K}[t]^! \simeq T(\mathbb{K})/\mathbb{K}^{\otimes^N}$ is the (unital) graded algebra generated in degree one by d with relation $d^N = 0$. Part (ii) of Theorem 1 is the very reason for the appearance of N-complexes in the present context, remembering the obvious fact that graded $\bigwedge_N\{d\}$-module and N-complexes are the same thing.

Theorem 2. *The functorial isomorphism in* **Vect**

$$\mathrm{Hom}_{\mathbb{K}}(E \otimes E', E'') \cong \mathrm{Hom}_{\mathbb{K}}\big(E, E'^* \otimes E''\big)$$

induces a corresponding functorial isomorphism

$$\mathrm{Hom}(\mathcal{A} \bullet \mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}(\mathcal{A}, \mathcal{B}^! \circ \mathcal{C})$$

in $\mathbf{H}_N\mathbf{Alg}$, *(setting* $\mathcal{A} = A(E, R)$, $\mathcal{B} = A(E', R')$ *and* $\mathcal{C} = A(E'', R'')$*).*

Again the proof is the same as for quadratic algebras [27]. It follows that the tensor category $(\mathbf{H}_N\mathbf{Alg}, \bullet)$ has an internal **Hom** [10] given by

$$\mathbf{Hom}(\mathcal{B}, \mathcal{C}) = \mathcal{B}^! \circ \mathcal{C} \tag{5}$$

for two N-homogeneous algebras $\mathcal{B}$ and $\mathcal{C}$. Setting $\mathcal{A} = A(E, R)$, $\mathcal{B} = A(E', R')$, and $\mathcal{C} = A(E'', R'')$ one verifies that the canonical linear mappings $(E^* \otimes E') \otimes E \to E'$ and $(E'^* \otimes E'') \otimes (E^* \otimes E') \to E^* \otimes E''$ induce products

$$\mu : \mathbf{Hom}(\mathcal{A}, \mathcal{B}) \bullet \mathcal{A} \to \mathcal{B}, \tag{6}$$

$$m : \mathbf{Hom}(\mathcal{B}, \mathcal{C}) \bullet \mathbf{Hom}(\mathcal{A}, \mathcal{B}) \to \mathbf{Hom}(\mathcal{A}, \mathcal{C}) \tag{7}$$

these internal products as well as their associativity properties follow more generally from the formalism of tensor categories [10].

Following [27], define $\mathbf{hom}(\mathcal{A}, \mathcal{B}) = \mathbf{Hom}(\mathcal{A}^!, \mathcal{B}^!)^! = \mathcal{A}^! \bullet \mathcal{B}$. Then one obtains by duality from (6) and (7) morphisms

$$\delta_\circ : \mathcal{B} \to \mathbf{hom}(\mathcal{A}, \mathcal{B}) \circ \mathcal{A}, \tag{8}$$

$$\Delta_\circ : \mathbf{hom}(\mathcal{A}, \mathcal{C}) \to \mathbf{hom}(\mathcal{B}, \mathcal{C}) \circ \mathbf{hom}(\mathcal{A}, \mathcal{B}) \tag{9}$$

satisfying the corresponding coassociativity properties from which one obtains by composition with the corresponding homomorphisms i the algebra homomorphisms

$$\delta : \mathcal{B} \to \mathbf{hom}(\mathcal{A}, \mathcal{B}) \otimes \mathcal{A}, \tag{10}$$

$$\Delta : \mathbf{hom}(\mathcal{A}, \mathcal{C}) \to \mathbf{hom}(\mathcal{B}, \mathcal{C}) \otimes \mathbf{hom}(\mathcal{A}, \mathcal{B}). \tag{11}$$

Theorem 3. *Let $\mathcal{A} = A(E, R)$ be a N-homogeneous algebra. Then the (N-homogeneous) algebra* $\mathbf{end}(\mathcal{A}) = \mathcal{A}^! \bullet \mathcal{A} = \mathbf{hom}(\mathcal{A}, \mathcal{A})$ *endowed with the coproduct Δ becomes a bialgebra with counit $\varepsilon : \mathcal{A}^! \bullet \mathcal{A} \to \mathbb{K}$ induced by the duality $\varepsilon = \langle \cdot, \cdot \rangle : E^* \otimes E \to \mathbb{K}$ and δ defines on $\mathcal{A}$ a structure of left* $\mathbf{end}(\mathcal{A})$*-comodule.*

4. The N-complexes $L(f)$ and $K(f)$

Let us apply Theorem 2 with $\mathcal{A} = \bigwedge_N \{d\}$ and use Theorem 1(ii). One has

$$\mathrm{Hom}(\mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}\left(\bigwedge_N \{d\}, \mathcal{B}^! \circ \mathcal{C}\right) \tag{12}$$

and we denote by $\xi_f \in \mathcal{B}^! \circ \mathcal{C}$ the image of d corresponding to the morphism $f \in \mathrm{Hom}(\mathcal{B}, \mathcal{C})$. One has $(\xi_f)^N = 0$ and by using the injective algebra homomorphism $i : \mathcal{B}^! \circ \mathcal{C} \to \mathcal{B}^! \otimes \mathcal{C}$ of Proposition 1 we let d be the left multiplication by $i(\xi_f)$ in $\mathcal{B}^! \otimes \mathcal{C}$. One has $d^N = 0$ so, equipped with the appropriate graduation, $(\mathcal{B}^! \otimes \mathcal{C}, d)$ is a N-complex which will be denoted by $L(f)$. In the case where $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and where f is the identity mapping $I_{\mathcal{A}}$ of $\mathcal{A}$ onto itself, this N-complex will be denoted by $L(\mathcal{A})$. These N-complexes are the generalizations of the Koszul complexes denoted by the same symbols for quadratic algebras and morphisms [27]. Note that $(\mathcal{B}^! \otimes \mathcal{C}, d)$ is a cochain N-complex of right $\mathcal{C}$-modules, i.e., $d : \mathcal{B}^!_n \otimes \mathcal{C} \to \mathcal{B}^!_{n+1} \otimes \mathcal{C}$ is $\mathcal{C}$-linear.

Similarly the Koszul complexes $K(f)$ associated with morphisms f of quadratic algebras generalize as N-complexes for morphisms of N-homogeneous algebras. Let $\mathcal{B} = A(E, R)$ and $\mathcal{C} = A(E', R')$ be two N-homogeneous algebras and let $f : \mathcal{B} \to \mathcal{C}$ be a morphism of N-homogeneous algebras ($f \in \mathrm{Hom}(\mathcal{B}, \mathcal{C})$). One can define the N-complex $K(f) = (\mathcal{C} \otimes \mathcal{B}^{!*}, d)$ by using partial dualization of the N-complex $L(f)$ generalizing thereby the construction of [26] or one can define $K(f)$ by generalizing the construction of [25,27].

The first way consists in applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{C})$ to each right $\mathcal{C}$-module of the N-complex $(\mathcal{B}^! \otimes \mathcal{C}, d)$. We get a chain N-complex of left $\mathcal{C}$-modules. Since $\mathcal{B}^!_n$ is a finite-dimensional vector space, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}^!_n \otimes \mathcal{C}, \mathcal{C})$ is canonically identified to the left module $\mathcal{C} \otimes (\mathcal{B}^!_n)^*$. Then we get the N-complex $K(f)$ whose differential d is easily described in terms of f. In the case $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and $f = I_{\mathcal{A}}$, this complex will be denoted by $K(\mathcal{A})$.

We shall follow hereafter the second more explicit way. We shall make use of the following slight elaboration of an ingredient of the presentation of [25].

Lemma 1. *Let A be an associative algebra with product denoted by m, let C be a coassociative coalgebra with coproduct denoted by Δ and let $\mathrm{Hom}_{\mathbb{K}}(C, A)$ be equipped with its structure of associative algebra for the convolution product $(\alpha, \beta) \mapsto \alpha * \beta = m \circ (\alpha \otimes \beta) \circ \Delta$. Then one defines an algebra-homomorphism $\alpha \mapsto d_\alpha$ of $\mathrm{Hom}_{\mathbb{K}}(C, A)$ into the algebra $\mathrm{End}_A(A \otimes C) = \mathrm{Hom}_A(A \otimes C, A \otimes C)$ of endomorphisms of the left A-module $A \otimes C$ by defining d_α as the composite*

$$A \otimes C \xrightarrow{I_A \otimes \Delta} A \otimes C \otimes C \xrightarrow{I_A \otimes \alpha \otimes I_C} A \otimes A \otimes C \xrightarrow{m \otimes I_C} A \otimes C$$

for $\alpha \in \mathrm{Hom}_{\mathbb{K}}(C, A)$.

The proof is straightforward, $d_\alpha \circ d_\beta = d_{\alpha * \beta}$ follows easily from the coassociativity of Δ and the associativity of m. As pointed out in [25] one obtains a graphical version ("electronic version") of the proof by using the usual graphical version of the coassociativity of Δ combined with the usual graphical version of the associativity of m. The left A-linearity of d_α is straightforward.

Let us associate with $f \in \mathrm{Hom}(\mathcal{B}, \mathcal{C})$ the homogeneous linear mapping of degree zero $\alpha : (\mathcal{B}^!)^* \to \mathcal{C}$ defined by setting $\alpha = f : E \to E'$ in degree 1 and $\alpha = 0$ in degrees different from 1. The dual $(\mathcal{B}^!)^*$ of $\mathcal{B}^!$ defined degree by degree is a graded coassociative counital coalgebra and one has

$$\alpha^{*N} = \underbrace{\alpha * \cdots * \alpha}_{N} = 0.$$

Indeed it follows from the definition that α^{*N} is trivial in degrees $n \neq N$. On the other hand in degree N, α^{*N} is the composition

$$R \xrightarrow{f^{\otimes N}} E'^{\otimes N} \to E'^{\otimes N}/R'$$

which vanishes since $f^{\otimes N}(R) \subset R'$. Applying Lemma 1 it is easily checked that the N-differential

$$d_\alpha : \mathcal{C} \otimes \mathcal{B}^{!*} \to \mathcal{C} \otimes \mathcal{B}^{!*}$$

coincides with d of the first way.

Let us give an even more explicit description of $K(f)$ and pay some attention to the degrees. Recall that by $(\mathcal{B}^!)^*$ we just mean here the direct sum $\bigoplus_n (\mathcal{B}^!_n)^*$ of the dual spaces $(\mathcal{B}^!_n)^*$ of the finite-dimensional vector spaces $\mathcal{B}^!_n$. On the other hand, with $\mathcal{B} = A(E, R)$ as above, one has

$$\mathcal{B}^!_n = \begin{cases} E^{*\otimes^n} & \text{if } n < N, \\ E^{*\otimes^n} \Big/ \sum_{r+s=n-N} E^{*\otimes^r} \otimes R^\perp \otimes E^{*\otimes^s} & \text{if } \quad n \geqslant N. \end{cases}$$

So one has for the dual spaces

$$\left(\mathcal{B}^!_n\right)^* \cong \begin{cases} E^{\otimes^n} & \text{if } n < N, \\ \bigcap_{r+s=n-N} E^{\otimes^r} \otimes R \otimes E^{\otimes^s} & \text{if } n \geqslant N. \end{cases} \tag{13}$$

In view of (13), one has canonical injections

$$\left(\mathcal{B}^!_n\right)^* \hookrightarrow \left(\mathcal{B}^!_k\right)^* \otimes \left(\mathcal{B}^!_\ell\right)^*$$

for $k+\ell=n$ and one sees that the coproduct Δ of $(\mathcal{B}^!)^*$ is given by

$$\Delta(x)=\sum_{k+\ell=n} x_{k\ell}$$

for $x\in(\mathcal{B}^!_n)^*$ where the $x_{k\ell}$ are the images of x into $(\mathcal{B}^!_k)^*\otimes(\mathcal{B}^!_\ell)^*$ under the above canonical injections.

If $f:\mathcal{B}\to\mathcal{C}=A(E',R')$ is a morphism of $\mathbf{H}_N\mathbf{Alg}$, one verifies that the N-differential d of $K(f)$ defined above is induced by the linear mappings

$$c\otimes(e_1\otimes e_2\otimes\cdots\otimes e_n)\mapsto cf(e_1)\otimes(e_2\otimes\cdots\otimes e_n) \tag{14}$$

of $\mathcal{C}\otimes E^{\otimes^n}$ into $\mathcal{C}\otimes E^{\otimes^{n-1}}$. One has $d(\mathcal{C}_s\otimes(\mathcal{B}^!_r)^*)\subset\mathcal{C}_{s+1}\otimes(\mathcal{B}^!_{r-1})^*$ so the N-complex $K(f)$ splits into subcomplexes

$$K(f)^n=\bigoplus_m \mathcal{C}_{n-m}\otimes\left(\mathcal{B}^!_m\right)^*,\quad n\in\mathbb{N}$$

which are homogeneous for the total degree. Using (13), (14) one can describe $K(f)^0$ as

$$\cdots\to 0\to\mathbb{K}\to 0\to\cdots \tag{15}$$

and $K(f)^n$ as

$$\cdots\to 0\to E^{\otimes^n}\xrightarrow{f\otimes I_E^{\otimes^{n-1}}} E'\otimes E^{\otimes^{n-1}}\to\cdots\xrightarrow{I_{E'}^{\otimes^{n-1}}\otimes f} E'^{\otimes^n}\to 0\to\cdots \tag{16}$$

for $1\leqslant n\leqslant N-1$ while $K(f)^N$ reads

$$\cdots 0\to R\xrightarrow{f\otimes I_E^{\otimes^{N-1}}} E'\otimes E^{\otimes^{N-1}}\to\cdots\to E'^{\otimes^{N-1}}\otimes E\xrightarrow{\text{can}}\mathcal{C}_N\to 0\cdots \tag{17}$$

where 'can' is the composition of $I_{E'}^{\otimes^{N-1}}\otimes f$ with canonical projection of $E'^{\otimes^N}$ onto $E'^{\otimes^N}/R'=\mathcal{C}_N$.

Let us seek for conditions of maximal acyclicity for the N-complex $K(f)$. Firstly, it is clear that $K(f)^0$ is not acyclic, one has ${}_pH_0(K(f)^0)=\mathbb{K}$ for $p\in\{1,\ldots,N-1\}$. Secondly if $N\geqslant 3$, it is straightforward that if $n\in\{1,\ldots,N-2\}$ then $K(f)^n$ is acyclic if and only if $E=E'=0$. Next comes the following lemma.

Lemma 2. *The N-complexes $K(f)^{N-1}$ and $K(f)^N$ are acyclic if and only if f is an isomorphism of N-homogeneous algebras.*

Proof. First $K(f)^{N-1}$ is acyclic if and only if f induces an isomorphism $f:E\xrightarrow{\simeq}E'$ of vector spaces as easily verified and then, the acyclicity of $K(f)^N$ is equivalent to $f^{\otimes^N}(R)=R'$ which means that f is an isomorphism of N-homogeneous algebras. □

It is worth noticing here that for $N \geqslant 3$ the nonacyclicity of the $K(f)^n$ for $n \in \{1, \dots, N-2\}$ whenever E or E' is nontrivial is easy to understand and to possibly cure. Let us assume that $K(f)^{N-1}$ and $K(f)^N$ are acyclic. Then by identifying through the isomorphism f the two N-homogeneous algebras, one can assume that $\mathcal{B} = \mathcal{C} = \mathcal{A} = A(E, R)$ and that f is the identity mapping $I_{\mathcal{A}}$ of $\mathcal{A}$ onto itself, that is with the previous notation that one is dealing with $K(f) = K(\mathcal{A})$. Trying to make $K(\mathcal{A})$ as acyclic as possible one is now faced to the following result for $N \geqslant 3$.

Proposition 2. *Assume that $N \geqslant 3$, then one has*

$$\operatorname{Ker}\bigl(d^{N-1} : \mathcal{A}_2 \otimes \bigl(\mathcal{A}^!_{N-1}\bigr)^* \to \mathcal{A}_{N+1}\bigr) = \operatorname{Im}\bigl(d : \mathcal{A}_1 \otimes \bigl(\mathcal{A}^!_N\bigr)^* \to \mathcal{A}_2 \otimes \bigl(\mathcal{A}^!_{N-1}\bigr)^*\bigr)$$

if and only if either $R = E^{\otimes^N}$ or $R = 0$.

Proof. One has

$$\mathcal{A}_2 \otimes \bigl(\mathcal{A}^!_{N-1}\bigr)^* = E^{\otimes^2} \otimes E^{\otimes^{N-1}} \simeq E^{\otimes^{N+1}}, \qquad \mathcal{A}_{N+1} \simeq E^{\otimes^{N+1}} \big/ E \otimes R + R \otimes E$$

and d^{N-1} identifies here with the canonical projection

$$E^{\otimes^{N+1}} \to E^{\otimes^{N+1}} \big/ E \otimes R + R \otimes E$$

so its kernel is $E \otimes R + R \otimes E$. On the other hand one has $\mathcal{A}_1 \otimes (\mathcal{A}^!_N)^* = E \otimes R$ and $d : E \otimes R \to E^{\otimes^{N+1}}$ is the inclusion. So $\operatorname{Im}(d) = \operatorname{Ker}(d^{N-1})$ is here equivalent to $R \otimes E = E \otimes R + R \otimes E$ and thus to $R \otimes E = E \otimes R$ since all vector spaces are finite-dimensional. It turns out that this holds if and only if either $R = E^{\otimes^N}$ or $R = 0$ (see Appendix A). □

Corollary 1. *Assume that $N \geqslant 3$ and let $\mathcal{A} = A(E, R)$ be a N-homogeneous algebra. Then the $K(\mathcal{A})^n$ are acyclic for $n \geqslant N - 1$ if and only if either $R = 0$ or $R = E^{\otimes^N}$.*

Proof. In view of Proposition 2, $R = 0$ or $R = E^{\otimes^N}$ is necessary for the acyclicity of $K(\mathcal{A})^{N+1}$; on the other hand, if $R = 0$ or $R = E^{\otimes^N}$ then the acyclicity of the $K(\mathcal{A})^n$ for $n \geqslant N - 1$ is obvious. □

Notice that $R = 0$ means that $\mathcal{A}$ is the tensor algebra $T(E)$ whereas $R = E^{\otimes^N}$ means that $\mathcal{A} = T(E^*)^!$. Thus the acyclicity of the $K(\mathcal{A})^n$ for $n \geqslant N - 1$ is stable by the duality $\mathcal{A} \mapsto \mathcal{A}^!$ as for quadratic algebras ($N = 2$). However, for $N \geqslant 3$ this condition does not lead to an interesting class of algebras contrary to what happens for $N = 2$ where it characterizes the Koszul algebras [29]. This is the very reason why another generalization of Koszulity has been introduced and studied in [8] for N-homogeneous algebras.

5. Koszul homogeneous algebras

Let us examine more closely the N-complex $K(\mathcal{A})$:

$$\cdots \to \mathcal{A} \otimes \left(\mathcal{A}_i^!\right)^* \xrightarrow{d} \mathcal{A} \otimes \left(\mathcal{A}_{i-1}^!\right)^* \to \cdots \to \mathcal{A} \otimes \left(\mathcal{A}_1^!\right)^* \xrightarrow{d} \mathcal{A} \to 0.$$

The $\mathcal{A}$-linear map $d : \mathcal{A} \otimes (\mathcal{A}_i^!)^* \to \mathcal{A} \otimes (\mathcal{A}_{i-1}^!)^*$ is induced by the canonical injection (see in last section)

$$\left(\mathcal{A}_i^!\right)^* \hookrightarrow \left(\mathcal{A}_1^!\right)^* \otimes \left(\mathcal{A}_{i-1}^!\right)^* = \mathcal{A}_1 \otimes \left(\mathcal{A}_{i-1}^!\right)^* \subset \mathcal{A} \otimes \left(\mathcal{A}_{i-1}^!\right)^*.$$

The degree i of $K(\mathcal{A})$ as N-complex has not to be confused with the total degree n. Recall that, when $N = 2$, the quadratic algebra $\mathcal{A}$ is said to be Koszul if $K(\mathcal{A})$ is acyclic at any degree $i > 0$ (clearly it is equivalent to saying that each complex $K(\mathcal{A})^n$ is acyclic for any total degree $n > 0$).

For any N, it is possible to contract the N-complex $K(\mathcal{A})$ into (2-)complexes by putting together alternately p or $N - p$ arrows d in $K(\mathcal{A})$. The complexes so obtained are the following ones:

$$\cdots \xrightarrow{d^{N-p}} \mathcal{A} \otimes \left(\mathcal{A}_{N+r}^!\right)^* \xrightarrow{d^p} \mathcal{A} \otimes \left(\mathcal{A}_{N-p+r}^!\right)^* \xrightarrow{d^{N-p}} \mathcal{A} \otimes \left(\mathcal{A}_r^!\right)^* \xrightarrow{d^p} 0,$$

which are denoted by $C_{p,r}$. All the possibilities are covered by the conditions $0 \leqslant r \leqslant N-2$ and $r+1 \leqslant p \leqslant N-1$. Note that the complex $C_{p,r}$ at degree i is $\mathcal{A} \otimes (\mathcal{A}_k^!)^*$, where $k = jN + r$ or $k = (j+1)N - p + r$, according to $i = 2j$ or $i = 2j+1$ ($j \in \mathbb{N}$).

In [8], the complex $C_{N-1,0}$ is called the *Koszul complex* of $\mathcal{A}$, and the homogeneous algebra $\mathcal{A}$ is said to be *Koszul* if this complex is acyclic at any degree $i > 0$. A motivation for this definition is that Koszul property is equivalent to a purity property of the minimal projective resolution of the trivial module. One has the following result [8,9]:

Proposition 3. *Let $\mathcal{A}$ be a homogeneous algebra of degree N. For $i = 2j$ or $i = 2j+1$, $j \in \mathbb{N}$, the graded vector space $\mathrm{Tor}_i^{\mathcal{A}}(\mathbb{K}, \mathbb{K})$ lives in degrees $\geqslant jN$ or $\geqslant jN+1$, respectively. Moreover, $\mathcal{A}$ is Koszul if and only if each $\mathrm{Tor}_i^{\mathcal{A}}(\mathbb{K}, \mathbb{K})$ is concentrated in degree jN or $jN+1$, respectively* (*purity property*).

When $N = 2$, it is exactly Priddy's definition [29]. Another motivation is that a certain cubic Artin–Schelter regular algebra has the purity property, and this cubic algebra is a good candidate for making noncommutative algebraic geometry [1,2]. Some other non-trivial examples are contained in [8].

The following result shows how the Koszul complex $C_{N-1,0}$ plays a particular role. Actually all the other contracted complexes of $K(\mathcal{A})$ are irrelevant as far as acyclicity is concerned.

Proposition 4. *Let $\mathcal{A} = A(E, R)$ be a homogeneous algebra of degree $N \geqslant 3$. Assume that (p, r) is distinct from $(N-1, 0)$ and that $C_{p,r}$ is exact at degree $i = 1$. Then $R = 0$ or $R = E^{\otimes N}$.*

Proof. Assume $r = 0$, hence $1 \leqslant p \leqslant N - 2$. Regarding $C_{p,0}$ at degree 1 and total degree $N + 1$, one gets the exact sequence

$$E \otimes R \xrightarrow{d^p} E^{\otimes N+1} \xrightarrow{d^{N-p}} E^{\otimes N+1} \big/ E \otimes R + R \otimes E,$$

where the maps are the canonical ones. Thus $E \otimes R = E \otimes R + R \otimes E$, leading to $R \otimes E = E \otimes R$. This holds only if $R = 0$ or $R = E^{\otimes N}$ (see Appendix A).

Assume now $1 \leqslant r \leqslant N - 2$ (hence $r + 1 \leqslant p \leqslant N - 1$). Regarding $C_{p,r}$ at degree 1 and total degree $N + r$, one gets the exact sequence

$$\left(\mathcal{A}^!_{N+r}\right)^* \xrightarrow{d^p} E^{\otimes N+r} \xrightarrow{d^{N-p}} E^{\otimes N+r} \big/ R \otimes E^{\otimes r},$$

where the maps are the canonical ones. Thus $(\mathcal{A}^!_{N+r})^* = R \otimes E^{\otimes r}$, and $R \otimes E^{\otimes r}$ is contained in $E^{\otimes r} \otimes R$. So $R \otimes E^{\otimes r} = E^{\otimes r} \otimes R$, which implies again $R = 0$ or $R = E^{\otimes N}$ (see Appendix A). □

It is easy to check that, if $R = 0$ or $R = E^{\otimes N}$, any $C_{p,r}$ is exact at any degree $i > 0$. On the other hand, for any R, one has

$$H_0(C_{p,r}) = \bigoplus_{0 \leqslant j \leqslant N-p-1} E^{\otimes j} \otimes E^{\otimes r},$$

which can be considered as a Koszul left $\mathcal{A}$-module if $\mathcal{A}$ is Koszul.

Appendix A. A lemma on tensor products

Lemma 3. *Let E be a finite-dimensional vector space. Let R be a subspace of $E^{\otimes N}$, $N \geqslant 1$. If $R \otimes E^{\otimes r} = E^{\otimes r} \otimes R$ holds for an integer $r \geqslant 1$, then $R = 0$ or $R = E^{\otimes N}$.*

Proof. Fix a basis $X = (x_1, \dots, x_n)$ of E, ordered by $x_1 < \cdots < x_n$. The set X^N of the words of length N in the letters $x_1, \dots, x_n$ is a basis of $E^{\otimes N}$ which is lexicographically ordered. Denote by S the X^N-reduction operator of $E^{\otimes N}$ associated to R [6,7]. This means the following properties:

(i) S is an endomorphism of the vector space $E^{\otimes N}$ such that $S^2 = S$;
(ii) for any $a \in X^N$, either $S(a) = a$ or $S(a) < a$ (the latter inequality means $S(a) = 0$, or otherwise any word occurring in the linear decomposition of $S(a)$ on X^N is $< a$ for the lexicographic ordering);
(iii) $\mathrm{Ker}(S) = R$.

Then $S \otimes I_{E^{\otimes r}}$ and $I_{E^{\otimes r}} \otimes S$ are the X^{N+r}-reduction operators of $E^{\otimes^{N+r}}$ associated respectively to $R \otimes E^{\otimes^r}$ and $E^{\otimes^r} \otimes R$. By assumption these endomorphisms are equal. In particular, one has

$$\mathrm{Im}(S) \otimes E^{\otimes^r} = E^{\otimes^r} \otimes \mathrm{Im}(S).$$

But the subspace $\mathrm{Im}(S)$ is monomial, i.e., generated by words. So it suffices to prove the lemma when R is monomial.

Assume that R contains the word $x_{i_1} \ldots x_{i_N}$. For any letters $x_{j_1}, \ldots, x_{j_r}$, the word $x_{i_1} \ldots x_{i_N} x_{j_1} \ldots x_{j_r}$ belongs to $E^{\otimes^r} \otimes R$. Since R is monomial $x_{i_{r+1}} \ldots x_{i_N} x_{j_1} \ldots x_{j_r}$ belongs to R. Continuing the process, we see that any word belongs to R. □

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