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Applied Mathematics Letters 17 (2004) 323–328

Applied Mathematics Letters
www.elsevier.com/locate/aml

Comparing Numerical Methods for the Solutions of Systems of Ordinary Differential Equations

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(Received July 2002; revised and accepted February 2003)

Abstract—In this article, we implement a relatively new numerical technique, the Adomian decomposition method, for solving linear and nonlinear systems of ordinary differential equations. The method in applied mathematics can be an effective procedure to obtain analytic and approximate solutions for different types of operator equations. In this scheme, the solution takes the form of a convergent power series with easily computable components. This paper will present a numerical comparison between the Adomian decomposition and a conventional method such as the fourth-order Runge-Kutta method for solving systems of ordinary differential equations. The numerical results demonstrate that the new method is quite accurate and readily implemented.

Keywords—Adomian decomposition method, Fourth-order Runge-Kutta method, System of ordinary differential equations.

1. INTRODUCTION

Many scientific and technological problems are modeled mathematically by systems of ordinary differential equations, for example, mathematical models of series circuits and mechanical systems involving several springs attached in series can lead to a system of differential equations. Furthermore, such systems are often encountered in chemical, ecological, biological, and engineering applications. A standard class of problems, for which considerable literature and software exists, is that of initial value problems for first-order systems of ordinary differential equations.

Most realistic systems of ordinary differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. Adomian's decomposition method [1,2] is a relatively new approach to provide an analytical approximation to linear and nonlinear problems, and it is particularly valuable as a tool for scientists and applied mathematicians, because

0893-9659/04/$ - see front matter Typeset by AMS-TEX
doi: 10.1016/S0893-9659(04)00014-X

it provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization [1,2]. In this paper, the Adomian decomposition method is implemented to derive analytic approximate solutions to general systems of ordinary differential equations, and then a numerical comparison with the Runge-Kutta method is demonstrated through some examples. In the literature, this method has been used to obtain approximate analytic solutions of a large class of linear or nonlinear differential equations [1–11]. Additionally, it is used to get numerical solutions in terms of the decomposition series [12,13]. Furthermore, Bellomo and Sarafyan [14], Bellomo and Monaco [15], and Wazwaz [16] implemented the method in a comparative way. In order to make a numerical comparison, Kaya [17] and Tonningen [18] have recently used the decomposition method to solve linear and nonlinear differential equations using computer. Shawagfeh and Adomian [3] implemented the decomposition method to solve the system of Lotka-Volterra equations.

2. AN ANALYSIS OF THE METHOD FOR THE NONLINEAR SYSTEM

Consider the initial value problem

$$L\mathbf{X} = B\mathbf{X} + \mathbf{F}(\mathbf{X}) + \mathbf{g}(t), \qquad \mathbf{X}(a) = \mathbf{C}, \tag{1}$$

where $L = \frac{d}{dt}$, B is $n \times n$ matrix function of t,

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{F}(\mathbf{X}) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix}, \qquad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}. \tag{2}$$

All vectors are assumed to be in Hilbert space H.

Applying the inverse operator L^{-1} on both sides, we obtain

$$\mathbf{X} = \mathbf{C} + L^{-1}(B\mathbf{X}) + L^{-1}(\mathbf{F}(\mathbf{X})) + L^{-1}(\mathbf{g}(\mathbf{t})). \tag{3}$$

Now we write the solution in the decomposition form

$$\mathbf{X} = \mathbf{X}_0 + \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_m + \cdots, \tag{4}$$

then $\mathbf{F}(\mathbf{X})$ can be expanded as

$$\mathbf{F}(\mathbf{X}) = \sum_{m=0}^{\infty} \mathbf{A}_m = \sum_{m=0}^{\infty} \begin{pmatrix} A_m^{(1)} \\ A_m^{(2)} \\ \vdots \\ A_m^{(n)} \end{pmatrix}, \tag{5}$$

where $A_m^{(j)}$ are Adomian polynomials for the function of several variables $f_j(x_1, x_2, \ldots, x_n)$ [1].

We define the components of the solution as follows:

$$\mathbf{X}_0 = \mathbf{C} + L^{-1}(\mathbf{g}(\mathbf{t})), \quad \mathbf{X}_m = L^{-1}(B\mathbf{X}_{m-1}) + L^{-1}(\mathbf{A}_{m-1}), \qquad m \geq 1. \tag{6}$$

The K-term approximate solution is

$$\mathbf{U}_K = \sum_{m=0}^{K-1} \mathbf{X}_m, \tag{7}$$

and the exact solution is

$$\mathbf{X} = \lim_{K \to \infty} \mathbf{U}_K. \tag{8}$$

The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques.

3. AN ANALYSIS OF THE METHOD FOR THE LINEAR SYSTEM

The system

$$L\mathbf{X} = B\mathbf{X} + \mathbf{g}(t),$$

where B is a constant matrix, is encountered in many applications in various different scientific fields. In this case, (6) gives

$$\mathbf{X}_0 = \mathbf{C} + L^{-1}(\mathbf{g}(\mathbf{t})), \quad \mathbf{X}_m = BL^{-1}(\mathbf{X}_{m-1}), \qquad m \geq 1.$$

Noting that $L^{-1} = \int_a^t (.)\, dt$, then recursively we have

$$\begin{aligned} \mathbf{X}_1 &= BL^{-1}(\mathbf{X}_0) = B\mathbf{C}(t-a) + BL^{-2}\mathbf{g}(t), \\ &\vdots \\ \mathbf{X}_m &= B^m\mathbf{C}\frac{(t-a)^m}{m!} + B^m L^{-(m+1)}\mathbf{g}(\mathbf{t}). \end{aligned} \tag{9}$$

Using the well-known identity

$$L^{-(m+1)}\mathbf{g}(t) = \int_a^t \int_a^t \cdots \int_a^t \mathbf{g}(t)\,(dt)^{m+1} = \frac{1}{\Gamma(m+1)} \int_a^t (t-\tau)^m \mathbf{g}(t)\, d\tau,$$

and summing up the components, we obtain that the exact solution is

$$\mathbf{X} = \sum_{m=0}^{\infty} B^m \mathbf{C}\frac{(t-a)^m}{m!} + \sum_{m=0}^{\infty} \frac{1}{m!} B^m \int_a^t (t-\tau)^m \mathbf{g}(t)\, d\tau$$

that can be written in the form

$$\mathbf{X} = \mathbf{C}e^{B(t-a)} + \int_a^t e^{B(t-\tau)}\mathbf{g}(t)\, d\tau,$$

which is the well-known exact solution.

4. EXAMPLES

To give a clear overview of the methodology as a numerical tool, we consider two examples in this section. We apply the fourth-order Runge-Kutta [19] and Adomian decomposition methods on these examples so that the comparisons are made numerically.

For numerical approximate solution we truncate the series solution (7) up to a few terms, and this will be enough to give an accurate result, as the following example shows.

EXAMPLE 1. For comparison purposes, we consider the simple system

$$\begin{aligned} \dot{x} &= x + y, & x(0) &= 0, \\ \dot{y} &= -x + y, & y(0) &= 1, \end{aligned} \tag{10}$$

whose exact solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}. \tag{11}$$

Comparing (10) with (1) we have

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{X}(\mathbf{0}) = \mathbf{C} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Table 1. The decompositions solution (ϕ_K and ψ_K, $K = 5, 10$) and the Runge-Kutta solution (ϕ_{RK} and ψ_{RK}).

t	$x - \phi_5$	$y - \psi_5$	$x - \phi_{10}$	$y - \psi_{10}$	$x - \phi_{\text{RK}}$	$y - \psi_{\text{RK}}$
0.2	7.31405E−07	2.13559E−08	1.63758E−14	1.47097E−14	1.13921E−05	1.06453E−05
0.4	4.80992E−05	2.87222E−06	3.35512E−11	3.59346E−11	3.24548E−05	1.99544E−05
0.6	5.62334E−04	5.15406E−05	2.89381E−09	3.21264E−09	6.55388E−05	2.40168E−05
0.8	3.23866E−03	4.05297E−04	6.82297E−08	7.86056E−08	1.122376E−04	1.71278E−05
1.0	1.26447E−02	2.02727E−03	7.89824E−07	9.45446E−07	1.73346E−04	8.45776E−06

thus (7) gives the K-term approximate solution of (10) for any initial point a,

$$\mathbf{U}_K = \begin{pmatrix} \phi_K \\ \psi_K \end{pmatrix} = \sum_{m=0}^{K-1} B^m C \frac{(t-a)^m}{m!}. \tag{12}$$

In Table 1, a comparison of the numerical results of the absolute errors obtained by using the fourth-order Runge-Kutta method (ϕ_{RK} and ψ_{RK}), and the decomposition series solution with two different number of terms (ϕ_K and ψ_K, $K = 5, 10$) with the exact solutions (11) are given for the same step sizes h.

For large values of t, we may increase the accuracy of the series solution by computing more terms, which is quite easy using one of the symbolic programming packages MATHEMATICA, MATLAB, MAPLE, etc.

Alternately, for large values of t, we can still use only the five-term approximate solution (12) by dividing the t interval into subintervals $[t_k, t_{k+1}]$ and then use the same formula (9) but taking $a = t_k$ and $\mathbf{C} = \mathbf{U}(t_k)$ to evaluate $\mathbf{U}(t_{k+1})$. A comparison with the exact solution $\mathbf{X}(\mathbf{t})$ in (9) and the numerical solution obtained using the fourth-order Runge-Kutta method and the approximate solution $\mathbf{U}_5$ in Table 2 demonstrates the accuracy of this approach.

Table 2. Comparison between the exact solution (x, y), the Runge-Kutta solution $(\phi_{\text{RK}}, \psi_{\text{RK}})$, and the decomposition solution (ϕ_5, ψ_5).

t	x	y	ϕ_{RK}	ψ_{RK}	ϕ_5	ψ_5
.05	.0525417	1.04996	.0525417	1.04996	.0525417	1.04996
.50	.790439	1.44689	.790439	1.44689	.790436	1.44688
1.0	2.28736	1.46869	2.28736	1.46869	2.28733	1.46868
1.5	4.47046	.317022	4.47046	.317021	4.4704	.317013
2.0	6.71885	−3.07493	6.71885	−3.07494	6.71872	−3.07488
2.5	7.29088	−9.75993	7.29088	−9.75994	7.29069	−9.7597
3.0	2.83447	−19.8845	2.83446	−19.8845	2.83435	−19.884
3.5	−11.6163	−31.0112	−11.6164	−31.0112	−11.616	−31.0101
4.0	−41.3200	−35.6877	−41.3201	−35.6877	−41.3185	−35.6862
4.5	−87.9945	−18.9752	−87.9946	−18.9752	−87.99907	−18.9741
5.0	−142.317	42.0992	−142.317	42.0994	−142.31	42.0977

EXAMPLE 2. As an application of the decomposition method to a system of nonlinear ordinary differential equations, we consider the system

$$\frac{dx}{dt} = \lambda(Tx - y) - \lambda x^3,$$
$$\frac{dy}{dt} = x, \tag{13}$$

$$x(0) = c_1, \qquad y(0) = c_2, \tag{14}$$

which represents the Zeeman model for the beating action of the human heart [20]. Here $x(t)$ is the length of the muscle fiber at time t, and $y(t)$ is the amount of the chemical control agent at time t. The constant λ is a positive constant of proportionality, and T is the constant tension in the muscle fiber.

Writing the system in the standard form (2), we have

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda T & -\lambda \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{F}(\mathbf{X}) = \begin{pmatrix} x^3 \\ 0 \end{pmatrix}.$$

Substituting in (3), considering a as initial point, we obtain the five-term approximate solution

$$\mathbf{U}_5(t) = \begin{pmatrix} \phi_K \\ \psi_K \end{pmatrix} = \mathbf{X}_0 + \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \mathbf{X}_4,$$

where

$$\mathbf{X}_0 = \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \mathbf{X}_1 = \begin{pmatrix} \gamma_1 \\ c_1 \end{pmatrix}(t-a), \qquad \mathbf{X}_2 = \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix}\frac{(t-a)^2}{2!},$$
$$\mathbf{X}_3 = \begin{pmatrix} \gamma_3 \\ \gamma_2 \end{pmatrix}\frac{(t-a)^3}{3!}, \qquad \mathbf{X}_4 = \begin{pmatrix} \gamma_4 \\ \gamma_3 \end{pmatrix}\frac{(t-a)^4}{4!},$$

where $\gamma_1 = \lambda(Tc_1 - c_2 - c_1^3)$, $\gamma_2 = \lambda(T\gamma_1 - c_1 - 3c_1^2\gamma_1)$, $\gamma_3 = \lambda(T\gamma_2 - \gamma_1 - 3c_1^2\gamma_2 - 6c_1\gamma_1^2)$, and $\gamma_4 = \lambda(T\gamma_3 - \gamma_2 - 3c_1^2\gamma_3 - 18c_1\gamma_1\gamma_2 - 6\gamma_1^3)$.

Actual physiological data [20] are the values $\lambda = 40$, $T = 0.1575$, $c_1 = 0.45$, and $c_2 = -0.02025$. In this case, we have $\gamma_1 = 0.0$, $\gamma_2 = -18$, $\gamma_3 = 324$, and $\gamma_4 = -5112$.

Substituting in the above formulas, we obtain the approximate solution

$$\phi_5 = 0.45 - 9t^2 + 54t^3 - 213t^4, \qquad \psi_5 = -0.02025 + 0.45t - 3t^3 + 13.5t^4.$$

For values of $t \leq 1$, this gives excellent agreement with the numerical solution. To accelerate the convergence for large values of t, we divide the t interval into subintervals $[t_k, t_{k+1}]$ and use the same approximate solution formula (7) by taking $\mathbf{C} = \mathbf{U}(t_k)$, $a = t_k$ to compute $\mathbf{U}(t_{k+1})$. Although this involves discretization, the nonlinearity is still preserved, and therefore the decomposition solution compares well with the Runge-Kutta numerical solution, which by definition is numerical linearization of the original system of equations. This is clear in Table 3, which shows a comparison between the two solutions in the interval [0,5].

Table 3. The Runge-Kutta solution $(\phi_{\mathrm{RK}}, \psi_{\mathrm{RK}})$ and the decomposition solution (ϕ_5, ψ_5).

t	ϕ_{RK}	ψ_{RK}	ϕ_5	ψ_5
.05	.432785	.00195655	.432919	.00195938
.50	−.455848	.0328329	−.457448	.0328475
1.0	.436235	−.0448053	.435876	−.0446845
1.5	−.390065	.0558809	−.388992	.0556261
2.0	.313953	−.0653096	.316124	−.0650413
2.5	−.216529	.0724026	−.2200838	.0722067
3.0	.11442	−.0768059	.118777	−.0767065
3.5	−.0207792	.078582	−.0247071	.0785796
4.0	−.0594135	−.0780391	−.0558401	−.0781363
4.5	.126576	.0755382	.1232700	.0757390
5.0	−.182987	−.0713976	−.1798830	−.0717079

5. CONCLUSIONS

In this paper, we compare between the Adomian decomposition method and the Runge-Kutta method applied to systems of ordinary differential equations. For illustration purposes, we consider two examples, one is linear and the other is nonlinear, to show the computational accuracy. It may be concluded that the Adomian methodology is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of systems of differential equations.

As expected, the numerical solutions in the tables clearly indicate that the decomposition scheme obtains efficient results much closer to the actual solutions and also easier to use than the conventional method. Numerical approximations show a high degree of accuracy, and in most cases $\mathbf{U}_K$, the K-term approximation, is accurate for quite low values of K. It is also worth noting that the advantage of the decomposition method is that it does not require linearization, discretization, or perturbation, and it does not need closure approximation, smallness assumptions, or physically unrealistic white noise assumption in the nonlinear stochastic case [1,2,21].

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