Vertex PI indices of four sums of graphs

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ABSTRACT

Suppose that e is an edge of a graph G. Denote by \( m_e(G) \) the number of vertices of G that are not equidistant from both ends of e. Then the vertex PI index of G is defined as the summation of \( m_e(G) \) over all edges e of G. In this paper we give the explicit expressions for the vertex PI indices of four sums of two graphs in terms of other indices of two individual graphs, which correct the main results in a paper published in Ars Combin. 98 (2011).

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1. Introduction

A topological index is a real number related to a molecular graph, which does not depend on the labeling or the pictorial representation of a graph. Several indices have been defined and have found applications as means for modeling chemical, pharmaceutical and other properties of molecules. The Wiener index, introduced in 1947 by Wiener as the path number for the characterization of alkanes, was the first topological index to be used in chemistry [18–20]. The Szeged index, introduced by Gutman [5], is closely related to the Wiener index [6,10,11,14]. Since the Szeged index takes into account how the vertices are distributed, it is natural to introduce an index that takes into account the distribution of edges. The PI index is a Szeged-like index that takes into account the distribution of edges and a unique topological index related to parallelism of edges too. The vertex PI index was introduced by Khalifeh et al. in [13]. Its definition is similar to that of the PI index, in that it is additive, but now the distances of vertices from edges are considered. All indices mentioned above have many chemical applications [1,3,7,9,16] and correlate with the physico-chemical properties and biological activities of a large number of diverse and complex compounds [8,12].

Wiener indices, and hyper-Wiener indices and reverse Wiener indices for four new sums of two graphs were computed in [4,17], respectively. Vertex PI indices of four sums of two graphs have been computed in [15], but the main results in [15] are wrong. In this paper we deal with the errors in [15] and give the correct expressions for their vertex PI indices in terms of other indices of two individual graphs.

2. Preliminaries

We first recall some operations on graphs in [2] (see Fig. 1).

Suppose that \( G = (V, E) \) is a connected graph, and refer to each vertex of V as a black vertex. Then we denote by \( S(G) \) the graph obtained from G by inserting an additional vertex which is referred to as the white vertex in each edge of G. Two black vertices in \( S(G) \) are related if they are adjacent in G; and two white vertices in \( S(G) \) are related if their corresponding edges in G are adjacent. Denote by \( R(G) \) and \( Q(G) \) the graphs obtained from \( S(G) \) by joining every pair of related black vertices

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Lemma 2.2. Let \( G \) be a connected graph and \( v = (v_1, v_2) \) be a vertex of \( G \). Then:

(a) If \( v_1 \in V_1 \) (that is \( v \) is a black vertex), then for all \( u = (u_1, u_2) \in V(G_1 +_F G_2) \) we have
\[
d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2).
\]

(b) If \( v_1 \in V_1 \), then for all \( u = (u_1, u_2) \in V(G_1 +_F G_2) \) with \( u_2 \neq v_2 \), \( u_1 = u_1^1v_1^1 \in E_1 \) and \( u_1^1, v_1^1 \in V_1 \) (that is \( v \) and \( u \) are white vertices in different copies of \( F(G_1) \)), we have
\[
d(u, v|G_1 +_F G_2) = 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, v_1^1|F(G_1)), d(u_1^1, v_1^1|F(G_1))\}.
\]

(c) If \( v_1 \in V_1 \), then for all \( u = (u_1, u_2) \in V(G_1 +_F G_2) \), where \( u_2 = v_2 \) and \( u_1 \in E_1 \) (that is \( v \) and \( u \) are white vertices in the same copy of \( F(G_1) \)), we have
\[
d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)).
\]

Lemma 2.2. Let \( G_1 \) and \( G_2 \) be two connected graphs, \( u_1, v_1 \in E_1, u_2, v_2 \in V_2 \) and \( F = S \) or \( R \). Then for \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) in \( G_1 +_F G_2 \) with \( u_2 \neq v_2 \), we have
\[
d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}
\]

Lemma 2.3. Let \( G_1 \) and \( G_2 \) be two connected graphs, \( u_1, v_1 \in E_1, u_2, v_2 \in V_2 \) and \( F = Q \) or \( T \). Then for \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) in \( G_1 +_F G_2 \) with \( u_2 \neq v_2 \), we have
\[
d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ 1 + d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}
\]
3. The main results

Let $e = uv$ be an edge of a connected graph $G$. Then we denote by $M_{uv}(e|G)$ (or $m_{uv}(e|G)$) the set of vertices of $G$ lying closer to the vertex $u$ (or $v$) than to $v$ (or $u$). If we denote by $|A|$ the cardinality of a set $A$, and suppose that $m_{uv}(e|G) = |M_{uv}(e|G)|$ and $m_{vu}(e|G) = |M_{vu}(e|G)|$, then the vertex $Pl$ index of $G$, $Pl_e$, is defined as the summation of $m_{uv}(e|G) + m_{vu}(e|G)$ over all edges of $e$. We denote by $M_{e}(G)$ the set of vertices of $G$ that are not equidistant from both ends of the edge $e$ and suppose that $m_e(G) = |M_e(G)|$. Then $m_e(G) = m_{uv}(e|G) + m_{vu}(e|G)$ and $Pl_e = \sum_{e \in G} m_e(G)$. In this section we will give the explicit expressions for $Pl_e(G_1 + G_2)$ in terms of other indices of $F(G_1)$ and $G_2$.

For convenience, we introduce the following notation. Set
\[ A := \{ e = uv \in E(G_1 + G_2) : u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2 \} \]
\[ B := \{ e = uv \in E(G_1 + G_2) : u = (u_1, u_2) \in V_1 \times V_2, \quad v = (v_1, v_2) \in E_1 \times V_2 \} \]
\[ C := \{ e = uv \in E(G_1 + G_2) : u = (u_1, u_2), \quad v = (v_1, v_2) \in E_1 \times V_2 \}. \]

Then $E(G_1 + G_2) = A \cup B \cup C$. Suppose that $A = \sum_{e \in A} m_e(G_1 + G_2)$, $B = \sum_{e \in B} m_e(G_1 + G_2)$ and $C = \sum_{e \in C} m_e(G_1 + G_2)$.

Suppose that $e$ is an edge of a graph $G$. Then we denote by $M_e(G)$ the set of vertices of $G$ that are equidistant from both ends of $e$, and suppose that $m_e(G) = |M_e(G)|$ and $\tilde{M}_e(G) = \{ e \in E(G) \}$. Then $\tilde{M}_e(G)$ is defined as the summation of $m_e(G_1 + G_2) = \sum_{e \in E(G)} m_e(G)$. Suppose that $e$ is an edge of a graph $G$. Then we denote by $M_e(G)$ the set of vertices of $G$ that are equidistant from both ends of $e$, and suppose that $m_e(G) = |M_e(G)|$ and $\tilde{M}_e(G) = \{ e \in E(G) \}$. Then $\tilde{M}_e(G)$ is defined as the summation of $m_e(G_1 + G_2) = \sum_{e \in E(G)} m_e(G)$.

**Theorem 3.1.** Let $G_1$ and $G_2$ be two connected graphs. Then
\[ Pl_e(G_1 + G_2) = (|V_1| + |E_1|)(|V_1|Pl_e(G_2) + 2|E_1| |V_2|^2). \]

**Proof.** By the definition of the $S$-sum, we know that $C = \emptyset$, and so $C = 0$. Next we only need to compute $A$ and $B$ to obtain $Pl_e(G_1 + G_2)$.

Suppose that $e = uv \in A$. Then, by the definition of the $S$-sum, we know that $u_1 = v_1$ and $u_2 = v_2 \in E_2(G)$.

For any $w = (w_1, w_2) \in V_1 \times V_2$, by Lemma 2.1(a), we have
\[ d(w, uG_1 + G_2) = d(w, uG_1) + d(w_2, u_2G_2) \]
\[ d(w, vG_1 + G_2) = d(w, vG_1) + d(w_2, v_2G_2). \]

From the above two equations, we know that $w \in \tilde{M}_e(G_1 + G_2)$ if and only if $w_2 \in \tilde{M}_e(G_2)$. Therefore, $m_e(G_1 + G_2) = (|V_1| + |E_1|)(|V_2| - \tilde{M}_e(G_2))$, and further we obtain
\[ A = |V_1|(|V_1| + |E_1|)Pl_e(G_2). \]

Suppose that $e = uv \in B$. Then, by the definition of the $S$-sum, we know that $u_2 = v_2$ and $u_1$ is an end vertex of $v_1$ in $G_1$. If $w = (w_1, w_2) \in V_1 \times V_2$ then, by Lemma 2.1(a), we have
\[ d(w, uG_1 + G_2) = d(w, uG_1) + d(w_2, u_2G_2). \]
\[ d(w, vG_1 + G_2) = d(w, vG_1) + d(w_2, v_2G_2). \]

Since $u_1$ is an end vertex of $v_1$ in $G_1$, $d(w_1, u_1|S(G_1)) \neq d(w_1, v_1|S(G_1))$. Note that $u_2 = v_2$. From the above two equations, we know that $w \not\in M_e(G_1 + G_2)$.

If $w \in E_1 \times V_2$ then, by Lemma 2.1(a) and 2.2, we have
\[ d(w, uG_1 + G_2) = d(w, uG_1) + d(w_2, u_2G_2); \]
\[ d(w, vG_1 + G_2) = \begin{cases} 2 + d(w_2, v_2G_2) & \text{if } w_1 = v_1, \\ 2 + d(w_2, v_2G_2) & \text{if } w_1 \neq v_1. \end{cases} \]

Note once more that $u_1$ is an edge of $v_1$ in $G_1$. We know that $d(w_1, u_1|S(G_1)) = 1$ if $w_1 = v_1$, and $d(w_1, u_1|S(G_1)) \neq d(w_1, v_1|S(G_1))$ otherwise. Since $u_2 = v_2$, from the above two equations, we can see that $w \not\in M_e(G_1 + G_2)$.

From the above argument we know that $M_e(G_1 + G_2) = \emptyset$ if $e \in B$. Therefore,
\[ B = |V_2|(|V_1| + |E_1|) \cdot 2|E_1| |V_2| \]
\[ = 2|E_1| |V_2|^2(|V_1| + |E_1|). \]

Hence we obtain
\[ Pl_e(G_1 + G_2) = A + B = (|V_1| + |E_1|)(|V_1|Pl_e(G_2) + 2|E_1| |V_2|^2). \]

**Theorem 3.2.** Let $G_1$ and $G_2$ be two connected graphs. Then
\[ Pl_e(G_1 + G_2) = |V_1|(|V_1| + |E_1|)Pl_e(G_2) + |V_2|^2Pl_e(R(G_1)). \]

**Proof.** By the definition of the $R$-sum, we know that $C = \emptyset$, and so $C = 0$. Next we only need to compute $A$ and $B$ to obtain $Pl_e(G_1 + G_2)$.
Suppose that \( e = uv \in \mathcal{A} \). Then we further set

\[
\mathcal{A}_1 := \{ e = uv : u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2, u_1 = v_1 \text{ and } e_2 = u_2v_2 \in E(G_2) \};
\]

\[
\mathcal{A}_2 := \{ e = uv : u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2, u_2 = v_2 \text{ and } e_1 = u_1v_1 \in E(R(G_1)) \}.
\]

Then \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \). Suppose that \( A_1 = \sum_{e \in \mathcal{A}_1} m_e(G_1 +_R G_2) \) and \( A_2 = \sum_{e \in \mathcal{A}_2} m_e(G_1 +_R G_2) \). Then \( A = A_1 + A_2 \).

As in the former proof of Theorem 3.1, we can see that \( A_1 = |V_1|(|V_1| + |E_1|)P_{1}(G_2) \).

For any \( e = uv \in \mathcal{A}_2 \) and \( w = (w_1, w_2) \in V(G_1 +_R G_2) \), by Lemma 2.1(a), we have

\[
d(w, u)|G_1 +_R G_2 = d(w_1, u_1|R(G_1)) + d(w_2, u_2|G_2);
\]

\[
d(w, v)|G_1 +_R G_2 = d(w_1, v_1|R(G_1)) + d(w_2, v_2|G_2).
\]

Since \( u_2 = v_2 \), we can easily see that \( w \in \tilde{M}_e(G_1 +_R G_2) \) if and only if \( w \in \tilde{M}_e(G_1) \). Therefore, \( m_e(G_1 +_R G_2) = |V_2|(|V_1| + |E_1|) - |V_2|^2 \tilde{m}_{e_1}(R(G_1)) \), and we further have

\[
A_2 = |V_1||V_2|^2(|V_1| + |E_1|) - |V_2|^2 \sum_{e_1 \in E(R(G_1))} \tilde{m}_{e_1}(R(G_1)).
\]

Hence, we obtain \( A = (|V_1| + |E_1|)(|V_1|P_{1}(G_2) + |E_1| |V_2|^2) - |V_2|^2 \sum_{e_1 \in E(R(G_1))} \tilde{m}_{e_1}(R(G_1)) \).

Suppose that \( e = uv \in \mathcal{B} \). Then, by the definition of the \( R \)-sum, we know that \( u_2 = v_2 \), \( e_1 = u_1v_1 \in E(R(G_1)) \) and \( u_1 \) is an end vertex of \( v_1 \) in \( G_1 \). If \( w = (w_1, w_2) \in V_1 \times V_2 \) then, by Lemma 2.1(a), we have

\[
d(w, u)|G_1 +_R G_2 = d(w_1, u_1|R(G_1)) + d(w_2, u_2|G_2);
\]

\[
d(w, v)|G_1 +_R G_2 = d(w_1, v_1|R(G_1)) + d(w_2, v_2|G_2).
\]

Since \( u_2 = v_2 \) and \( u_1 \) is an end of \( v_1 \) in \( G_1 \), from the above equations, we know that \( w \in \tilde{M}_e(G_1 +_R G_2) \) if and only if \( w_1 \in \tilde{M}_e(G_1) \).

If \( w \in \mathcal{B} \) then, by Lemmas 2.1(a) and 2.2, we have

\[
d(w, u)|G_1 +_R G_2 = d(w_1, u_1|R(G_1)) + d(w_2, u_2|G_2);
\]

\[
d(w, v)|G_1 +_R G_2 = \begin{cases} |V_2|(|V_1| + |E_1|) - |V_2|^2 \sum_{e_1 \in E(R(G_1))} \tilde{m}_{e_1}(R(G_1)). \end{cases}
\]

Therefore, we have

\[
B_e(G_1 +_R G_2) = A + B = |V_1|(|V_1| + |E_1|)P_{1}(G_2) + |V_2|^2P_{1}(R(G_1)). \]

Suppose that \( e = uv \in \mathcal{A}_2 \). Then, as in the former proof of Theorem 3.1, we can obtain

\[
A = |V_1|(|V_1| + |E_1|)P_{1}(G_2).
\]

Suppose that \( e = uv \in \mathcal{B} \). Then, by the definition of the \( Q \)-sum, we know that \( u_2 = v_2 \), \( e_1 = u_1v_1 \in E(Q(G_1)) \) and \( u_1 \) is an end of \( v_1 \) in \( G_1 \). If \( w = (w_1, w_2) \in V_1 \times V_2 \) then, by Lemma 2.1(a), we have

\[
d(w, u)|G_1 +_Q G_2 = d(w_1, u_1|Q(G_1)) + d(w_2, u_2|G_2);
\]

\[
d(w, v)|G_1 +_Q G_2 = d(w_1, v_1|Q(G_1)) + d(w_2, v_2|G_2).
\]

Since \( u_2 = v_2 \), from the above equations, we observe that \( w \in \tilde{M}_e(G_1 +_Q G_2) \) if and only if \( w_1 \in \tilde{M}_e(G_1) \).
Claim 1. Suppose that $v_1 = u_i u_i'$ is an edge of a connected graph $G_1$, and suppose that $\tilde{\mathcal{E}}_1 = u_1 v_1 \in E(Q(G_1))$. Then, for $w_1 \in V$, we have $w_1 \in \tilde{\mathcal{M}}_1((Q(G_1)))$ if and only if $w_1 \in M_{v_i u_i}(v_1 | G_1) \setminus \{u_i\}$ or $w_1 \in \tilde{\mathcal{M}}_1(G_1)$.

Proof. Suppose that $w_1 \in \tilde{\mathcal{M}}_1((Q(G_1)))$. If $w_1 \in M_{v_i u_i}(v_1 | G_1)$ and we let $P_i = x_1 x_2 \cdots x_t$ be the shortest path between $w_1$ and $u_i$ in $G_1$, then $P_{i+1} = x_{1} x_{2} \cdots x_{t} x_{t+1}$ is the shortest path between $w_1$ and $u_i$ in $G_1$, where $x_i = w_1$, $x_t = u_i$ and $x_{t+1} = u_i$. Thus, by the definition of $Q(G_1)$, we can see that $P_{i+1} = w_1 y_2 \cdots y_{t-1} v_i$ is the shortest path between $w_1$ and $v_i$ in $G_1$ and that $P_{i+2} = w_1 y_1 y_2 \cdots y_{t-1} v_i$ is the shortest path between $w_1$ and $u_i$ in $G_1$, where $y_i = x_i x_{t+1}$. $i = 1, 2, \ldots, t - 1$. This contradicts $w_1 \in \tilde{\mathcal{M}}_1((Q(G_1)))$, and so $w_1 \in M_{v_i u_i}(v_1 | G_1) \setminus \{u_i\}$ or $w_1 \in \tilde{\mathcal{M}}_1(G_1)$.

If $w_1 \in M_{v_i u_i}(v_1 | G_1) \setminus \{u_i\}$, and we let $P_k = q_1 q_2 \cdots q_k$ be the shortest path between $w_1$ and $u_i$ in $G_1$, then $P_{k+1} = q_1 q_2 \cdots q_k q_{k+1}$ is the shortest path between $w_1$ and $u_i$ in $G_1$, where $q_1 = w_1$, $q_k = u_i$ and $q_{k+1} = u_i$. Thus, by the definition of $Q(G_1)$, we can see that $P_{k+1} = w_1 z_1 z_2 \cdots z_{k-1} v_i$ is the shortest path between $w_1$ and $v_i$ in $Q(G_1)$, where $z_i = q_i q_{i+1} \in E_1$, $i = 1, 2, \ldots, k - 1$. This implies that $w_1 \in \tilde{\mathcal{M}}_1(Q(G_1))$. Similarly, if $w_1 \in \tilde{\mathcal{M}}_1(G_1)$, then we can prove that $w_1 \in \tilde{\mathcal{M}}_1((Q(G_1)))$. Hence, Claim 1 is complete.

Claim 2. Suppose that $v_1 = u_i u_i'$ is an edge of a connected graph $G_1$, and suppose that $\tilde{\mathcal{E}}_1 = u_1 v_1 \in E(Q(G_1))$. Then, for $w_1 \in V$, we have $w_1 \in \tilde{\mathcal{M}}_1((Q(G_1)))$ if and only if $w_1 \in N_{v_i u_i}(v_1 | G_1)$ or $w_1 \in \tilde{\mathcal{N}}_1(G_1) \setminus \{v_1\}$, where $\tilde{\mathcal{N}}_1(G_1)$ is the set of edges of $G_1$ that are equidistant from both ends of $v_1$.

Proof. By a method similar to that of the proof of Claim 1, we can prove that Claim 2 is true.
Case 3. Suppose that \( w_1 \neq v_1 \) and \( w_1 \neq u_1 \). Then, by Lemma 2.3, we have
\[
d(w, u|G_1 +_Q G_2) = 1 + d(w_1, u_1|Q(G_1)) + d(w_2, u_2|G_2);
\]
\[
d(w, v|G_1 +_Q G_2) = 1 + d(w_1, v_1|Q(G_1)) + d(w_2, v_2|G_2).
\]
We can easily see that \( w \in \tilde{M}_{e}(G_1 +_Q G_2) \) if and only if \( w_1 \in \tilde{M}_{e_1}(Q(G_1)) \).
Therefore, by the above argument, we know that for \( e \in \mathcal{R} \)
\[
m_{e}(G_1 +_Q G_2) = |V_2|(|V_1| + |E_1|) - 2(|V_2| - 1) - |V_2|\tilde{m}_{e_1}(Q(G_1)).
\]

Hence, we finally obtain
\[
PL_e(G_1 +_Q G_2) = A + B + C = |V_1|(|V_1| + |E_1|)PL_e(G_2) - 2|V_2|(|V_2| - 1)PL(G_1) + |V_2|^2PL_e(Q(G_1)) + 2|E_1|(|V_2| - 1) - |V_2|(|V_2| - 1)M_1(G_1) - 2|E_1|.
\]

Theorem 3.4. Let \( G_1 \) and \( G_2 \) be two connected graphs. Then
\[
PL_e(G_1 +_T G_2) = |V_1|(|V_1| + |E_1|)PL_e(G_2) - 2|V_2|(|V_2| - 1)PL(G_1) + |V_2|^2PL_e(T(G_1)) - |V_2|(|V_2| - 1)M_1(G_1)
\]
\[
+ 2|E_1|^2 |V_2|(|V_2| - 1) - |V_2|(|V_2| - 1)M_1(G_1).
\]

Proof. Suppose that \( e = u_1v_1 \in \mathcal{R} \), and suppose that \( e_1 = u_1v_1 \). Then, as in the former proof of Theorem 3.2, we have
\[
A = |V_1|(|V_1| + |E_1|)|V_1|PL_e(G_2) - 2|V_2|(|V_2| - 1)PL(G_1) + |V_2|^2PL_e(T(G_1)) - |V_2|(|V_2| - 1)M_1(G_1)
\]
\[
+ |V_1|(|V_1| + |E_1|)PL_e(G_2) + |V_2|^2\sum_{e_1 \in E(T(G_1))} \tilde{m}_{e_1}(T(G_1)).
\]

Claim 3. Suppose that \( v_1 = u_1u_1' \) is an edge of a connected graph \( G_1 \), and suppose that \( \tilde{e}_1 = u_1v_1 \in E(T(G_1)) \). Then, for \( v_1 \in V_1 \), we have \( w_1 \in \tilde{M}_{e_1}(T(G_1)) \) if and only if \( w_1 \in \tilde{M}_{e_1}(T(G_1)). \)

Proof. By a method similar to that of the proof of Claim 1, we can prove that Claim 3 is true.

Claim 4. Suppose that \( v_1 = u_1u_1' \) is an edge of a connected graph \( G_1 \), and suppose that \( \tilde{e}_1 = u_1v_1 \in E(T(G_1)) \). Then, for \( v_1 \in V_1 \), we have \( w_1 \in \tilde{M}_{e_1}(T(G_1)) \) if and only if \( w_1 \in \tilde{M}_{e_1}(T(G_1)) \) or \( w_1 \in \tilde{N}_{u_1u_1'}(v_1|G_1) \).

Proof. By a method similar to that of the proof of Claim 1, we can prove that Claim 4 is true.

Hence, we finally obtain

\[ \text{PI}_v(G_1 + T G_2) = A + 2B + C \]

\[ = |V_1|(|V_1| + |E_1|)\text{PI}_v(G_2) - 2|V_2|(|V_2| - 1)\text{PI}(G_1) + |V_2|^2\text{PI}_v(T(G_1)) - |V_2|(|V_2| - 1)M_1(G_1) \]

\[ + 2|E_1|^2|V_2|(|V_2| - 1). \]

\[ \square \]

4. Postscript

The draft manuscript of a paper was published in Ars Combin. 98 (2011) without the present authors being aware of this. Now we give a simple example to show that the main results in [15] are wrong. Let \( G_1 \) and \( G_2 \) be the paths on three and two vertices, respectively. Then, by the definition of the vertex PI index, we can easily see that \( \text{PI}_v(G_1 + Q G_2) = 106 \) and \( \text{PI}_v(G_1 + T G_2) = 122 \). But using the corresponding formulae in [15], we have \( \text{PI}_v(G_1 + Q G_2) = 110 \) and \( \text{PI}_v(G_1 + T G_2) = 222 \).

References