EXTREMAL THEOREMS ON DIVISORS OF A NUMBER*

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Received 21 December 1976
Revised 7 March 1978

For every positive integer N, we determine the maximum sizes of collections C and C' of divisors of N subject to the following restrictions. If any two numbers in C are coprime, then their least common multiple must be N. If any k numbers (k arbitrary) in C' have the greatest common divisor 1, then their least common multiple must be N. For the special case that N is square free, the maximum sizes of C and C' have previously been determined. Since every number can be thought of as a multiset of primes, this work can be regarded as an extension of theorems on families of finite sets to families of multisets.

1. Introduction

Many extremal theorems on intersection and union of sets have been extended to more general kinds of lattices. Among the most important examples is the lattice of divisors of a fixed number, ordered by the relation of divisibility. Since every positive integer can be thought of as a multiset of primes, the lattice structure of the subsets of a finite set coincides with that of divisors of a square free number. One would naturally like to generalize results on subsets for divisors. A first such result is DeBruijn, Van Tengbergen and Kruijswijk's theorem [2] on the maximum size of a collection of divisors, none of which dividing another. This is a generalization of Sperner's theorem [10] on subsets. Some other results in this direction are contained in [3, 5, 6, 7, 9].

Let S denote the set of integers 1, 2, ..., n. Consider four kinds of families $F_1$, $F_2$, $F_3$, and $F_4$ of subsets of S subject to the following restrictions:

1. If $A$, $B$ are members of $F_1$, then $A \cap B \neq \emptyset$.
2. If $A_1, A_2, \ldots, A_k$ (k ≥ 2, arbitrary) are any member of $F_2$, then $A_1 \cap A_2 \cap \ldots \cap A_k \neq \emptyset$. In other words, there is an element common to all members of $F_2$.
3. If $A$, $B$ are members of $F_3$ with $A \cap B = \emptyset$, then $A \cup B = S$.
4. If $A_1, A_2, \ldots, A_k$ (k ≥ 2, arbitrary) are any members of $F_4$ with $A_1 \cap A_2 \cap \ldots \cap A_k = \emptyset$, then $A_1 \cup A_2 \cup \ldots \cup A_k = S$.

The maximum sizes of these families are all known results and are listed in

* Research supported in part by NSF Grant MPS 77-013533.
We let $s$ be the positive integer with prime decomposition $p_1^{e_1}p_2^{e_2}\cdots p_n^{e_n}$, where $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$. We consider four kinds of collections $f_1$, $f_2$, $f_3$, and $f_4$ of divisors of $s$ subject to conditions analogous to the above:

1. If $a$, $b \in f_1$, then g.c.d. $(a, b) \neq 1$.
2. If $a_1, a_2, \ldots, a_k \in f_2$, $k \geq 2$, then g.c.d. $(a_1, a_2, \ldots, a_k) \neq 1$. In other words, there is a common factor to all numbers of $f_2$.
3. If $a, b \in f_3$ with g.c.d. $(a, b) = 1$, then l.c.m. $(a, b) = s$.
4. If $a_1, a_2, \ldots, a_k \in f_4$, $k \geq 2$, and g.c.d. $(a_1, a_2, \ldots, a_k) = 1$, then l.c.m. $(a_1, a_2, \ldots, a_k) = s$.

The main purpose of this paper is to determine the maximum sizes for $f_3$ and $f_4$ (see Theorem 2.5 and Theorem 3.12, respectively). For $f_1$ and $f_2$, the maximum sizes are well-known; but they are also being discussed here because of two reasons. First, the above eight conditions are closely related to one another with some of them implying others. Secondly, we intend to contrast the size differences among the $f$'s with the corresponding differences among the $F$'s.

Another purpose of this paper is the exposition of a new graph-theoretical method of extremal enumerations. This is contained in Section 3.

Table 1. The maximum number of subsets or divisors in each family

| $F_1$ | $2^{n-1}$ |
| $f_1$ | $\frac{1}{2}\sum_{i=2}^{n} \max\left\{\prod_{i \in S} e_i \prod_{i \in S^c} e_i\right\}$ |
| $F_2$ | $2^{n-1}$ |
| $f_2$ | $e_1 \prod_{i \neq 2} (e_i + 1)$ |
| $F_3$ | $2^{n-1} + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right)$ |
| $f_3$ | $\frac{1}{2}\sum_{i=2}^{n} \max\left\{\prod_{i \in S} e_i \prod_{i \in S^c} e_i\right\}$ when $s$ is not square free |
| $F_4$ | $2^{n-1} + 1$ |
| $f_4$ | $e_1 \prod_{i \neq 2} (e_i + 1)$ when $s$ is not square free |

The maximum size of $F_1$ is well-known. Since $F_1$ does not contain any pair of complementary subsets, $|F_1|$ can be $\frac{1}{2}$ of the number of all subsets. This upper bound of $|F_1|$ can be achieved by taking all subsets containing a fixed element.

Similarly $f_1$ can not contain a pair of divisors $a, b$ such that $\{i : p_i \text{ divides } a_i\}$ and $\{i : p_i \text{ divides } b_i\}$ are complementary subsets in $s$. Therefore the table value for $|f_1|$ is an upper bound; and this upper bound can easily be achieved (see Example 2.2). This result was first mentioned in [5].

The same arguments for $F_1$ hold for $F_2$. 

Table 1. The maximum number of subsets or divisors in each family
For all numbers in \( f_2 \), there exists a common prime factor, say, \( p_1 \). Thus
\[
|f_2| \leq e_1 \prod_{i \neq 1} (e_i + 1)
\]
\[
= \prod_{i=1}^{n} (e_i + 1) \left/ \left(1 + \frac{1}{e_1}\right)\right.
\]
\[
\leq \prod_{i=1}^{n} (e_i + 1) \left/ \left(1 + \frac{1}{e_1}\right)\right.
\]
\[
= e_1 \prod_{i=2}^{n} (e_i + 1).
\]

This bound can be achieved by taking all numbers divisible by \( p_1 \). The sizes of \( F_1 \) and \( F_2 \) have the same bound, but \( |f_2| \) is more limited than \( |f_1| \) except when \( e_1 \geq \prod_{i=2}^{n} e_i \). It is an interesting coincidence that \( e_1 \prod_{i=2}^{n} (e_i + 1) \) is also the smallest size of a maximal \( f_1 \) (see [3, Theorem 4] and [6]).

Erdős, Ko and Rado [4] showed that, if \( F \) is a Sperner family of pairwise intersecting subsets of \( S \) with the sizes of all members bounded by a fixed number \( k \leq \frac{1}{2}n \), then \( F \) consists of at most \( \binom{n}{k-1} \) members. Therefore a Sperner system consisting of pairs of complementary subsets can have the size no more than \( 2\binom{n}{k-1} \). Actually this statement is a special case of a result in [1]. From this, Kleitman observed that the maximum size of \( F_3 \) is \( 2^{n-1} + \left(\binom{n}{n-1}\right) \). This number substantially exceeds the bound \( 2^{n-1} \) for \( |F_1| \). As a contrast the extremum size of \( f_3 \) is identical with that of \( f_1 \) except, of course, for the degenerate case when \( s \) is square free. This fact is proved in the next section.

Finally the maximum size \( 2^{n-1} + 1 \) for \( F_4 \) was proved by Li [8] with a method of graph representation. A similar method will be adapted in Section 3 for determining the maximum size of \( f_4 \). When \( s \) is not square free, the bound of \( |F_4| \) turns out to be no greater than that of \( |f_3| \).

All notations in this section will be employed throughout this article. Unless specifically indicated, we shall assume that \( e_1 > 1 \) henceforth.

2. The maximum size of \( f_3 \)

**Definition 2.1.** (1) For every divisor \( d \) of \( s \), define the *support* of \( d \) to be the set \( \{i : p_i \text{ divides } d\} \). Clearly this is a subset of \( S \).

(2) For every subset \( I \) of \( S \), denote \( e_i = \prod_{i \in I} e_i \) and \( p^I = \prod_{i \in I} p_i^e_i \). So \( p^I \) is a divisor of \( s \).

(3) A *proper* divisor of a number means a divisor which \( s \) not the number itself.

**Example 2.2.** Let \( f_3 \) be the collection of all divisors of \( s \) which are supported on those sets \( I \) with the following property:

\(* e_i > e_S \quad \text{or} \quad e_i = e_S \quad \text{and} \quad I \neq I. \)
Between any pair of complementary subsets of $S$, exactly one of them satisfies (*). Hence the size of $f$ is equal to

$$\frac{1}{2} \sum_{I \subseteq S} \max \{e_i, e_{S \setminus I}\},$$

which is the value given in Table 1 for $|f_1|$ and $|f_3|$. Clearly $f$ is a collection satisfying the conditions (1') and (3'). We shall see that $f$ has the maximum size among all possible collections $f_3$.

**Lemma 2.3.** Among all possible collections $f_3$ with the maximum size, there is one satisfying the following property: For every subset $I$ subject to the condition (*), if $p^{S \setminus I} \in f_3$, then $f_3$ contains a proper divisor of $p^{S \setminus I}$.

**Proof.** To start with, let $f$ be any maximum sized $f_3$. We shall argue inductively on the number of subsets $I$ satisfying (*) such that $p^{S \setminus I} \in f$ but $f$ contains no proper divisor of $p^{S \setminus I}$.

If no such subsets exist, we have nothing to prove. If there exists such an $I$, define the new collection

$$f' = \left( \bigcap_{I \in f} p_i \right) \cup f \setminus \{p^{S \setminus I}\}.$$

We want to show that $f'$ satisfies (3') and that $|f'| = |f|$. From the assumption $e_i > 1$ and the (*) property of $I$, there exists $i \in I$ with $e_i > 1$. So the two numbers $\prod_{i \in I} p_i$ and $p^{S \setminus I}$ can not be simultaneously contained in any collection $f_3$. This shows the equality between $|f|$ and $|f'|$. On the other hand, since no member of $f \setminus \{p^{S \setminus I}\}$ is a divisor of $p^{S \setminus I}$, the new collection $f'$ satisfies (3'). The proof can now be completed by induction.

**Lemma 2.4.** Among all possible $f_3$ with the maximum size, there is one consisting of only those divisors with supports satisfying (*).

**Proof.** Let $f$ be a maximum sized $f_3$ with the property stated in the preceding lemma. We shall argue inductively on the number of those subsets of $S$ which satisfy (*) and are complementary to supports of members of $f$.

If no such subsets exist, we have nothing to prove. So we assume there exist such subsets and let $I$ be a maximal one among them. At least the following informations are known about $I$ so far.

(i) $e_I = e_{S \setminus I}$.
(ii) if $p^{S \setminus I} \in f$, then $f$ contains a proper divisor of $p^{S \setminus I}$.
(iii) there exists a number in $f$ with support $S \setminus I$.
(iv) no number in $f$ has the support strictly contained in $S \setminus I$.

From (ii), (iii) and (iv), there exists a number in $f$ with support $S \setminus I$ which is not $p^{S \setminus I}$. Hence no member of $f$ has the support $I$. If we remove all the numbers with
the support $S \setminus I$ from the collection $f$ and add into it all the numbers with support $I$, the resulting collection still satisfies $(3')$ because of $(iv)$. Also because of $(iv)$, the new collection has the size at least $|f| - e_{S \setminus I} + e_I$, which is no less than $|f|$ by $(i)$. The proof can now be finished by induction.

We conclude the results of Example 2.2 and Lemma 2.4 in the following:

**Theorem 2.5.** Let $N$ be the positive integer with prime decomposition $N = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where $e_1 \geq e_2 \geq \cdots \geq e_r \geq 1$. Consider the collections of divisors of $N$ such that any two coprime number in the same collection must have their least common multiple equal to $N$. The maximum size of such a collection is $2^{e_1} + \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)$ when $N$ is square free and is otherwise equal to

$$\frac{1}{2} \sum_{t \in \{1, \ldots, n\}} \max \left\{ \prod_{t \in f} c_t, \prod_{t \in \{1, \ldots, n\} \setminus t} c_t \right\}.$$ 

3. **Covering problem on directed graphs**

With respect to the prime decomposition of the number $s$, we give a definition of coverings of directed graphs. It will be proved that graphs of a certain shape have the largest possible number of coverings. The proof depends mainly on gradually deforming any given graph into the optimal shape so that the number of coverings is nondecreasing on each step of the deformation. On the other hand, the determination of the maximum size of a collection $f_d$ will be shown to be equivalent to the search for a function graph on $n$ vertices which has the maximum number of coverings. As a consequence the bound of $|f_d|$ can be determined.

Given a collection $f_d$, we associate with it a directed graph with the vertices $1, 2, \ldots, n$ and the edge prescribed in below. For every $i, j \in S$, there is an edge pointing from $i$ to $j$ if and only if every number in $f_d$ is divisible by $p_i$ or $p_j$. Claim that for every $i \in S$, there is an edge with $i$ as its initial point. Thus consider all numbers in $f_d$ that are not divisible by $p_i$. Since $s/p_i$ is a common multiple of these numbers, they must have a common factor, say, $p_j$. Then $(i, j)$ is an edge in the graph. Note that we do not require $i$ and $j$ to be distinct from each other.

The above consideration of graphs leads us to the following:

**Definition 3.1.** By a function graph on $S$ (or simply a graph on $S$) we shall refer to a directed graph with vertices $1, 2, \ldots, n$ such that every vertex is the initial point of exactly one edge. In a graph so defined there is a unique cycle in every connected component (see Fig. 1 on next page). A divisor $d$ of $s$ is said to cover an edge $(i, j)$ in a graph on $S$ if $d$ is divisible by $p_i$ or $p_j$. A divisor of $s$ is called a covering of a graph on $S$ if it covers all the edges in the graph.
For every collection $f_A$, there exists a graph on $S$ such that every number in $f_A$ is a covering. Conversely any collection of coverings of a graph on $S$ is a collection $f_A$. Therefore the maximum size of a collection $f_A$ is equal to the maximum number of coverings of a graph on $S$.

**Proof.** The first statement follows from the above discussion. To prove the converse, let $G$ be a graph on $S$ and $a_1, a_2, \ldots, a_k$ be coverings. Suppose that $p_i \not\mid \text{l.c.m.} (a_1, a_2, \ldots, a_k)$ for some $i$. Choose $j$ so that $(i, j)$ is an edge in $G$. Then we have $p_i \mid \text{g.c.d.} (a_1, a_2, \ldots, a_k)$.

**Example 3.3.** Consider the graph on $S$ with the edges $(1, 1)$, $(2, 1)$, \ldots, and $(n, 1)$. A divisor $d$ of $s$ is a covering of this graph if and only if $d$ is divisible by $p_i$. Therefore there are $\prod_{i=2}^{n} (e_i + 1)$ coverings. This number of coverings will be shown to be the largest possible among all graphs on $S$.

Let $G$ be a graph on $S$. Suppose $(j, k)$ is an edge on a cycle in $G$ with $j \neq k$. Denote by $J$ the set of all vertices in the same component of $j$ which shall be disconnected from $k$ when $j$ is removed from the graph. $J$ may possibly be empty. Let $N$ be the number of divisors of $\prod_{i=1}^{k} p_i$ that cover all the edges on $J$, and let $\tilde{N}$ be the number of those covering all the edges on $J \cup \{j\}$. With these notations, we have the following lemma which is of some interest in its own right.

**Lemma 3.4.** Let $G$ be a graph on $S$ and $C$ be a connected component containing a cycle of length $a$ least 3. Among all the vertices on this cycle, choose $j$ to maximize $\tilde{N}/N$. Suppose $(h, j)$ and $(j, i)$ are edges on the cycle. Then the graph resulting from the replacement of the edge $(i, j)$ by $(h, k)$ has at least as many coverings as $G$ does.

**Proof.** Let $G'$ denote the graph resulting from the replacement. The proof is by counting the coverings of $G'$ which do not cover $G'$ and vice versa.

Denote by $K$ the set of vertices in $C$ which shall be disconnected from $j$ when $k$ is removed from $C$. Set $I = \mathbb{N} \setminus (I \cup K \cup \{h, j, k\})$. If a number $\prod_{i=1}^{n} p_i$ covers $G$ but
not $G'$, this number must meet the following requirements:

(i) $f_e \leq e_k - 1$, $f_k = 0$, and $f_j = e_j$;
(ii) $p_k \prod_{i \in J} p_i^e$ covers all the edges on $I \cup \{h, k\}$;
(iii) $\prod_{i \in J} p_i^e$ covers all the edges on $J$;
(iv) $\prod_{i \in K} p_i^e$ covers all the edges on $K \cup \{k\}$.

Let $M$ denote the number of divisors of $p_k^{-1} \prod_{i \in J} p_i^e$ which cover all the edges on $I \cup \{h, k\}$. Then there are exactly $MN_jN_k$ coverings of $G$ that do not cover $G'$.

On the other hand, we claim there are at least $MN_jN_k$ coverings of $G'$ that do not cover $G$. Respectively let $d_i$, $d_j$, $d_K$ be divisors of $p_k^{-1} \prod_{i \in J} p_i^e$, $\prod_{i \in J} p_i^e$, $\prod_{i \in K} p_i^e$, which cover all the edges on $I \cup \{h, k\}$, $J \cup \{j\}$, $K$. Then $d_i d_j d_K p_k^e$ is a covering of $G'$ but not of $G$. This justifies the claim.

Now by the choice of the vertex $j$ on the cycle, we know that $\bar{N}_j/N_k \geq \bar{N}_j/N_k$. Hence $MN_jN_k \geq MN_jN_k$.

**Corollary 3.5.** Among the graphs on $S$ with the maximum number of coverings, there is one such that the cycle in every of its components is of length 1 or 2.

**Lemma 3.6.** Let $G$ be a graph on $S$ and $(j, k)$ be an edge. Define $J = \{i: (i, j) \text{ is an edge in } G\}$. Let $G'$ be the graph resulting from the replacement of the edge $(i, j)$ by $(i, k)$ for every $i \in J$. Suppose no element in $J$ is the terminal point of any edge in $G$. Then there are at least as many coverings of $G'$ as of $G$.

**Proof.** We may of course assume that $J$ is not empty. The proof proceeds in the same manner as in the proof of the previous lemma.

Write $K = S \setminus (J \cup \{j, k\})$. If a number $d = \prod_{i \in J} p_i^e$ covers $G$ but not $G'$, this number must meet the following requirements.

(i) $f_j = e_j$, and $f_k = 0$;
(ii) $\prod_{i \in J} p_i^e$ is a proper divisor of $\prod_{i \in J} p_i^e$;
(iii) $\prod_{i \in K} p_i^e$ covers all the edges on $K \cup \{k\}$.

Therefore $dp_K^e/p_i^e$ is a covering of $G'$ but not of $G$. Hence there are at least as many coverings of $G'$ as of $G$.

The result of Corollary 3.5 can now be sharpened as:

**Corollary 3.7.** Among the graphs on $S$ with the maximum number of coverings, there is one such that the cycle in every of its components is of length 1 or 2 and also that the terminal point of every edge is on a cycle.

The next lemma follows from the same arguments that we used in determining the maximum size of $f_2$ in Section 1.

**Lemma 3.8.** Suppose that, in a graph on $S$, all the edges have the same terminal point. Then there are at most $e_i \prod_{i = 1}^e - 1$ coverings.
Lemma 3.9. In a connected graph on $S$ with a cycle of length 2, if the two vertices on the cycle are the only terminal points of edges, then there are at most $1 + (e_1e_2 + 1) \prod_{i=3}^{n} (e_i + 1)$ coverings.

Proof. Let $G$ be such a graph and $j, k$ be the two vertices on the cycle in $G$. Define

$$J = \{ i : i \neq k, (i, j) \text{ is an edge in } G \}$$

and

$$K = \{ i : i \neq j, (i, k) \text{ is an edge in } G \}.$$ 

Clearly $S = J \cup K \cup \{j, k\}$.

Let $\prod_{i=1}^{n} p_i^j$ be any covering of $G$. Then either

$$f_{f_k} > 0,$$

or

$$p_j^k \prod_{i \in J} p_i^j = p_k^j \prod_{i \in J} p_i^j,$$

or

$$p_j^k \prod_{i \in K} p_i^j = p_k^j \prod_{i \in K} p_i^j.$$

Therefore, the number of coverings of $G$

$$= e_je_k \left[ \prod_{i \in J \cup K} (e_i + 1) + \prod_{i \in K} (e_i + 1) + \prod_{i \in J} (e_i + 1) \right]$$

$$= 1 + (e_je_k + 1) \prod_{i \in J \cup K} (e_i + 1) - \left( \prod_{i \in J} (e_i + 1) - 1 \right) \left( \prod_{i \in K} (e_i + 1) - 1 \right)$$

$$= 1 + (e_je_k + 1) \prod_{i \in J \cup K} (e_i + 1)$$

$$= 1 + \frac{e_je_k + 1}{(e_j + 1)(e_k + 1)} \prod_{i=3}^{n} (e_i + 1)$$

$$= 1 + \left( \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2}{e_j + 1} \right) \left( 1 - \frac{2}{e_k + 1} \right) \right) \prod_{i=3}^{n} (e_i + 1)$$

$$= 1 + \left( \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{2}{e_1 + 1} \right) \left( 1 - \frac{2}{e_2 + 1} \right) \right) \prod_{i=3}^{n} (e_i + 1)$$

$$= 1 + (e_1e_2 + 1) \prod_{i=3}^{n} (e_i + 1).$$
The standing assumption that \( e_1 > 1 \) implies the inequality

\[
e_1 \prod_{i=2}^{n} (e_i + 1) \geq 1 + (e_1 e_2 + 1) \prod_{i=3}^{n} (e_i + 1).
\]

So we can combine the results in the above two lemmas together as:

**Corollary 3.10.** If a connected graph on \( S \) with a cycle of length 1 or 2, if the terminal point of every edge is on the cycle, then there are at most \( e_1 \prod_{i=2}^{n} (e_i + 1) \) coverings.

**Lemma 3.11.** Let \( G \) be a graph on \( S \). Assume that the cycle in every component of \( G \) has length 1 or 2. Also assume that the terminal point of every edge in \( G \) is on a cycle. Then there are at most \( e_1 \prod_{i=2}^{n} (e_i + 1) \) coverings of \( G \).

**Proof.** From the above corollary, we need only to consider the case when \( G \) is not a connected graph. Assume, say, \( G \) is the disjoint union of two subgraphs \( G_1 \) and \( G_2 \). Let \( I_1 \) and \( I_2 \) denote the sets of vertices in these two subgraphs, respectively. Without loss of generality, we assume that \( 1 \in I_1 \) and that \( j \) is the smallest element in \( I_2 \).

Denote by \( \pi_1 \) the product of \( e_i + 1 \) over all \( i \in I_1 \setminus \{1\} \). Similarly let \( \pi_2 \) denote the product of \( e_i + 1 \) over all \( i \in I_2 \setminus \{j\} \). Applying the induction on \( n \) to Corollary 3.10, we know, when \( e_1 > 1 \), there are at most \( e_1 \pi_1 \pi_2 \) coverings of \( G \). While when \( e_1 = 1 \), there are at most \( e_1 \pi_1 (1 + \pi_2) \) coverings. In either case, the number of coverings of \( G \) is no more than \( e_1 \pi_1 e_i \pi_2 + e_1 \pi_1 \pi_2 \), which is equal to \( e_1 \prod_{i=2}^{n} (e_i + 1) \).

The results of Corollary 3.7 and Lemma 3.11 show that the graph in Example 3.3 has the maximum number of coverings among all graphs on \( S \). We conclude this section in the following.

**Theorem 3.12.** Let \( N \) be the positive integer with prime decomposition \( p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n} \), where \( e_1 \geq e_2 \geq \cdots \geq e_n \geq 1 \). Let \( C \) be a collection of divisors of \( N \) such that, if any \( k \) numbers (\( k \) arbitrary) in \( C \) have the greatest common divisor equal to 1, their least common multiple must be \( N \). When \( N \) is square free, the size of \( C \) is bounded by \( 1 + 2^{n-1} \). Otherwise it is bounded by \( e_1 \prod_{i=2}^{n} (e_i + 1) \). These bounds can be achieved when \( C \) has been properly chosen.

Corollary 3.7 and Lemma 3.9 also apply to graphs without loops, i.e., graphs without edges of the type \((v, v)\). We state the following.

**Theorem 3.13.** For a graph on \( S \) without loops the number of coverings is at most

\[
1 + (e_1 e_2 + 1) \prod_{i=3}^{n} (e_i + 1).
\]
Proof. First, this number of coverings is achieved by the graph with the edges 
\((2, 1), (3, 1), \ldots, (n, 1), \text{ and } (1, 2)\). We want to show that this number is the maximum. Because of Corollary 3.7, we need only to consider graphs in which the cycle in every component has length 2 and the terminal point of every edge is on a cycle. If a graph of this type is also connected, then Lemma 3.9 gives the upper bound on the number of coverings. If the graph is disconnected, say, it is the disjoint union of the subgroups \(G_1\) and \(G_2\), then we apply induction hypothesis on \(n \rightarrow G_1\) and \(G_2\). The arguments proceed in the same way as the proof of Lemma 3.11.

Since the paper was written, W.-C. Li has found a shortcut in the proof of that 
e_1 \cdot \prod_{i=2}^{n} (e_i + 1)\) is an upper bound of \(|f_2|\). The arguments generally follow the method in [8]. An outline is as follows. Given a collection \(f_4\), choose \(j\) such that every number in the collection is divisible by \(p_{i1}\) or \(b_{i}\). Also choose \(k\) such that every number in the collection is divisible by \(p_{i,k}\) or \(p_{k}\). We may assume that \(j \neq 1\) and \(j \neq k\), otherwise the problem can be reduced to the determination of the maximum size of \(f_2\). Now if \(k = 1\), then \(|f_2| \leq (e_1 e_1 + 2) \cdot \prod_{i=1}^{n} (e_i + 1)\). While if \(k \neq 1\), then \(|f_2| \leq (e_1 e_1 e_1 + e_1 e_1 + e_1 + e_1 + 1) \cdot \prod_{i=1}^{n} (e_i + 1)\). In either case, we obtain the bound \(e_1 \cdot \prod_{i=1}^{n} (e_i + 1)\).

References