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Polynomials of Meixner's type in infinite dimensions—Jacobi fields and orthogonality measures

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Abstract

The classical polynomials of Meixner's type—Hermite, Charlier, Laguerre, Meixner, and Meixner–Pollaczek polynomials—are distinguished through a special form of their generating function, which involves the Laplace transform of their orthogonality measure. In this paper, we study analogs of the latter three classes of polynomials in infinite dimensions. We fix as an underlying space a (non-compact) Riemannian manifold X and an intensity measure σ on it. We consider a Jacobi field in the extended Fock space over $L^2(X; \sigma)$, whose field operator at a point $x \in X$ is of the form $\partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x + \partial_x^\dagger \partial_x \partial_x$, where λ is a real parameter. Here, ∂_x and ∂_x^\dagger are, respectively, the annihilation and creation operators at the point x . We then realize the field operators as multiplication operators in $L^2(\mathcal{D}'; \mu_\lambda)$, where \mathcal{D}' is the dual of $\mathcal{D} = C_0^\infty(X)$, and μ_λ is the spectral measure of the Jacobi field. We show that μ_λ is a gamma measure for $|\lambda| = 2$, a Pascal measure for $|\lambda| > 2$, and a Meixner measure for $|\lambda| < 2$. In all the cases, μ_λ is a Lévy noise measure. The isomorphism between the extended Fock space and $L^2(\mathcal{D}'; \mu_\lambda)$ is carried out by infinite-dimensional polynomials of Meixner's type. We find the generating function of these polynomials and using it, we study the action of the operators ∂_x and ∂_x^\dagger in the functional realization.

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1. Introduction

In his classical work [26], Meixner considered the following problem: Suppose that functions $f(z)$ and $\Psi(z)$ can be expanded in a formal power series of $z \in \mathbb{C}$ and suppose that $f(0) = 1$, $\Psi(0) = 0$, and $\Psi'(0) = 1$. Then, the equation

$$G(x, z) := \exp(x\Psi(z))f(z) = \sum_{n=0}^{\infty} \frac{P^{(n)}(x)}{n!} z^n \tag{1.1}$$

generates a system of polynomials $P^{(n)}(x)$, $n \in \mathbb{Z}_+$, with leading coefficient 1. (These polynomials are now called Sheffer polynomials.) Find all polynomials of such type which are orthogonal with respect to some probability measure μ on \mathbb{R} . To solve this problem, Meixner essentially used the two following facts. First, by the Favard theorem, a system of polynomials $P^{(n)}(x)$, $n \in \mathbb{Z}_+$, with leading coefficient 1 is orthogonal if and only if these polynomials satisfy the recurrence formula

$$\begin{aligned} xP^{(n)}(x) &= P^{(n+1)}(x) + a_n P^{(n)}(x) + b_n P^{(n-1)}(x), \\ n \in \mathbb{Z}_+, P^{(-1)}(x) &:= 0 \end{aligned} \tag{1.2}$$

with real numbers a_n and positive numbers b_n ; or equivalently, the polynomials $P^{(n)}(x)$ determine the infinite Jacobi matrix with the elements a_n on the main diagonal and the elements $\sqrt{b_n}$ on the upper and lower diagonals. And second, as follows from (1.1),

$$\Psi^{-1}(D)P^{(n)}(x) = nP^{(n-1)}(x), \quad n \in \mathbb{N}, \tag{1.3}$$

where Ψ^{-1} is the inverse function of Ψ and $D := \frac{d}{dx}$. Meixner showed that the solution of this problem is completely determined by the equations

$$\lambda = a_n - a_{n-1}, \quad n \in \mathbb{N}, \quad \kappa = \frac{b_n}{n} - \frac{b_{n-1}}{n-1}, \quad n \geq 2,$$

where λ and κ are some parameters. If $\kappa = 0$, we have to distinguish the two following cases:

- (I) $\lambda = 0$, without loss of generality, we then get $a_n = 0$ and $b_n = n$ in (1.2), $P^{(n)}(x)$ are the Hermite polynomials, μ is the standard Gaussian distribution on \mathbb{R} .
- (II) $\lambda \neq 0$, so that $a_n = \lambda n$, $b_n = n$, $P^{(n)}(x)$ are the Charlier polynomials, μ is a centered Poisson distribution on \mathbb{R} .

Let now $\kappa \neq 0$ and we set $\kappa = 1$ for simplicity of notations. We then get $a_n = \lambda n$ and $b_n = n^2$. We introduce two quantities α and β through the equation

$$1 + \lambda z + z^2 = (1 - \alpha z)(1 - \beta z). \tag{1.4}$$

We now have to distinguish the three following cases:

- (III) $|\lambda| = 2$, so that $\alpha = \beta = \pm 1$, $P^{(n)}(x)$ are the Laguerre polynomials, μ is a centered gamma distribution.
- (IV) $|\lambda| > 2$, so that $\alpha \neq \beta$, both real, $P^{(n)}(x)$ are the Meixner polynomials (of the first kind), which are orthogonal with respect to a centered Pascal (negative binomial) distribution.
- (V) $|\lambda| < 2$, so that $\alpha \neq \beta$, both complex conjugate, $P^{(n)}(x)$ are the Meixner polynomials of the second kind, or the Meixner–Pollaczek polynomials in other terms. These are orthogonal with respect to a measure μ obtained by centering a probability measure of the form $C \exp(ax) |\Gamma(1 + imx)|^2 dx$, where $a \in \mathbb{R}$, $m > 0$, and C is the normalizing constant. We will call it a Meixner measure, though there seems to be no commonly accepted name for it.

In all the above cases, the generating function $G(x, z)$ defined in (1.1) can be represented as $G(x, z) = \exp(x\Psi(z))/L_\mu(\Psi(z))$, where $L_\mu(z) := \int_{\mathbb{R}} e^{zx} \mu(dx)$ is the extension of the Laplace transform of the measure μ defined in a neighborhood of zero in \mathbb{C} .

In the present paper, we will study analogs of polynomials of Meixner's type and their orthogonality measures in infinite dimensions. In the case of the Gaussian and Poisson measures, such a theory is, of course, well studied; we refer to e.g. [6,17] for the Gaussian case and to e.g. [20,22] for the Poisson case. Notice that, in both cases, the Fock space and the corresponding Jacobi fields of operators in it play a fundamental role (see [3,7,25] for the notion of a Jacobi field in the Fock space). In particular, the field operator at a point $x \in X$, where X is an underlying space, has the form $\partial_x^\dagger + \partial_x$ in the Gaussian case, and $\partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x$ in the Poisson case. Here, ∂_x and ∂_x^\dagger are the annihilation and creation operators at the point x , respectively.

Concerning the gamma case (III), an infinite-dimensional analog of the Laguerre polynomials and the corresponding Jacobi field was studied in [21,23]. The polynomials are now orthogonal with respect to the (infinite-dimensional) gamma measure, which is a special case of a compound Poisson measure. Since such a measure does not possess the chaotic decomposition property, instead of the usual Fock space one has to use the so-called extended Fock space. This space, on the one hand, extends the usual Fock space and, on the other hand, still has some similarities with it. The field operator at a point $x \in X$ has the form $\partial_x^\dagger + 2\partial_x^\dagger \partial_x + \partial_x + \partial_x^\dagger \partial_x \partial_x$. In [8], the structure of the extended Fock space was discussed in detail, and in [9], it was shown that the extended Fock space decomposition of the Gamma process can be thought of as an expansion of any L^2 -random variable in multiple integrals constructed by using a family of resolutions of the identity in the extended Fock space. We also refer to the recent paper [30] and the references therein, where many other properties of the gamma measure are discussed in detail.

As for cases (IV) and (V), the role of the orthogonality measure should be played by (infinite-dimensional) Pascal and Meixner measures (processes). These processes in the case $X = \mathbb{R}_+$, both Lévy, were introduced in [11,29], respectively. In [16], the Meixner process was proposed for a model for risky asserts and an analog of the Black–Sholes formula was established. In [27] (see also the recent book [28]), the gamma, Pascal, and Meixner processes served as main examples of a chaotic representation for every square-integrable random variable in terms of the orthogonalized Teugels martingales related to the process. Though the one-dimensional polynomials of Meixner’s type were used in this work in order to carry out the orthogonalization procedure of the Teugels martingales (which, in turn, are the centered power jump processes related to the original process), infinite-dimensional polynomials corresponding to these processes have not appeared in this work; furthermore, they were mentioned as an open problem in [28].

The contents of the present paper is as follows. In Section 2, we fix as an underlying space X a smooth (non-compact) Riemannian manifold and an intensity measure σ on it. We consider a Jacobi field in the extended Fock space over $L^2(X; \sigma)$, whose field operator at a point $x \in X$ has the form $\partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x + \partial_x^\dagger \partial_x \partial_x$, where λ is a fixed real parameter. Using ideas of [7,21,25], we construct via the projection spectral theorem [6] a Fourier transform in generalized joint eigenvectors of the Jacobi field. This gives us a unitary operator I_λ between the extended Fock space and the space $L^2(\mathcal{D}'; \mu_\lambda)$, where \mathcal{D}' is the dual space of $\mathcal{D} := C_0^\infty(X)$ with respect to the zero space $L^2(X; \sigma)$, and μ_λ is the spectral measure of the Jacobi field, i.e., the image of any field operator under I_λ is a multiplication operator in $L^2(\mathcal{D}'; \mu_\lambda)$. The μ_λ is a gamma measure for $|\lambda| = 2$, a Pascal measure for $|\lambda| > 2$, and a Meixner measure for $|\lambda| < 2$, in the sense that, for any bounded $A \subset X$, the (naturally defined) random variable $\langle \cdot, \chi_A \rangle$ has a corresponding one-dimensional distribution. Furthermore, for $|\lambda| \geq 2$ μ_λ is a compound Poisson measure, and for $|\lambda| < 2$ μ_λ is a Lévy noise measure. In particular, for $X = \mathbb{R}$ we obtain the gamma, Pascal, and Meixner processes, respectively.

Next, under the unitary I_λ , the image of any vector $f^{(i)} \in \mathcal{D}_\mathbb{C}^{\otimes i}$ is a continuous polynomial $\langle : \omega^{\otimes i} :_\lambda, f^{(i)} \rangle$ of the variable $\omega \in \mathcal{D}'$, which may be understood as an infinite-dimensional polynomial of Meixner’s type, since $\langle : \omega^{\otimes i} :_\lambda, \chi_A^{\otimes i} \rangle = P_{\lambda,A}^{(i)}(\langle \omega, \chi_A \rangle)$, where $P_{\lambda,A}^{(i)}(\cdot)$ is a one-dimensional polynomial of Meixner’s type.

In Section 3, we identify the generating function $G_\lambda(\omega, \varphi) := \sum_{n=0}^\infty \frac{1}{n!} \langle : \omega^{\otimes n} :_\lambda, \varphi^{\otimes n} \rangle$, $\omega \in \mathcal{D}'$, $\varphi \in \mathcal{D}_\mathbb{C}$, and show that $G_\lambda(\omega, \varphi) = \exp(\langle \omega, \Psi_\lambda(\varphi) \rangle) / \ell_\lambda(\Psi_\lambda(\varphi))$, where ℓ_λ is the extension of the Laplace transform of the measure μ_λ defined in a neighborhood of zero in $\mathcal{D}_\mathbb{C}$, and Ψ_λ is the same function as Ψ in (1.1).

Finally, in Section 4, using results of [22,24], we introduce a test space (\mathcal{D}_λ) consisting of those functions on \mathcal{D}' which may be extended to entire functions on $\mathcal{D}'_\mathbb{C}$ of first order of growth and of minimal type. This space is densely and continuously embedded into $L^2(\mathcal{D}'; \mu_\lambda)$. We then study the action of the operators $\partial_x : (\mathcal{D}_\lambda) \rightarrow (\mathcal{D}_\lambda)$ and $\partial_x^\dagger : (\mathcal{D}_\lambda) \rightarrow (\mathcal{D}_\lambda)^*$, where $(\mathcal{D}_\lambda)^*$ is the dual of (\mathcal{D}_λ) . We note that,

analogously to (1.3), we have $\partial_x = \Psi_\lambda^{-1}(\nabla_x)$, where ∇_x is the Gâteaux derivative in direction δ_x . We obtain explicit formulas for the operators ∂_x and $\int_X \sigma(dx) \xi(x) \partial_x^\dagger$, $\xi \in \mathcal{D}$. It should be stressed that, for the latter operator in the case $|\lambda| < 2$, the possibility of a (unique) extension of a test function on \mathcal{D}' to a function on \mathcal{D}'_C plays a principle role.

In a forthcoming paper, we will study a connection between the extended Fock space decomposition of $L^2(\mathcal{D}'; \mu_\lambda)$ obtained in this paper and the chaotic decomposition of this space in the case $X = \mathbb{R}$ as in [27]. Finally, we note that one can also study a more general model where, in the field operator at a point $x \in X$, the value of the parameter λ depends on x . Then, the corresponding noise will be with independent values and at each point $x \in X$ its properties will be the same as the properties of the noise at the point x under $\mu_{\lambda(x)}$.

2. Meixner’s Jacobi field and its spectral measures

Let X be a complete, connected, oriented C^∞ (non-compact) Riemannian manifold and let $\mathcal{B}(X)$ be the Borel σ -algebra on X . Let σ be a Radon measure on $(X, \mathcal{B}(X))$ that is non-atomic, i.e., $\sigma(\{x\}) = 0$ for every $x \in X$ and non-degenerate, i.e., $\sigma(O) > 0$ for any open set $O \subset X$. (We note the assumption of the non-degeneracy of σ is non-essential and the results below may be generalized to the case of a degenerate σ .) Note that $\sigma(A) < \infty$ for each $A \in \mathcal{O}_c(X)$ —the set of all open sets in X with compact closure.

We denote by \mathcal{D} the space $C_0^\infty(X)$ of all real-valued infinite differentiable functions on X with compact support. This space may be naturally endowed with a topology of a nuclear space, see e.g. [10] for the case $X = \mathbb{R}^d$ and e.g. [14] for the case of a general Riemannian manifold. We recall that

$$\mathcal{D} = \text{proj lim}_{\tau \in T} \mathcal{H}_\tau. \tag{2.1}$$

Here, T denotes the set of all pairs (τ_1, τ_2) with $\tau_1 \in \mathbb{Z}_+$ and $\tau_2 \in C^\infty(X)$, $\tau_2(x) \geq 1$ for all $x \in X$, and $\mathcal{H}_\tau = \mathcal{H}_{(\tau_1, \tau_2)}$ is the Sobolev space on X of order τ_1 weighted by the function τ_2 , i.e., the scalar product in \mathcal{H}_τ , denoted by $(\cdot, \cdot)_\tau$, is given by

$$(f, g)_\tau = \int_X \left(f(x)g(x) + \sum_{i=1}^{\tau_1} \langle \nabla^i f(x), \nabla^i g(x) \rangle_{T_x(X)^{\otimes i}} \right) \tau_2(x) dx, \tag{2.2}$$

where ∇^i denotes the i th (covariant) gradient, and dx is the volume measure on X . For $\tau, \tau' \in T$, we will write $\tau' \geq \tau$ if $\tau'_1 \geq \tau_1$ and $\tau'_2(x) \geq \tau_2(x)$ for all $x \in X$.

The space \mathcal{D} is densely and continuously embedded into $L^2(X; \sigma)$. As easily seen, there always exists $\tau_0 \in T$ such that \mathcal{H}_{τ_0} is continuously embedded into $L^2(X; \sigma)$. We denote $T' := \{\tau \in T : \tau \geq \tau_0\}$ and (2.1) holds with T replaced by T' . In what follows, we will just write T instead of T' . Let $\mathcal{H}_{-\tau}$ denote the dual space of \mathcal{H}_τ with respect

to the zero space $\mathcal{H} := L^2(X; \sigma)$. Then $\mathcal{D}' = \text{ind} \lim_{\tau \in T} \mathcal{H}_{-\tau}$ is the dual of \mathcal{D} with respect to \mathcal{H} , and we thus get the standard triple

$$\mathcal{D}' \supset \mathcal{H} \supset \mathcal{D}.$$

The dual pairing between any $\omega \in \mathcal{D}'$ and $\xi \in \mathcal{D}$ will be denoted by $\langle \omega, \xi \rangle$.

Following [21], we define, for each $n \in \mathbb{N}$, an n -particle extended Fock space over \mathcal{H} , denoted by $\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})$. Under a loop κ connecting points x_1, \dots, x_m , $m \geq 2$, we understand a class of ordered sets $(x_{\pi(1)}, \dots, x_{\pi(m)})$, where π is a permutation of $\{1, \dots, n\}$, which coincide up to a cyclic permutation. We put $|\kappa| = m$. We will also interpret a set $\{x\}$ as a “one-point” loop κ , i.e., a loop that comes out of x , $|\kappa| = 1$. Let $\alpha_n = \{\kappa_1, \dots, \kappa_{|\alpha_n|}\}$ be a collection of loops κ_j that connect points from the set $\{x_1, \dots, x_n\}$ so that every point $x_i \in \{x_1, \dots, x_n\}$ goes into one loop $\kappa_j = \kappa_{j(i)}$ from α_n . Here, $|\alpha_n|$ denotes the number of the loops in α_n , evidently $n = \sum_{j=1}^{|\alpha_n|} |\kappa_j|$. Let A_n stand for the set of all possible collections of loops α_n over the points $\{x_1, \dots, x_n\}$. (We note that the set A_n contains $n!$ elements [21, Remark 2.1].) Every $\alpha_n \in A_n$ generates the following continuous mapping

$$\begin{aligned} \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n} \ni f^{(n)} &= f^{(n)}(x_1, \dots, x_n) \\ &\mapsto f_{\alpha_n}^{(n)}(\underbrace{x_1, \dots, x_1}_{|\kappa_1| \text{ times}}, \underbrace{x_2, \dots, x_2}_{|\kappa_2| \text{ times}}, \dots, \underbrace{x_{|\alpha_n|}, \dots, x_{|\alpha_n|}}_{|\kappa_{|\alpha_n|}| \text{ times}}) \in \mathcal{D}_{\mathbb{C}}^{\otimes |\alpha_n|}, \end{aligned} \tag{2.3}$$

where the lower index \mathbb{C} denotes complexification of a real space and the symbol $\hat{\otimes}$ stands for the symmetric tensor power. We define a scalar product on $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$ by

$$(f^{(n)}, g^{(n)})_{\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})} = \sum_{\alpha_n \in A_n} \int_{X^{|\alpha_n|}} (\overline{f^{(n)}} g^{(n)})_{\alpha_n} d\sigma^{\otimes |\alpha_n|}, \tag{2.4}$$

where $\overline{f^{(n)}}$ is the complex conjugate of $f^{(n)}$. Let $\overline{\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})}$ be the closure of $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$ in the norm generated by (2.4).

The extended Fock space $\mathcal{F}_{\text{ext}}(\mathcal{H})$ over \mathcal{H} is defined as a weighted direct sum of the spaces $\overline{\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})}$:

$$\mathcal{F}_{\text{ext}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \overline{\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})} n!, \tag{2.5}$$

where $\overline{\mathcal{F}_{\text{ext}}^{(0)}(\mathcal{H})} = \mathbb{C}$ and $0! = 1$. That is, $\mathcal{F}_{\text{ext}}(\mathcal{H})$ consists of sequences $f = (f^{(0)}, f^{(1)}, f^{(2)}, \dots)$ such that $f^{(n)} \in \overline{\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})}$ and

$$\|f\|_{\mathcal{F}_{\text{ext}}(\mathcal{H})}^2 = \sum_{n=0}^{\infty} \|f^{(n)}\|_{\overline{\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})}}^2 n! < \infty.$$

We will always identify any $f^{(n)} \in \mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})$ with the element $(0, \dots, 0, f^{(n)}, 0, \dots)$ of $\mathcal{F}_{\text{ext}}(\mathcal{H})$.

Note that the usual Fock space $\mathcal{F}(\mathcal{H})$ can be considered as a subspace of $\mathcal{F}_{\text{ext}}(\mathcal{H})$ generated by functions $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$ such that $f^{(n)}(x_1, \dots, x_n) = 0$ if $x_i = x_j$ for some $i, j \in \{1, \dots, n\}$, $i \neq j$.

Let $\mathcal{F}_{\text{fin}}(\mathcal{D})$ denote the topological direct sum of the spaces $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$, i.e., $\mathcal{F}_{\text{fin}}(\mathcal{D})$ consists of all sequences $f = (f^{(0)}, f^{(1)}, \dots, f^{(m)}, 0, 0, \dots)$ such that $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$ and the convergence in $\mathcal{F}_{\text{fin}}(\mathcal{D})$ means uniform finiteness and the coordinate-wise convergence. Since each space $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$ is nuclear, so is $\mathcal{F}_{\text{fin}}(\mathcal{D})$. As easily seen, the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is densely and continuously embedded into $\mathcal{F}_{\text{ext}}(\mathcal{H})$.

For each $\xi \in \mathcal{D}$, let $a^+(\xi)$ be the standard creation operator defined on $\mathcal{F}_{\text{fin}}(\mathcal{D})$:

$$a^+(\xi)f^{(n)} = \xi \hat{\otimes} f^{(n)}, \quad f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}, \quad n \in \mathbb{Z}_+.$$

A simple calculation shows that the adjoint operator of $a^+(\xi)$ in $\mathcal{F}_{\text{ext}}(\mathcal{H})$, restricted to $\mathcal{F}_{\text{fin}}(\mathcal{D})$, is given by the formula

$$a^-(\xi) = (a^+(\xi))^* \upharpoonright \mathcal{F}_{\text{fin}}(\mathcal{D}) = a_1^-(\xi) + a_2^-(\xi),$$

where $a_1^-(\xi)$ is the standard annihilation operator:

$$(a_1^-(\xi)f^{(n)})(x_1, \dots, x_{n-1}) = n \int_X \xi(x)f^{(n)}(x, x_1, \dots, x_{n-1})\sigma(dx),$$

and $a_2^-(\xi)$ is given by

$$(a_2^-(\xi)f^{(n)})(x_1, \dots, x_{n-1}) = n(n-1)(\xi(x_1)f^{(n)}(x_1, x_1, x_2, \dots, x_{n-1}))^\sim,$$

where $(\cdot)^\sim$ denotes the symmetrization of a function.

Finally, we define on $\mathcal{F}_{\text{fin}}(\mathcal{D})$ the neutral operator $a^0(\xi)$, $\xi \in \mathcal{D}$, in a standard way:

$$(a^0(\xi)f^{(n)})(x_1, \dots, x_n) = n(\xi(x_1)f^{(n)}(x_1, \dots, x_n))^\sim.$$

One easily checks that $a^0(\xi)$ is a Hermitian operator in $\mathcal{F}_{\text{ext}}(\mathcal{H})$.

Now, we fix a parameter $\lambda \in [0, \infty)$ and define operators

$$a_\lambda(\xi) := a^+(\xi) + \lambda a^0(\xi) + a^-(\xi) + c_\lambda \langle \xi \rangle \text{id}, \quad \xi \in \mathcal{D}.$$

Here, $\langle \xi \rangle := \int_X \xi(x)\sigma(dx)$, id denotes the identity operator, and the constant $c_\lambda > 0$ is given by

$$c_\lambda := \begin{cases} \lambda/2 & \text{if } \lambda \in [0, 2], \\ 2/(\lambda + \sqrt{\lambda^2 - 4}) & \text{if } \lambda > 2. \end{cases} \tag{2.6}$$

(The special choice of this constant will become clear later on, however it is not of a real importance.) Each $a_\lambda(\xi)$ with domain $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is a Hermitian operator in $\mathcal{F}_{\text{ext}}(\mathcal{H})$.

By construction, the family of operators $(a_\lambda(\xi))_{\xi \in \mathcal{D}}$ has a Jacobi field structure in the extended Fock space $\mathcal{F}_{\text{ext}}(\mathcal{H})$ (cf. [3,7,25]). We will call this family Meixner’s Jacobi field corresponding to the parameter λ .

Lemma 2.1. *The operators $a_\lambda(\xi)$, $\xi \in \mathcal{D}$, with domain $\mathcal{F}_{\text{fin}}(\mathcal{D})$ are essentially self-adjoint in $\mathcal{F}_{\text{ext}}(\mathcal{H})$, and their closures $a_\lambda^\sim(\xi)$ constitute a family of commuting self-adjoint operators, where the commutation is understood in the sense of the commutation of their resolutions of the identity.*

Proof. The lemma follows directly from [3, Theorem 4.1] whose proof admits a direct generalization to the case of the extended Fock space. \square

Theorem 2.1. *For each $\lambda \in [0, \infty)$, there exist a unique probability measure μ_λ on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$, where $\mathcal{C}_\sigma(\mathcal{D}')$ is the cylinder σ -algebra on \mathcal{D}' , and a unique unitary operator*

$$I_\lambda : \mathcal{F}_{\text{ext}}(\mathcal{H}) \rightarrow L^2(\mathcal{D}'; \mu_\lambda)$$

such that, for each $\xi \in \mathcal{D}$, the image of $a_\lambda^\sim(\xi)$ under I_λ is the operator of multiplication by the function $\langle \cdot, \xi \rangle$ in $L^2(\mathcal{D}'; \mu_\lambda)$ and $I_\lambda \Omega = 1$, where $\Omega := (1, 0, 0, \dots)$. The Fourier transform of the measure μ_λ is given, in a neighborhood of zero, by the following formula: for $\lambda = 2$

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} d\mu_2(\omega) = \exp \left[- \int_X \log(1 - i\varphi(x)) \sigma(dx) \right],$$

$$\varphi \in \mathcal{D}, \quad \|\varphi\|_\infty := \sup_{x \in X} |\varphi(x)| < 1, \tag{2.7}$$

and for $\lambda \neq 2$

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} d\mu_\lambda(\omega)$$

$$= \exp \left[- \frac{1}{\alpha\beta} \int_X \log \left(\frac{\alpha e^{-i\beta\varphi(x)} - \beta e^{-i\alpha\varphi(x)}}{\alpha - \beta} \right) \sigma(dx) + ic_\lambda \langle \varphi \rangle \right] \tag{2.8}$$

for all $\varphi \in \mathcal{D}$ satisfying

$$\left\| \frac{\alpha(e^{-i\beta\varphi} - 1) - \beta(e^{-i\alpha\varphi} - 1)}{\alpha - \beta} \right\|_\infty < 1, \tag{2.9}$$

α, β defined by (1.4). The unitary operator I_λ is given on the dense set $\mathcal{F}_{\text{fin}}(\mathcal{D})$ by the formula

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) \ni f = (f^{(n)})_{n=0}^\infty \mapsto I_\lambda f = (I_\lambda f)(\omega) = \sum_{n=0}^\infty \langle : \omega^{\otimes n} :_\lambda, f^{(n)} \rangle$$

(the series is, in fact, finite), where $: \omega^{\otimes n} :_\lambda \in \mathcal{D}'^{\hat{\otimes} n}$ is defined by the recurrence formula

$$\begin{aligned} : \omega^{\otimes(n+1)} :_\lambda &= : \omega^{\otimes n+1} :_\lambda(x_1, \dots, x_{n+1}) = (: \omega^{\otimes n} :_\lambda(x_1, \dots, x_n) \omega(x_{n+1}))^\sim \\ &- n (: \omega^{\otimes(n-1)} :_\lambda(x_1, \dots, x_{n-1}) \delta(x_{n+1} - x_n))^\sim \\ &- n(n-1) (: \omega^{\otimes(n-1)} :_\lambda(x_1, \dots, x_{n-1}) \delta(x_n - x_{n-1}) \delta(x_{n+1} - x_n))^\sim \\ &- \lambda n (: \omega^{\otimes n} :_\lambda(x_1, \dots, x_n) \delta(x_{n+1} - x_n))^\sim \\ &- c_\lambda (: \omega^{\otimes n} :_\lambda(x_1, \dots, x_n) 1(x_{n+1}))^\sim, \\ &: \omega^{\otimes 0} :_\lambda = 1, : \omega^{\otimes 1} :_\lambda = \omega - c_\lambda. \end{aligned} \tag{2.10}$$

Remark 2.1. It can be shown that μ_λ is the spectral measure of the commutative family of self-adjoint operators $a_\lambda^\sim(\xi)$, $\xi \in \mathcal{D}$ (see [6, Chapter 3] for the notion of a spectral measure).

Remark 2.2. Note that taking a parameter $\lambda < 0$ would lead us to the measure μ_λ obtained from the measure $\mu_{-\lambda}$ by the transformation $\omega \mapsto -\omega$ of the space \mathcal{D}' , which is why we have excluded this choice.

Proof of Theorem 2.1. As easily seen, for any $\xi \in \mathcal{D}$ and $n \in \mathbb{N}$, the operators $a^+(\xi)$, $a^0(\xi)$, and $a^-(\xi)$ act continuously from $\mathcal{D}_\mathbb{C}^{\hat{\otimes} n}$ into $\mathcal{D}_\mathbb{C}^{\hat{\otimes}(n+1)}$, $\mathcal{D}_\mathbb{C}^{\hat{\otimes} n}$, $\mathcal{D}_\mathbb{C}^{\hat{\otimes}(n-1)}$, respectively. Therefore, for any $\xi \in \mathcal{D}$, $a_\lambda(\xi)$ acts continuously on $\mathcal{F}_{\text{fin}}(\mathcal{D})$. Furthermore, for any fixed $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$, the mapping $\mathcal{D} \ni \xi \mapsto a_\lambda(\xi) f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ is linear and continuous. Finally, the vacuum Ω is evidently a cyclic vector for the operators $a^+(\xi)$, $\xi \in \mathcal{D}$, in $\mathcal{F}_{\text{fin}}(\mathcal{D})$, and hence in $\mathcal{F}_{\text{ext}}(\mathcal{H})$. Then, using the Jacobi field structure of $a_\lambda(\xi)$, it is easy to show that Ω is a cyclic vector for $a_\lambda(\xi)$, $\xi \in \mathcal{D}$, in $\mathcal{F}_{\text{ext}}(\mathcal{H})$. Thus, analogously to [7, 21, Theorem 3.1, 25], we deduce, by using the projection spectral theorem [6, Chapter 3, Theorem 2.7, Subsection 3.3.1], the existence of a unique probability measure μ_λ on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$ and a unique unitary operator $I_\lambda : \mathcal{F}_{\text{ext}}(\mathcal{H}) \rightarrow L^2(\mathcal{D}'; \mu_\lambda)$ such that, for each $\xi \in \mathcal{D}$, the image of $a_\lambda^\sim(\xi)$ under I_λ is the operator of multiplication by the function $\langle \cdot, \xi \rangle$ and $I_\lambda \Omega = 1$.

Let us dwell upon the explicit form of I_λ . Let $\mathcal{F}_{\text{fin}}^*(\mathcal{D})$ denote the dual space of $\mathcal{F}_{\text{fin}}(\mathcal{D})$. $\mathcal{F}_{\text{fin}}^*(\mathcal{D})$ is the topological direct sum of the dual spaces $(\mathcal{D}_\mathbb{C}^{\hat{\otimes} n})^*$ of $\mathcal{D}_\mathbb{C}^{\hat{\otimes} n}$. It will be convenient for us to realize each $(\mathcal{D}_\mathbb{C}^{\hat{\otimes} n})^*$ as the dual of $\mathcal{D}_\mathbb{C}^{\hat{\otimes} n}$ with respect to

the zero space $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n} n!$, so that $(\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n})^*$ becomes $\mathcal{D}'_{\mathbb{C}}^{\hat{\otimes} n}$. Thus, $\mathcal{F}_{\text{fin}}^*(\mathcal{D})$ consists of infinite sequences $F = (F^{(n)})_{n=0}^{\infty}$, where $F^{(n)} \in \mathcal{D}'_{\mathbb{C}}^{\hat{\otimes} n}$, and the dualization with $f = (f^{(n)})_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ is given by

$$\langle\langle F, f \rangle\rangle = \sum_{n=0}^{\infty} \langle \overline{F^{(n)}}, f^{(n)} \rangle n!, \tag{2.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the dualization generated by the scalar product in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$, which is supposed to be linear in both dots.

Next, according to the projection spectral theorem, for μ_{λ} -a.e. $\omega \in \mathcal{D}'$, there exists a generalized joint vector $P(\omega) = (P^{(n)}(\omega))_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}}^*(\mathcal{D})$ of the family $a(\xi)$, $\xi \in \mathcal{D}$:

$$\forall f \in \overline{\mathcal{F}_{\text{fin}}(\mathcal{D})} : \quad \langle\langle P(\omega), a(\xi)f \rangle\rangle = \langle \omega, \xi \rangle \langle\langle P(\omega), f \rangle\rangle, \tag{2.12}$$

and for each $f = (f^{(n)})_{n=0}^{\infty} \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ the action of I_{λ} onto f is given by

$$I_{\lambda}f = (I_{\lambda}f)(\omega) = \langle\langle P(\omega), f \rangle\rangle = \sum_{n=0}^{\infty} \langle P^{(n)}(\omega), f^{(n)} \rangle n!. \tag{2.13}$$

We denote $: \omega^{\otimes n} :_{\lambda} := P^{(n)}(\omega)n!$, which is an element of $\mathcal{D}'_{\mathbb{C}}^{\hat{\otimes} n}$ for μ_{λ} -a.e. $\omega \in \mathcal{D}'$. By (2.12) and (2.13), we have, for μ_{λ} -a.e. $\omega \in \mathcal{D}'$,

$$\begin{aligned} \langle \omega, \xi \rangle \langle : \omega^{\otimes n} :_{\lambda}, \xi^{\otimes n} \rangle &= \langle : \omega^{\otimes(n+1)} :_{\lambda}, \xi^{\otimes(n+1)} \rangle \\ &+ \langle : \omega^{\otimes n} :_{\lambda}, \lambda n(\xi^2) \hat{\otimes} \xi^{\otimes(n-1)} + c_{\lambda} \langle \xi \rangle \xi^{\hat{\otimes} n} \rangle \\ &+ \langle : \omega^{\otimes(n-1)} :_{\lambda}, n \langle \xi^2 \rangle \xi^{\otimes(n-1)} + n(n-1) \langle \xi^3 \rangle \hat{\otimes} \xi^{\otimes(n-2)} \rangle \end{aligned} \tag{2.14}$$

for all $\xi \in \mathcal{D}$. Therefore, for μ_{λ} -a.e. $\omega \in \mathcal{D}'$, the $: \omega^{\otimes n} :_{\lambda}$'s are given by the recurrence relation (2.10). As easily seen, $: \omega^{\otimes n} :_{\lambda}$ is even well defined as an element of $\mathcal{D}'_{\mathbb{C}}^{\hat{\otimes} n}$ for each $\omega \in \mathcal{D}'$.

Let us calculate the Fourier transform of μ_{λ} . Let $\Delta \in \mathcal{O}_c(X)$ and let χ_{Δ} denote the indicator of Δ . One easily checks that each of the vectors $\chi_{\Delta}^{\otimes n}$, $n \in \mathbb{Z}_+$, $\chi_{\Delta}^{\otimes 0} = \Omega$, belongs to $\mathcal{F}_{\text{ext}}(\mathcal{H})$, and let \mathcal{K}_{Δ} be the subspace of $\mathcal{F}_{\text{ext}}(\mathcal{H})$ spanned by these vectors. Let $a_{\lambda}^{\sim}(\chi_{\Delta})$ denote the operator in $\mathcal{F}_{\text{ext}}(\mathcal{H})$ whose image under the unitary I_{λ} is the operator of multiplication by the function

$$\langle \omega, \chi_{\Delta} \rangle := \langle : \omega^{\otimes 1} :_{\lambda}, \chi_{\Delta} \rangle + c_{\lambda} \sigma(\Delta) = (I_{\lambda} \chi_{\Delta})(\omega) + c_{\lambda} \sigma(\Delta).$$

We approximate the indicator function χ_{Δ} by functions $\varphi_n \in \mathcal{D}$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} \text{supp } \varphi_n$ is precompact in X , φ_n 's are uniformly bounded, and $\varphi_n(x) \rightarrow \chi_{\Delta}(x)$ as $n \rightarrow \infty$ for each $x \in X$. By using the definitions of $\mathcal{F}_{\text{ext}}(\mathcal{H})$ and $a_{\lambda}^{\sim}(\varphi)$ and the

majorized convergence theorem, we get

$$a_{\tilde{\lambda}}(\chi_{\Delta})\chi_{\Delta}^{\otimes n} = \chi_{\Delta}^{\otimes(n+1)} + (\lambda n + c_{\lambda}\sigma(\Delta))\chi_{\Delta}^{\otimes n} + n(n - 1 + \sigma(\Delta))\chi_{\Delta}^{\otimes(n-1)}.$$

Therefore, \mathcal{H}_{Δ} is an invariant subspace for the operator $a_{\tilde{\lambda}}(\chi_{\Delta})$. Hence, analogous to [21, pp. 315–316], we may conclude that the distribution of the random variable $\langle \cdot, \chi_{\Delta} \rangle$ is the probability measure $\mu_{\lambda, \Delta}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ whose orthogonal polynomials $(P_{\lambda, \Delta}^{(n)}(x))_{n=0}^{\infty}$ with leading coefficient 1 satisfy the recurrence relation

$$\begin{aligned} xP_{\lambda, \Delta}^{(n)}(x) &= P_{\lambda, \Delta}^{(n+1)}(x) + (\lambda n + c_{\lambda}\sigma(\Delta))P_{\lambda, \Delta}^{(n)}(x) \\ &\quad + n(n - 1 + \sigma(\Delta))P_{\lambda, \Delta}^{(n-1)}(x) \end{aligned} \tag{2.15}$$

(the measure $\mu_{\lambda, \Delta}$ being defined uniquely through this condition). Thus, $(P_{\lambda, \Delta}^{(n)}(\cdot))_{n=0}^{\infty}$ is a system of Laguerre polynomials for $\lambda = 2$, and a system of Meixner polynomials for $\lambda \neq 2$. We will now consider only the case $\lambda \neq 2$, the case $\lambda = 2$ being considered analogously (see also [21]). By Meixner [26], the Fourier transform of the measure $\mu_{\lambda, \Delta}$ in a neighborhood of zero in \mathbb{R} is given by

$$\int_{\mathbb{R}} e^{iux} \mu_{\lambda, \Delta}(dx) = \left(\frac{\alpha - \beta}{\alpha e^{-i\beta u} - \beta e^{-izu}} \right)^{\sigma(\Delta)/(\alpha\beta)} \exp[ic_{\lambda}u\sigma(\Delta)]. \tag{2.16}$$

Therefore, for any $u \in \mathbb{R}$ satisfying

$$\left| \frac{\alpha(e^{-i\beta u} - 1) - \beta(e^{izu} - 1)}{\alpha - \beta} \right| < 1, \tag{2.17}$$

we get

$$\begin{aligned} \int_{\mathcal{D}'} e^{iu\langle \omega, \chi_{\Delta} \rangle} \mu_{\lambda}(d\omega) &= \exp \left[-\frac{\sigma(\Delta)}{\alpha\beta} \log \left(\frac{\alpha e^{-i\beta u} - \beta e^{-izu}}{\alpha - \beta} \right) + ic_{\lambda}u\sigma(\Delta) \right] \\ &= \exp \left[-\frac{1}{\alpha\beta} \int_X \log \left(\frac{\alpha e^{-i\beta u\chi_{\Delta}(x)} - \beta e^{-izu\chi_{\Delta}(x)}}{\alpha - \beta} \right) \sigma(dx) \right. \\ &\quad \left. + ic_{\lambda} \int_X u\chi_{\Delta}(x)\sigma(dx) \right]. \end{aligned} \tag{2.18}$$

Now, let $\Delta_1, \dots, \Delta_n \in \mathcal{O}_c(X)$ be disjoint. Then the spaces $\mathcal{H}_{\Delta_1} \ominus \mathcal{F}_0(H), \dots, \mathcal{H}_{\Delta_n} \ominus \mathcal{F}_0(\mathcal{H})$ are orthogonal in $\mathcal{F}_{\text{ext}}(\mathcal{H})$. Therefore, the random variables $\langle \cdot, \chi_{\Delta_1} \rangle, \dots, \langle \cdot, \chi_{\Delta_n} \rangle$ are independent. Hence, for any step function $\varphi = \sum_{i=1}^n u_i \chi_{\Delta_i}$ such that all u_i 's satisfy (2.17) with $u = u_i$, formula (2.8) holds.

Finally, fix any $\varphi \in \mathcal{D}$ satisfying (2.8). Choose any sequence of step functions $\{\varphi_n\}_{n \in \mathbb{N}}$ such that

$$\sup_{n \in \mathbb{N}, x \in X} \left| \frac{\alpha(e^{-i\beta\varphi_n(x)} - 1) - \beta(e^{i\alpha\varphi_n(x)} - 1)}{\alpha - \beta} \right| < 1,$$

$\bigcup_{n \in \mathbb{N}} \text{supp } \varphi_n \in \mathcal{O}_c(X)$, and φ_n converges pointwisely to φ as $n \rightarrow \infty$. Then, by the majorized convergence theorem, we conclude that the right-hand side of (2.8) with $\varphi = \varphi_n$ converges to the right-hand side of (2.8). On the other hand, $\langle \cdot, \varphi_n \rangle$ converges to $\langle \cdot, \varphi \rangle$ in $L^2(\mathcal{D}'; \mu)$, and therefore also in probability. Hence, again by the majorized convergence theorem, the left-hand side of (2.8) with $\varphi = \varphi_n$ converges to the left-hand side of (2.8). \square

Corollary 2.1. *For each $\Delta \in \mathcal{O}_c(X)$, the distribution $\mu_{\lambda, \Delta}$ of the random variable $\langle \cdot, \chi_\Delta \rangle$ under μ_λ is given as follows: For $\lambda > 2$, $\mu_{\lambda, \Delta}$ is the negative binomial (Pascal) distribution*

$$\mu_{\lambda, \Delta} = (1 - p_\lambda)^{\sigma(\Delta)} \sum_{k=0}^{\infty} \frac{(\sigma(\Delta))_k}{k!} p_\lambda^k \delta_{\sqrt{\lambda^2 - 4} k}, \tag{2.19}$$

where

$$p_\lambda := \frac{\lambda - \sqrt{\lambda^2 - 4}}{\lambda + \sqrt{\lambda^2 - 4}},$$

and for $r > 0$ $(r)_0 := 1$, $(r)_k := r(r + 1) \cdots (r + k - 1)$, $k \in \mathbb{N}$. For $\lambda = 2$, $\mu_{2, \Delta}$ is the Gamma distribution

$$\mu_{2, \Delta}(ds) = \frac{s^{\sigma(\Delta) - 1} e^{-s}}{\Gamma(\sigma(\Delta))} \chi_{[0, \infty)}(s) ds. \tag{2.20}$$

Finally, for $\lambda \in [0, 2)$,

$$\begin{aligned} \mu_{\lambda, \Delta}(ds) &= \frac{(4 - \lambda^2)^{(\sigma(\Delta) - 1)/2}}{2\pi\Gamma(\sigma(\Delta))} \\ &\times |\Gamma(\sigma(\Delta)/2 + i(4 - \lambda^2)^{-1/2}s)|^2 \\ &\times \exp[-s2(4 - \lambda^2)^{-1/2} \arctan(\lambda(4 - \lambda^2)^{-1/2})] ds. \end{aligned} \tag{2.21}$$

Proof. The result follows from (the proof of) [12, Theorem 2.1 Chapter VI, Section 3; 26] (see also [28, Subsection 4.3.5, 29]). \square

For $f^{(n)} \in \mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})$, let $\langle : \cdot^{\otimes n} :_{\lambda, f^{(n)}} \rangle$ denote the element of $L^2(\mathcal{D}'; \mu_\lambda)$ defined as $I_\lambda f^{(n)}$.

Corollary 2.2. For any $\Delta \in \mathcal{O}_c(X)$ we have

$$\langle \cdot : \omega^{\otimes n} :_{\lambda, \chi_{\Delta}}^{\otimes n} \rangle = P_{\lambda, \Delta}^{(n)}(\langle \omega, \chi_{\Delta} \rangle) \quad \mu_{\lambda}\text{-a.a. } \omega \in \mathcal{D}', \tag{2.22}$$

where $(P_{\lambda, \Delta}^{(n)})_{n=0}^{\infty}$ are the polynomials on \mathbb{R} as in the proof of Theorem 2.1.

Proof. This result directly follows from the proof of Theorem 2.1. \square

Remark 2.3. Let us state the one-dimensional analog of the results of Theorem 2.1 and Corollaries 2.1, 2.2. We consider the weighted ℓ_2 -space $\mathcal{F} = \ell_2(((n!)^2)_{n=0}^{\infty})$ consisting of all sequences $f = (f^{(n)})_{n=0}^{\infty}$, $f^{(n)} \in \mathbb{C}$, such that $\|f\|_{\mathcal{F}}^2 := \sum_{n=0}^{\infty} |f^{(n)}|^2 (n!)^2 < \infty$. Let \mathcal{F}_{fin} denote the set of all finite sequences in \mathcal{F} . For each $\lambda \in [0, \infty)$, we define a linear Hermitian operator a_{λ} in \mathcal{F} with domain \mathcal{F}_{fin} by setting

$$a_{\lambda} = a^+ + \lambda a^0 + a^- + c_{\lambda} \text{ id},$$

where $(a^+ f^{(n)})^{(k)} = \delta_{k, n+1} f^{(n)}$, $(a^0 f^{(n)})^{(k)} = \delta_{k, n} n f^{(n)}$, $(a^- f^{(n)})^{(k)} = \delta_{k, n-1} n^2 f^{(n)}$. Here $\delta_{i, j} = 1$ if $i = j$ and $\delta_{i, j} = 0$ otherwise, and c_{λ} is given by (2.6). We note that, under the natural unitary mapping of the weighted ℓ_2 -space \mathcal{F} onto the usual ℓ_2 , the operator a_{λ} goes over into the operator defined by the infinite Jacobi matrix $J = (\alpha_{m, n})_{m, n=0}^{\infty}$ with the elements $\alpha_{n, n} = \lambda n + c_{\lambda}$, $n \in \mathbb{Z}_+$, $\alpha_{n, n+1} = \alpha_{n+1, n} = n + 1$, $n \in \mathbb{Z}_+$, and $\alpha_{m, n} = 0$ for $|m - n| > 1$.

The operator a_{λ} is essentially self-adjoint, and let a_{λ}^{\sim} denote its closure. By the spectral theory of infinite Jacobi matrices (e.g. [2, Chapter VII, Section 1]), there exist a unique probability measure m_{λ} on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and a unique unitary operator $\mathcal{I}_{\lambda} : \mathcal{F} \rightarrow L^2(\mathbb{R}; m_{\lambda})$ such that the image of the operator a_{λ}^{\sim} under \mathcal{I}_{λ} is the operator of multiplication by the variable x and $\mathcal{I}_{\lambda}(1, 0, 0, \dots) = 1$. The mapping \mathcal{I}_{λ} is given on the dense set \mathcal{F}_{fin} by

$$\mathcal{F}_{\text{fin}} \ni f = (f^{(n)})_{n=0}^{\infty} \mapsto \mathcal{I}_{\lambda} f = \sum_{n=0}^{\infty} f^{(n)} P_{\lambda}^{(n)}(x).$$

Here, $(P_{\lambda}^{(n)})_{n=0}^{\infty}$ is the system of polynomials on \mathbb{R} satisfying the following recurrence relation: for $n \in \mathbb{Z}_+$

$$x P_{\lambda}^{(n)}(x) = P_{\lambda}^{(n+1)}(x) + (\lambda n + c_{\lambda}) P_{\lambda}^{(n)}(x) + n^2 P_{\lambda}^{(n-1)}(x), \quad P_{\lambda}^{(-1)}(x) = 0, \\ P_{\lambda}^{(1)}(x) = 1,$$

and m_{λ} is the unique probability measure on \mathbb{R} with respect to which the polynomials $(P_{\lambda}^{(n)})_{n=0}^{\infty}$ are orthogonal. By Meixner [26], the Fourier transform of the measure m_{λ}

in a neighborhood of zero in \mathbb{R} is given by

$$\int_{\mathbb{R}} e^{iux} m_{\lambda}(dx) = \begin{cases} e^{ic_{\lambda}u} \left(\frac{\alpha-\beta}{\alpha e^{-i\beta u} - \beta e^{-i\alpha u}}\right)^{1/(\alpha\beta)}, & \lambda \neq 2, \\ (1-iu)^{-1}, & \lambda = 2. \end{cases} \tag{2.23}$$

Furthermore, the measure m_{λ} is explicitly given by the right-hand side of formula (2.19), resp. (2.20), resp. (2.21), with $\sigma(\Delta) = 1$. Thus m_{λ} is a Pascal distribution for $\lambda \in [0, 2)$, a Gamma distribution for $\lambda = 2$, and a Meixner distribution for $\lambda > 2$.

We will now show that each μ_{λ} is a compound Poisson, respectively, Lévy noise measure.

Corollary 2.3. *For each $\lambda \geq 2$, μ_{λ} is a compound Poisson measure on $(\mathcal{D}', \mathcal{C}_{\sigma}(\mathcal{D}'))$ whose Lévy–Khintchine representation of the Fourier transform reads as follows:*

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \mu_{\lambda}(d\omega) = \exp \left[\int_{X \times \mathbb{R}_+} (e^{is\varphi(x)} - 1) \sigma(dx) v_{\lambda}(ds) \right], \quad \varphi \in \mathcal{D}, \tag{2.24}$$

where

$$v_2(ds) = \frac{e^{-s}}{s} ds,$$

$$v_{\lambda}(ds) = \sum_{k=1}^{\infty} \frac{p_{\lambda}^k}{k} \delta_{\sqrt{\lambda^2 - 4k}}, \quad \lambda > 2. \tag{2.25}$$

In particular, each μ_{λ} is concentrated on the set of all Radon measures on $(X, \mathcal{B}(X))$ of the form $\sum_{n=1}^{\infty} s_n \delta_{x_n}$, $\{x_n\} \subset X$, $s_n > 0$, $n \in \mathbb{N}$.

Proof. It follows from the general theory of compound Poisson measures (e.g. [19]) that there exists a compound Poisson measure $\tilde{\mu}_{\lambda}$ whose Fourier transform is given by (2.24) with v_{λ} given by (2.25), and which is concentrated on the set of those Borel measures on X as in the formulation of the theorem. Furthermore, it follows from the general theory that, for any disjoint $\Delta_1, \dots, \Delta_n \in \mathcal{O}_c(X)$, the random variables are independent. Thus, it suffices to show that, for any fixed $\Delta \in \mathcal{O}_c(X)$, the distributions of the random variable $\langle \omega, \chi_{\Delta} \rangle$ under μ_{λ} and $\tilde{\mu}_{\lambda}$ coincide. But this can be easily done by calculating the Fourier transform

$$\int_{\mathcal{D}'} e^{iu\langle \omega, \chi_{\Delta} \rangle} \tilde{\mu}_{\lambda}(d\omega) = \exp \left[\sigma(\Delta) \int_{\mathbb{R}_+} (e^{ius} - 1) v_{\lambda}(ds) \right], \quad u \in \mathbb{R},$$

and comparing it with the Fourier transform of the measure $\mu_{\lambda, \Delta}$ (see (2.16) for the case $\lambda > 2$). \square

In the case $\lambda \in [0, 2)$, the situation is a little bit more complicated, since the corresponding Lévy measure ν_λ to be identified does not have the first moment finite.

Corollary 2.4. For $\lambda \in [0, 2)$, μ_λ is the Lévy noise measure whose Lévy–Khintchine representation of the Fourier transform reads as follows:

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \mu_\lambda(d\omega) = \exp \left[\int_{X \times \mathbb{R}} (e^{is\varphi(x)} - 1 - is\varphi(x)) \sigma(dx) \nu_\lambda(ds) + ic_\lambda \langle \varphi \rangle \right],$$

$$\varphi \in \mathcal{D}, \tag{2.26}$$

where

$$\nu_\lambda(ds) = \frac{\sqrt{4 - \lambda^2}}{2\pi} \times |\Gamma(1 + i(4 - \lambda^2)^{-1/2}s)|^2 \exp[-s2(4 - \lambda^2)^{-1/2} \times \arctan(\lambda(4 - \lambda^2)^{-1/2})] \frac{1}{s^2} ds. \tag{2.27}$$

Proof. The existence of a probability measure $\tilde{\mu}_\lambda$ on \mathcal{D}' whose Fourier transform is given by the right-hand side of (2.26) with ν_λ given by (2.27) follows by, e.g. the Bochner–Minlos theorem. Furthermore, as easily seen, for each $A \in \mathcal{O}_c(X)$, one can naturally define a random variable $\langle \cdot, \chi_A \rangle$ as an element of $L^2(\mathcal{D}'; \tilde{\mu}_\lambda)$. By (2.26), for any disjoint $A_1, \dots, A_n \in \mathcal{O}_c(X)$, the random variables $\langle \cdot, \chi_{A_1} \rangle, \dots, \langle \cdot, \chi_{A_n} \rangle$ are independent. Analogous to the proof of Corollary 2.3, we conclude the statement by calculating the integral $\int_{\mathbb{R}} (e^{ius} - 1 - ius) \nu_\lambda(ds)$, $u \in \mathbb{R}$, using [28, Subsection 4.3.5], see also [29]. \square

Remark 2.4. It follows from Corollaries 2.3 and 2.4 that $s^2 \nu_\lambda(ds)$ is a Meixner distribution for $\lambda \in [0, 2)$, gamma distribution for $\lambda = 2$, and Pascal distribution for $\lambda > 2$. Furthermore, for each $\lambda \geq 0$, $s^2 \nu_\lambda(ds)$ is a probability measure on \mathbb{R} whose orthogonal polynomials $(Q_\lambda^{(n)})_{n=0}^\infty$ with leading coefficient 1 satisfy the following recurrence relation:

$$sQ_\lambda^{(n)}(s) = Q_\lambda^{(n+1)}(s) + \lambda(n + 1)Q_\lambda^{(n)}(s) + n(n + 1)Q_\lambda^{(n-1)}(s), \quad n \in \mathbb{Z}_+,$$

$$Q_\lambda^{-1}(s) := 0.$$

We denote by $\mathcal{P}(\mathcal{D}')$ the set of continuous polynomials on \mathcal{D}' , i.e., functions on \mathcal{D}' of the form

$$F(\omega) = \sum_{i=0}^n \langle \omega^{\otimes i}, f^{(i)} \rangle, \quad f^{(i)} \in \mathcal{D}'^{\hat{\otimes} i}, \quad \omega^{\otimes 0} := 1, \quad i \in \mathbb{Z}_+.$$

The greatest number i for which $f^{(i)} \neq 0$ is called the power of a polynomial. We denote by $\mathcal{P}_n(\mathcal{D}')$ the set of continuous polynomials of power $\leq n$.

Corollary 2.5. *For each $\lambda \geq 0$, we have $I_\lambda(\mathcal{F}_{\text{fin}}(\mathcal{D})) = \overline{\mathcal{P}(\mathcal{D}'})$. In particular, $\mathcal{P}(\mathcal{D}')$ is a dense subset of $L^2(\mathcal{D}'; \mu_\lambda)$. Furthermore, let $\mathcal{P}_{\lambda,n}(\mathcal{D}')$ denote the closure of $\mathcal{P}_n(\mathcal{D}')$ in $L^2(\mathcal{D}'; \mu_\lambda)$, and let $(L^2_{\lambda,n})$ denote the orthogonal difference $\mathcal{P}_{\lambda,n}(\mathcal{D}') \ominus \mathcal{P}_{\lambda,n-1}(\mathcal{D}')$ in $L^2(\mathcal{D}'; \mu_\lambda)$. Then,*

$$L^2(\mathcal{D}'; \mu_\lambda) = \bigoplus_{n=0}^{\infty} (L^2_{\lambda,n}). \tag{2.28}$$

Finally, let $P_{\lambda,n}$ denote the orthogonal projection of $L^2(\mathcal{D}'; \mu_\lambda)$ onto $(L^2_{\lambda,n})$. Then, for any $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$,

$$P_{\lambda,n}(\langle \cdot^{\otimes n}, f^{(n)} \rangle) = \langle : \cdot^{\otimes n} :_{\lambda}, f^{(n)} \rangle \quad \mu_\lambda\text{-a.e.} \tag{2.29}$$

and

$$I_\lambda(\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})) = (L^2_{\lambda,n}). \tag{2.30}$$

Proof. Using recurrence relation (2.10), we obtain by induction the inclusion $I_\lambda(\mathcal{F}_{\text{fin}}(\mathcal{D})) \subset \mathcal{P}(\mathcal{D}')$ and moreover, the equality

$$\langle : \omega^{\otimes n} :_{\lambda}, f^{(n)} \rangle = \langle \omega^{\otimes n}, f^{(n)} \rangle + p_{n-1}(\omega), \quad f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}, \tag{2.31}$$

where $p_{n-1} \in \mathcal{P}_{n-1}(\mathcal{D}')$. Using (2.31), we then obtain by induction also the inverse inclusion $\mathcal{P}(\mathcal{D}') \subset I_\lambda(\overline{\mathcal{F}_{\text{fin}}(\mathcal{D})})$. That $\mathcal{P}(\mathcal{D}')$ is dense in $L^2(\mathcal{D}'; \mu_\lambda)$ follows from the fact that $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is dense in $\mathcal{F}_{\text{ext}}(\mathcal{H})$. Decomposition (2.28) now becomes evident. Finally, (2.29) follows by (2.31), and (2.30) is a consequence of (2.29). \square

3. The generating function

Now, we will identify the generating function of the polynomials $\langle : \omega^{\otimes n} :_{\lambda}, \varphi^{\otimes n} \rangle$, $\varphi \in \mathcal{D}$.

Let $\mathcal{M}(X)$ denote the set of signed Radon measures on $(X, \mathcal{B}(X))$. We can evidently identify any measure $\omega \in \mathcal{M}(X)$ with an element $\tilde{\omega} \in \mathcal{D}'$ by setting

$$\langle \tilde{\omega}, \varphi \rangle := \int_X \varphi(x) \omega(dx), \quad \varphi \in \mathcal{D}.$$

In what follows, we will just write ω instead of $\tilde{\omega}$. Then, for $\Delta \in \mathcal{O}_c(X)$, the writing $\langle \omega, \chi_\Delta \rangle$ will mean $\omega(\Delta)$.

Proposition 3.1. *We have, for $\lambda \neq 2$,*

$$\begin{aligned}
 G_\lambda(\omega, \varphi) &:= \sum_{n=0}^\infty \frac{1}{n!} \langle : \omega^{\otimes n} :_\lambda, \varphi^{\otimes n} \rangle \\
 &= \exp \left[-\frac{1}{\alpha - \beta} \left\langle \log \left(\frac{(1 - \beta\varphi)^{1/\beta}}{(1 - \alpha\varphi)^{1/\alpha}} \right) \right\rangle \right. \\
 &\quad \left. + \frac{1}{\alpha - \beta} \left\langle \omega - c_\lambda, \log \left(\frac{1 - \beta\varphi}{1 - \alpha\varphi} \right) \right\rangle \right] \tag{3.1}
 \end{aligned}$$

and for $\lambda = 2$

$$\begin{aligned}
 G_2(\omega, \varphi) &:= \sum_{n=0}^\infty \frac{1}{n!} \langle : \omega^{\otimes n} :_2, \varphi^{\otimes n} \rangle \\
 &= \exp \left[-\langle \log(1 + \varphi) \rangle + \left\langle \omega, \frac{\varphi}{\varphi + 1} \right\rangle \right]. \tag{3.2}
 \end{aligned}$$

Formulas (3.1), (3.2) hold for each $\omega \in \mathcal{M}(X)$ and for each $\varphi \in \mathcal{D}_\mathbb{C}$ satisfying $\|\varphi\|_\infty < (\max(|\alpha|, |\beta|))^{-1}$ for (3.1) and $\|\varphi\|_\infty < 1$ for (3.2). More generally, for each fixed $\tau \in T$, there exists a neighborhood of zero in $\mathcal{D}_\mathbb{C}$ (depending on λ), denoted by \mathcal{O}_τ , such that (3.1), respectively (3.2), holds for all $\omega \in \mathcal{H}_{-\tau}$ and all $\varphi \in \mathcal{O}_\tau$.

Remark 3.1. In the one-dimensional case (see Remark 2.3), the generating function of the polynomials $(P_\lambda^{(n)}(\cdot))_{n=0}^\infty$ is given by (cf. [26])

$$\begin{aligned}
 \mathcal{G}_\lambda(x, u) &:= \sum_{n=0}^\infty \frac{u^n}{n!} P_\lambda^{(n)}(x) \\
 &= \left(\frac{(1 - \beta u)^{1/\beta}}{(1 - \alpha u)^{1/\alpha}} \right)^{-1/(\alpha - \beta)} \left(\frac{1 - \beta u}{1 - \alpha u} \right)^{(x - c_\lambda)/(\alpha - \beta)}, \quad \lambda \neq 2,
 \end{aligned}$$

$$\mathcal{G}_2(x, u) := \sum_{n=0}^\infty \frac{u^n}{n!} P_2^{(n)}(x) = \frac{1}{1 + u} e^{ux/(u+1)}$$

for u from a neighborhood of zero in \mathbb{C} . □

Proof of Proposition 3.1. We only prove formula (3.1), corresponding to the case $\lambda \neq 2$. Let us fix $\omega \in \mathcal{M}(X)$. Then, as easily seen from (2.10), $: \omega^{\otimes n} :_\lambda \in \mathcal{M}(X^n)$ for each $n \in \mathbb{N}$ (if we identify $\mathcal{M}(X^n)$ as a subset of $\mathcal{D}'^{\otimes n}$). Fix $\Delta \in \mathcal{O}_c(X)$. As follows from the proof of Theorem 2.1, the equality in (2.22) holds for each

$\omega \in \mathcal{M}(X)$. Then,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \omega^{\otimes n} :_{\lambda}, (u\chi_{\Delta})^{\otimes n} \rangle &= \sum_{n=0}^{\infty} \frac{u^n}{n!} \langle : \omega^{\otimes n} :_{\lambda}, \chi_{\Delta}^{\otimes n} \rangle \\ &= \sum_{n=0}^{\infty} \frac{u^n}{n!} P_{\lambda, \Delta}^{(n)}(\langle \omega, \chi_{\Delta} \rangle). \end{aligned}$$

Hence, it follows from [26] (see also Remark 3.1) that (3.2) holds with $\varphi = u\chi_{\Delta}$ and $u \in \mathbb{C}$ such that $|u| < (\max(|\alpha|, |\beta|))^{-1}$.

We next prove the following lemma.

Lemma 3.1. *For any $\omega \in \mathcal{M}(X)$ and any disjoint $\Delta_1, \dots, \Delta_l \in \mathcal{O}_c(X)$,*

$$\begin{aligned} \langle : \omega^{\otimes n} :_{\lambda}, \chi_{\Delta_1}^{\otimes k_1} \hat{\otimes} \dots \hat{\otimes} \chi_{\Delta_l}^{\otimes k_l} \rangle &= \prod_{i=1}^l \langle : \omega^{\otimes k_i} :_{\lambda}, \chi_{\Delta_i}^{\otimes k_i} \rangle \\ &= \prod_{i=1}^l P_{\lambda, \Delta_i}^{(k_i)}(\langle \omega, \chi_{\Delta_i} \rangle), \end{aligned} \tag{3.3}$$

where $k_1, \dots, k_l \in \mathbb{N}$, $k_1 + \dots + k_l = n$.

Proof. We prove this lemma by induction in $n \in \mathbb{N}$. For $n = 1$, formula (3.3) trivially holds. Now, suppose that (3.3) holds for all $n \leq N$. Let $k_1, \dots, k_l \in \mathbb{N}$, $k_1 + \dots + k_l = N + 1$. Applying recurrence formula (2.10) to $: \omega^{\otimes(N+1)} :_{\lambda}$, we express $\langle : \omega^{\otimes(N+1)} :_{\lambda}, \chi_{\Delta_1}^{\otimes k_1} \hat{\otimes} \dots \hat{\otimes} \chi_{\Delta_l}^{\otimes k_l} \rangle$ through $\langle \omega, \chi_{\Delta_j} \rangle$, $\langle : \omega^{\otimes N} :_{\lambda}, \chi_{\Delta_1}^{\otimes k_1} \hat{\otimes} \dots \hat{\otimes} \chi_{\Delta_j}^{\otimes(k_j-1)} \hat{\otimes} \dots \hat{\otimes} \chi_{\Delta_l}^{\otimes k_l} \rangle$, $\langle : \omega^{\otimes(N-1)} :_{\lambda}, \chi_{\Delta_1}^{\otimes k_1} \hat{\otimes} \dots \hat{\otimes} \chi_{\Delta_j}^{\otimes(k_j-2)} \hat{\otimes} \dots \hat{\otimes} \chi_{\Delta_l}^{\otimes k_l} \rangle$, and $\sigma(\Delta_j)$, $j = 1, \dots, l$. Applying formula (3.3) with $n = N$ and $n = N - 1$ and then using the recurrence relation (2.15) for the polynomials $P_{\lambda, \Delta}^{(n)}$, we conclude the statement. \square

Fix any disjoint $\Delta_1, \dots, \Delta_m \in \mathcal{O}_c(X)$ and any $u_1, \dots, u_m \in \mathbb{C}$ satisfying $|u_i| < \max(|\alpha|, |\beta|)^{-1}$, $i = 1, \dots, m$, and set $f := \sum_{i=1}^m u_i \chi_{\Delta_i}$. By Lemma 3.1, we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle : \omega^{\otimes n} :_{\lambda}, f^{\otimes n} \rangle = \prod_{i=1}^m \left(\sum_{n=0}^{\infty} \frac{u_i^n}{n!} \langle : \omega^{\otimes n} :_{\lambda}, \chi_{\Delta_i}^{\otimes n} \rangle \right).$$

Hence, (3.2) holds with $\varphi = f$.

Using (2.10), one easily shows by induction that, for each fixed $\Delta \in \mathcal{O}_c(X)$, there exists a constant $C_{\Delta, \omega}$ such that

$$\forall n \in \mathbb{N} : \quad | : \omega^{\otimes n} :_{\lambda} \upharpoonright \Delta^n | \leq n! C_{\Delta, \omega}^n, \tag{3.4}$$

where $|\langle \omega^{\otimes n} :_\lambda \uparrow A^n \rangle|$ denotes the full variation of the signed measure $\omega^{\otimes n} :_\lambda$ on A^n . Fix any $\varphi \in \mathcal{D}$ such that $\text{supp } \varphi \subset A$ and $\|\varphi\|_\infty < C_{A,\omega}^{-1}$. Let $\{f_k, k \in \mathbb{N}\}$ be a sequence of step functions on X such that

$$C := \sup_{k \in \mathbb{N}, x \in X} |f_k(x)| < C_{A,\omega}^{-1},$$

$\bigcup_{k \in \mathbb{N}} \text{supp } f_k \subset A$ and $f_k \rightarrow \varphi$ as $k \rightarrow \infty$ uniformly on X . We then get by (3.4)

$$\begin{aligned} & \left| \sum_{n=0}^{\infty} \frac{1}{n!} \langle \omega^{\otimes n} :_\lambda, f_k^{\otimes n} \rangle - \sum_{n=0}^{\infty} \frac{1}{n!} \langle \omega^{\otimes n} :_\lambda, \varphi^{\otimes n} \rangle \right| \\ & \leq \sum_{n=0}^{\infty} \frac{1}{n!} |\langle \omega^{\otimes n} :_\lambda \uparrow A^n \rangle| \sup_{(x_1, \dots, x_n) \in X^n} |f_k^{\otimes n}(x_1, \dots, x_n) - \varphi^{\otimes n}(x_1, \dots, x_n)| \\ & \leq \sum_{n=0}^{\infty} C_{A,\omega}^n n \max(\|\varphi\|_\infty, C)^{n-1} \sup_{x \in X} |f_k(x) - \varphi(x)| \rightarrow 0 \\ & \text{as } k \rightarrow \infty. \end{aligned} \tag{3.5}$$

Let $\tilde{G}_\lambda(\omega, \varphi)$ denote the right-hand side of (3.2). Then, if

$$\max(\|\varphi\|_\infty, C) < \max(|\alpha|, |\beta|)^{-1},$$

by the majorized convergence theorem, $\tilde{G}_\lambda(\omega, f_k) \rightarrow \tilde{G}_\lambda(\omega, \varphi)$ as $k \rightarrow \infty$. Thus, (3.2) holds for any $\varphi \in \mathcal{D}$ such that $\text{supp } \varphi \subset A$ and

$$\|\varphi\|_\infty < \min(C_{A,\omega}^{-1}, \max(|\alpha|, |\beta|)^{-1}).$$

Let us show that (3.2) holds for any $\varphi \in \mathcal{D}$ such that $\|\varphi\|_\infty < \max(|\alpha|, |\beta|)^{-1}$. Fix such $\varphi \in \mathcal{D}$. Denote

$$O_\varphi := \{z \in \mathbb{C} : |z| < 1 + \frac{1}{2}(\max(|\alpha|, |\beta|)^{-1} - \|\varphi\|_\infty)\},$$

and consider the analytic function

$$O_\varphi \ni z \mapsto g(z) := \tilde{G}_\lambda(\omega, z\varphi).$$

For all $z \in \mathbb{C}$ such that $|z| \leq 1$, we have the Taylor expansion

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n. \tag{3.6}$$

But it follows from the proved above that

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \omega^{\otimes n} :_\lambda, \varphi^{\otimes n} \rangle z^n \tag{3.7}$$

for all $z \in \mathbb{C}$ from some neighborhood of zero. Comparing (3.6) and (3.7), we get

$$g^{(n)}(0) = \langle : \omega^{\otimes n} :_{\lambda}, \varphi^{\otimes n} \rangle, \tag{3.8}$$

and thus, by (3.6) and (3.8) we have proved the proposition in the case where $\omega \in \mathcal{M}(X)$.

Let us consider the general case. Fix $\tau \in T$. Without loss of generality, we can suppose that the inclusion $\mathcal{H}_{\tau} \hookrightarrow L^2(X; \sigma)$ is quasi-nuclear,

$$\|\varphi\|_{\infty} \leq C_{\tau} \|\varphi\|_{\tau}, \quad \varphi \in \mathcal{D}, \quad C_{\tau} > 0,$$

and $1 \in \mathcal{H}_{-\tau}$. Let

$$\mathcal{O}_{\tau} := \{ \varphi \in \mathcal{D}_{\mathbb{C}} : \|\varphi\|_{\tau} \leq (2C_{\tau} \max(|\alpha|, |\beta|))^{-1} \}.$$

It follows from (2.2) that

$$\sup_{\varphi \in \mathcal{O}_{\tau}} \max(\|\log(1 - \alpha\varphi)\|_{\tau}, \|\log(1 - \beta\varphi)\|_{\tau}) < \infty.$$

Then, for each fixed $\omega \in \mathcal{D}_{-\tau}$, the function $\tilde{G}_{\lambda}(\omega, \cdot)$ is \mathbf{G} -holomorphic and bounded on \mathcal{O}_{τ} . Hence, by e.g. [15], $\tilde{G}_{\lambda}(\omega, \cdot)$ is holomorphic on \mathcal{O}_{τ} . The Taylor decomposition of $\tilde{G}_{\lambda}(\omega, \cdot)$ and the kernel theorem (cf. [24, Subsection 4.1]) yield

$$\tilde{G}_{\lambda}(\omega, \varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle G_{\lambda}^{(n)}(\omega), \varphi^{\otimes n} \rangle, \tag{3.9}$$

where $G_{\lambda}^{(n)}(\omega) \in \mathcal{H}_{-\tau}^{\hat{\otimes} n}$ is given through

$$\langle G_{\lambda}^{(n)}(\omega), \varphi^{\otimes n} \rangle = \left. \frac{d^n}{dt^n} \right|_{t=0} \tilde{G}_{\lambda}(\omega, t\varphi).$$

Next, for each $\omega \in \mathcal{M}(X) \cap \mathcal{H}_{-\tau}$, we have, by the proved above,

$$\langle G_{\lambda}^{(n)}(\omega), \varphi^{\otimes n} \rangle = \langle : \omega^{\otimes n} :_{\lambda}, \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{D}. \tag{3.10}$$

Differentiating $\tilde{G}_{\lambda}(\omega, t\varphi)$ in t , we conclude that $\left. \frac{d^n}{dt^n} \right|_{t=0} \tilde{G}_{\lambda}(\omega, t\varphi)$ depends continuously on $\omega \in \mathcal{H}_{-\tau}$. On the other hand, $\langle : \omega^{\otimes n} :_{\lambda}, \varphi^{\otimes n} \rangle$ does also depend continuously on $\omega \in \mathcal{H}_{-\tau}$. Since $\mathcal{M}(X) \cap \mathcal{H}_{-\tau}$ is dense in $\mathcal{H}_{-\tau}$ ($\mathcal{M}(X) \cap \mathcal{H}_{-\tau}$, in particular, contains \mathcal{D}), we conclude that (3.10) holds for all $\omega \in \mathcal{H}_{-\tau}$, and hence also (3.9) holds for all $\omega \in \mathcal{H}_{-\tau}$. \square

Corollary 3.1. *For each $\lambda \geq 0$, the function*

$$\varphi \mapsto L_{\lambda}(\varphi) := \int_{\mathcal{D}} e^{\langle \omega, \varphi \rangle} \mu_{\lambda}(d\omega)$$

is well defined and holomorphic on some neighborhood of zero in $\mathcal{D}_{\mathbb{C}}$. Furthermore, for each fixed $\tau \in T$, there exists a neighborhood of zero in $\mathcal{D}_{\mathbb{C}}$, denoted by O_{τ} , such that, for all $\omega \in \mathcal{H}_{-\tau}$ and all $\varphi \in O_{\tau}$, we have

$$G_{\lambda}(\omega, \varphi) = \frac{e^{\langle \omega, \Psi_{\lambda}(\varphi) \rangle}}{L_{\lambda}(\Psi_{\lambda}(\varphi))}. \tag{3.11}$$

Here,

$$\Psi_{\lambda}(\varphi) := \frac{1}{\alpha - \beta} \log\left(\frac{1 - \beta\varphi}{1 - \alpha\varphi}\right), \quad \lambda \neq 2,$$

$$\Psi_2(\varphi) := \frac{\varphi}{\varphi + 1}$$

is a holomorphic $\mathcal{D}_{\mathbb{C}}$ -valued function defined in a neighborhood of zero in $\mathcal{D}_{\mathbb{C}}$ which is invertible and satisfies $\Psi_{\lambda}(0) = 0$.

Remark 3.2. Corollary 3.1 shows that the system of polynomials $\langle : \omega^{\otimes n} :_{\lambda, f^{(n)}} \rangle$, $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{Z}_+$, is a generalized Appell system in terms of [22], see also [1, 13, 18, 24] and the references therein. We also refer to [4] and the references therein for the study of the Appell systems via the theory of generalized translation operators.

Remark 3.3. Note that

$$\Psi_{\lambda}^{-1}(\varphi) := \frac{e^{\alpha\varphi} - e^{\beta\varphi}}{\alpha e^{\alpha\varphi} - \beta e^{\beta\varphi}}, \quad \lambda \neq 2,$$

$$\Psi_2^{-1}(\varphi) := \frac{\varphi}{1 - \varphi}.$$

Remark 3.4. In the one-dimensional case (Remarks 2.3 and 3.1), the function

$$z \mapsto \mathcal{L}_{\lambda}(z) := \int_{\mathbb{R}} e^{zx} m_{\lambda}(dx)$$

is well defined and holomorphic on a neighborhood of zero in \mathbb{C} (for the explicit formula, replace iu in formula (2.23) with z). Furthermore, for all $x \in \mathbb{R}$ and z from the neighborhood of zero, we have [26]

$$\mathcal{G}_{\lambda}(x, z) = \frac{e^{x\Psi_{\lambda}(z)}}{\mathcal{L}_{\lambda}(\Psi_{\lambda}(z))},$$

where $\Psi_{\lambda}(z) = \frac{1}{\alpha - \beta} \log\left(\frac{1 - \beta z}{1 - \alpha z}\right)$, $\lambda \neq 2$, $\Psi_2(z) = \frac{z}{z + 1}$.

Proof of Corollary 3.1. Fix $\lambda \geq 0$. Let $\tilde{L}_\lambda(\varphi)$ denote the right-hand side of (2.7), respectively (2.8), with $i\varphi$ replaced with φ . It follows from the proof of Theorem 2.1 and [6, Section 3.2] that there exists $\tau_0 \in T$ such that $\mu_\lambda(\mathcal{H}_{-\tau_0}) = 1$. By Proposition 3.1, there exists a neighborhood of zero in $\mathcal{D}_\mathbb{C}$, denoted by O_{τ_0} , such that for all $\varphi \in O_{\tau_0}$ and all $\omega \in \mathcal{H}_{-\tau_0}$

$$e^{\langle \omega, \varphi \rangle} = \tilde{L}_\lambda(\varphi) G_\lambda(\omega, \Psi_\lambda^{-1}(\varphi)). \tag{3.12}$$

Since the number of the summands in the sum on the right-hand side of (2.4) is $n!$, we easily conclude that

$$\|G(\cdot, \Psi_\lambda^{-1}(\varphi))\|_{L^2(\mathcal{D}'; \mu_\lambda)}^2 = \sum_{n=0}^\infty \left(\frac{1}{n!}\right)^2 \|(\Psi_\lambda^{-1}(\varphi))^{\otimes n}\|_{\mathcal{F}_{\text{ext}}^{(n)}(\mathcal{H})}^2 n! < \infty \tag{3.13}$$

for all φ from some (other) neighborhood of zero in $\mathcal{D}_\mathbb{C}$, denoted by O . Then, for all $\varphi \in \tilde{O}_{\tau_0} := O_{\tau_0} \cap O$, we get by (3.12) and (3.13):

$$\begin{aligned} \int_{\mathcal{D}'} e^{\langle \omega, \varphi \rangle} \mu_\lambda(d\omega) &= \int_{\mathcal{H}_{-\tau}} e^{\langle \omega, \varphi \rangle} \mu_\lambda(d\omega) \\ &= \tilde{L}_\lambda(\varphi) \int_{\mathcal{H}_{-\tau}} G_\lambda(\omega, \Psi_\lambda^{-1}(\varphi)) \mu_\lambda(d\omega) \\ &= \tilde{L}_\lambda(\varphi). \end{aligned} \tag{3.14}$$

Formula (3.11) in the case of $\mathcal{H}_{-\tau_0}$ follows from (3.12) and (3.14). The general case now easily follows from Proposition 3.1. \square

4. Operators ∂_x and ∂_x^\dagger

For each $\tau \in T$, we introduce on $\mathcal{P}(\mathcal{D}')$ a Hilbertian norm $\|\cdot\|_{\lambda, \tau}$ as follows: for any $\phi \in \mathcal{P}(\mathcal{D}')$ of the form $\phi(\omega) = \sum_{n=0}^N \langle : \omega^{\otimes n} :_{\lambda, \tau}, f^{(n)} \rangle$ (cf. Corollary 2.5), we set

$$\|\phi\|_{\lambda, \tau}^2 := \sum_{n=0}^N \|f^{(n)}\|_{\tau}^2 n!.$$

Let $(\mathcal{D}_\lambda)_\tau$ denote the Hilbert space obtained by closing $\mathcal{P}(\mathcal{D}')$ in this norm. By Kondratiev et al. [22, Theorem 34], there exists $\tau_0 \in T$ such that the Hilbert space $(\mathcal{D}_\lambda)_{\tau_0}$ is densely and continuously embedded into $L^2(\mathcal{D}'; \mu_\lambda)$. Just as in Section 2, we first set $T' := \{\tau \in T : \tau \geq \tau_0\}$ and then re-denote $T := T'$. Thus, each $(\mathcal{D}_\lambda)_\tau$, $\tau \in T$, consists of $(\mu_\lambda$ -classes of) functions on \mathcal{D}' of the form

$$\phi(\omega) = \sum_{n=0}^\infty \langle : \omega^{\otimes n} :_{\lambda, \tau}, f^{(n)} \rangle$$

with $f^{(n)} \in \mathcal{H}_\tau^{\otimes n}$ and

$$\|\phi\|_{\lambda, \tau}^2 := \sum_{n=0}^{\infty} \|f^{(n)}\|_\tau^2 (n!)^2 < \infty.$$

Let

$$(\mathcal{D}_\lambda) := \text{proj lim}_{\tau \in T} (\mathcal{D}_\lambda)_\tau, \tag{4.1}$$

which is a nuclear space [22, Theorem 32]. (We note that, though only the case of a nuclear space that is the projective limit of a countable family of Hilbert space is considered in [22], all the results we cite from this paper admit a straightforward generalization to the case of a general nuclear space.)

Denote by $\mathcal{E}_{\min}^1(\mathcal{D}'_{\mathbb{C}})$ the set of all entire functions on $\mathcal{D}'_{\mathbb{C}}$ of first order of growth and of minimal type, i.e., a function ϕ entire on $\mathcal{D}'_{\mathbb{C}}$ belongs to $\mathcal{E}_{\min}^1(\mathcal{D}'_{\mathbb{C}})$ if and only if

$$\forall \tau \in T, \forall \varepsilon > 0 \exists C > 0 : \forall \omega \in \mathcal{H}_{-\tau, \mathbb{C}} : |\phi(\omega)| \leq C e^{\varepsilon \|\omega\|_{-\tau}}.$$

Denote by $\mathcal{E}_{\min}^1(\mathcal{D}')$ the set of restrictions to \mathcal{D}' of functions from $\mathcal{E}_{\min}^1(\mathcal{D}'_{\mathbb{C}})$. Following [22, 24], we then introduce norms on $\mathcal{E}_{\min}^1(\mathcal{D}'_{\mathbb{C}})$, and hence on $\mathcal{E}_{\min}^1(\mathcal{D}')$, as follows. For each $\phi \in \mathcal{E}_{\min}^1(\mathcal{D}'_{\mathbb{C}})$ and for any $\tau \in T$ and $q \in \mathbb{N}$, we set

$$n_{\tau, q}(\phi) := \sup_{z \in \mathcal{H}_{-\tau, \mathbb{C}}} (|\phi(z)| \exp(-2^{-q} \|z\|_{-\tau})).$$

Next, each $\phi \in \mathcal{E}_{\min}^1(\mathcal{D}'_{\mathbb{C}})$ can be uniquely represented in the form $\phi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, f^{(n)} \rangle$, and we set, for any $\tau \in T$ and $q \in \mathbb{N}$,

$$N_{\tau, q}(\phi) := \sum_{n=0}^{\infty} \|f^{(n)}\|_\tau^2 (n!)^2 2^{nq}.$$

By Kondratiev et al. [22, Theorems 2.5, 3.8, Section 6.2], the three systems of norms on (\mathcal{D}_λ) :

$$(\|\cdot\|_{\lambda, \tau}, \tau \in T), \quad (n_{\tau, q}(\cdot), \tau \in T, q \in \mathbb{N}), \quad (N_{\tau, q}(\cdot), \tau \in T, q \in \mathbb{N}),$$

are equivalent, and hence determine the same topology on (\mathcal{D}_λ) .

As easily seen, for each $\varphi \in \mathcal{D}$, $I_\lambda a_1^-(\xi) I_\lambda^{-1}$ can be extended to a continuous operator on (\mathcal{D}_λ) . We denote this operator $A_1^-(\xi)$. Next, for each $x \in X$, we denote by ∂_x the linear continuous operator on (\mathcal{D}_λ) defined by

$$\partial_x \langle : \omega^{\otimes n} :_{\lambda, f^{(n)}} \rangle := n \langle : \omega^{\otimes (n-1)} :_{\lambda, f^{(n)}}(x, \cdot) \rangle, \quad f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\otimes n}.$$

Lemma 4.1. For any $\phi \in (\mathcal{D}_\lambda)$, we have

$$\forall \omega \in \mathcal{D}' : (A_1^-(\xi)\phi)(\omega) = \int_X \xi(x)(\partial_x \phi)(\omega) \sigma(dx).$$

Proof. Let $\tau_0 \in T$ be such that $\|\delta_x\|_{\tau_0} \leq 1$ for all $x \in X$. Fix $\omega \in \mathcal{H}_{-\tau}$, $\tau \geq \tau_0$. Let $\tau' > \tau$ be such that the inclusion $\mathcal{H}_{-\tau_0} \hookrightarrow \mathcal{H}_{-\tau}$ is quasi-nuclear. By Kondratiev et al. [22, Proposition 22], we have for any $\varepsilon > 0$

$$\| : \omega^{\otimes n} :_\lambda \|_{-\tau'} \leq n! C_\tau^n \exp(\varepsilon \|\omega\|_{-\tau}), \quad C_\tau > 0. \tag{4.2}$$

Hence, we may estimate

$$\begin{aligned} \sum_{n=1}^\infty n | \langle : \omega^{\otimes(n-1)} :_\lambda, f^{(n)}(x, \cdot) \rangle | &\leq \sup_{x \in X} \|\delta_x\|_{-\tau'} \exp(\varepsilon \|\omega\|_{-\tau}) \\ &\times \sum_{n=1}^\infty (n-1)! C_\tau^{n-1} n \|f^{(n)}\|_{\tau'}. \end{aligned}$$

Therefore, for each $\phi = \sum_{n=0}^\infty \langle : \cdot^{\otimes n} :_\lambda, f^{(n)} \rangle \in (\mathcal{D}_\lambda)$, we get, by the majorized convergence theorem,

$$\int_X \xi(x) \sum_{n=0}^\infty n \langle : \omega^{\otimes(n-1)} :_\lambda, f^{(n)}(x, \cdot) \rangle \sigma(dx) = \sum_{n=0}^\infty \langle : \omega^{\otimes(n-1)} :_\lambda, n \langle f^{(n)}, \xi \rangle \rangle,$$

where

$$\langle f^{(n)}, \xi \rangle(x_1, \dots, x_{n-1}) := \int_X f^{(n)}(x, x_1, \dots, x_{n-1}) \xi(x) \sigma(dx).$$

From here, the lemma follows. \square

Remark 4.1. Let $\tau, \tau' \in T$ be as in proof of Lemma 4.1. Note that the operator ∂_x acts continuously in $(\mathcal{D}_\lambda)_\tau$. By (4.2), we then have for all $\omega \in \mathcal{H}_{-\tau}$ and all $\varphi \in \mathcal{D}_\mathbb{C}$ such that $\|\varphi\|_{\tau'} < \min(1, C_\tau^{-1})$:

$$\partial_x \sum_{n=0}^\infty \frac{1}{n!} \langle : \omega^{\otimes n} :_\lambda, \varphi^{\otimes n} \rangle = \varphi(x) \sum_{n=0}^\infty \frac{1}{n!} \langle : \omega^{\otimes n} :_\lambda, \varphi^{\otimes n} \rangle.$$

By Corollary 3.1, we get for all $\omega \in \mathcal{H}_{-\tau}$ and all φ from a neighborhood of zero in $\mathcal{D}_\mathbb{C}$:

$$\partial_x e^{\langle \omega, \Psi_\lambda(\varphi) \rangle} = \varphi(x) e^{\langle \omega, \Psi_\lambda(\varphi) \rangle},$$

and consequently

$$\partial_x e^{\langle \omega, \varphi \rangle} = (\Psi_\lambda^{-1}(\varphi))(x) e^{\langle \omega, \varphi \rangle}. \tag{4.3}$$

Let ∇_x denote the Gâteaux derivative of a function defined on \mathcal{D}' in direction δ_x , i.e., $\nabla_x F(\omega) := \frac{d}{dt} \Big|_{t=0} F(\omega + t\delta_x)$. Clearly,

$$\nabla_x e^{\langle \omega, \varphi \rangle} = \varphi(x) e^{\langle \omega, \varphi \rangle}. \tag{4.4}$$

Comparing (4.3) and (4.4), we get (at least informally):

$$\partial_x = \Psi_\lambda^{-1}(\nabla_x). \tag{4.5}$$

Remark 4.2. In the one-dimensional case (Remarks 2.3, 3.1, 3.4), we define a linear operator A by

$$AP_\lambda^{(n)}(\cdot) = nP_\lambda^{(n-1)}(\cdot).$$

By Remarks 3.1 and 3.4, one then gets (cf. [26])

$$A = \Psi_\lambda^{-1}(D), \tag{4.6}$$

where $D = \frac{d}{dx}$. Thus, (4.5) is an infinite-dimensional counterpart of (4.6).

Theorem 4.1. For each $\lambda \geq 0$ and for all $\phi \in (\mathcal{D}_\lambda)$ and $\omega \in \mathcal{D}'$:

$$\begin{aligned} (\partial_x \phi)(\omega) &= \int_{\mathbb{R}} (\phi(\omega + s\delta_x) - \phi(\omega)) s v_\lambda(ds), \\ (A_1^-(\xi)\phi)(\omega) &= \int_{X \times \mathbb{R}} (\phi(\omega + s\delta_x) - \phi(\omega)) s \xi(x) \sigma(dx) v_\lambda(ds), \end{aligned} \tag{4.7}$$

where $x \in X$, $\xi \in \mathcal{D}$, and v_λ is the Lévy measure of μ_λ (see Corollaries 2.3 and 2.4).

Proof. By Lemma 4.1, it suffices to prove the statement only for ∂_x . Fix $\tau \geq \tau_0$ as in proof of Lemma 4.1. Then, for all $\omega \in \mathcal{H}_{-\tau}$ and all φ from a neighborhood of zero in $\mathcal{D}_\mathbb{C}$, we have

$$\begin{aligned} \int_{\mathbb{R}} (e^{\langle \omega + s\delta_x, \varphi \rangle} - e^{\langle \omega, \varphi \rangle}) s v_\lambda(ds) &= e^{\langle \omega, \varphi \rangle} \int_{\mathbb{R}} (e^{s\varphi(x)} - 1) s v_\lambda(ds) \\ &= e^{\langle \omega, \varphi \rangle} (\Psi_\lambda^{-1}(\varphi))(x), \end{aligned} \tag{4.8}$$

where Ψ_λ^{-1} is given by (3.3). The latter equality in (4.8) can be derived by differentiating in θ the following equality:

$$\int_{\mathbb{R}} (e^{s\theta} - 1 - s\theta)v_\lambda(ds) = \int_{\mathbb{R}} \sum_{n=2}^{\infty} \frac{s^{n-2}\theta^n}{n!} s^2 v_\lambda(ds) = \begin{cases} -\frac{1}{\alpha\beta} \log\left(\frac{\alpha e^{-\beta\theta} - \beta e^{-\alpha\theta}}{\alpha - \beta}\right), & \lambda \neq 2, \\ -\log(1 - \theta) - \theta, & \lambda = 2, \end{cases}$$

which holds for all θ from a neighborhood of zero in \mathbb{C} (this equality has been already used in course of the proof of Corollaries 2.3 and 2.4). Therefore, by Remark 4.1, we have (4.7) for all $x \in X$, $\omega \in \mathcal{H}_{-\tau}$ and $\phi = G_\lambda(\cdot, \varphi)$, where φ runs through a neighborhood of zero in $\mathcal{D}_{\mathbb{C}}$, denoted by U_τ .

As easily seen, there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} e^{\varepsilon|s|} s^2 v_\lambda(ds) < \infty \tag{4.9}$$

(in the case $\lambda < 2$, see e.g. [12, p. 180]). Choose $q \in \mathbb{N}$ such that $2^{-q/2} < \varepsilon$. Choose $\tau_1, \tau_2 \in T$ and $q_1 \in \mathbb{N}$ such that $\tau < \tau_1 < \tau_2$,

$$CN_{\tau,q}(\cdot) \leq C_1 n_{\tau_1,q_1}(\cdot) \leq \|\cdot\|_{\lambda,\tau_2}, \quad C, C_1 > 0. \tag{4.10}$$

Fix any $\phi \in (\mathcal{D}_\lambda)$. For each $k \in \mathbb{N}$, let ϕ_k be a linear combination of functions $G_\lambda(\cdot, \varphi)$ with $\varphi \in U_\tau$ and let $\|\phi_k - \phi\|_{\lambda,\tau_2} \rightarrow 0$ as $k \rightarrow \infty$ (evidently, such a sequence $\{\phi_k\}$ always exists). By (4.10), we then have

$$n_{\tau_1,q_1}(\phi_k - \phi) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{4.11}$$

For a fixed $\omega \in \mathcal{H}_{-\tau} \subset \mathcal{H}_{-\tau_1}$, we define

$$\tilde{\phi}_k(z) := \phi_k(\omega + z) - \phi_k(\omega), \quad \tilde{\phi}(z) := \phi(\omega + z) - \phi(\omega)$$

for $z \in \mathcal{H}_{-\tau_1, \mathbb{C}}$. One easily checks that (4.11) implies

$$n_{\tau_1,q_1}(\tilde{\phi}_k - \tilde{\phi}) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and hence, by (4.10)

$$N_{\tau,q}(\tilde{\phi}_k - \tilde{\phi}) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We have

$$\tilde{\phi}_k(z) = \sum_{n=1}^{\infty} \langle z^{\otimes n}, f_k^{(n)} \rangle$$

(note that $\tilde{\phi}_k(0) = 0$). Since $\|\delta_x\|_{-\tau} \leq \|\delta_x\|_{-\tau_0} \leq 1$ for all $x \in X$, we get, using the Cauchy inequality ([24, proof of Lemma 2.7])

$$\begin{aligned} |\tilde{\phi}_k(s\delta_x)| |s|^{-1} &\leq \sum_{n=1}^{\infty} |s|^{n-1} \|f_k^{(n)}\|_{\tau_1} \\ &\leq \sum_{n=1}^{\infty} \max(1, |s|)^n \|f_k^{(n)}\|_{\tau_1} \\ &\leq N_{\tau,q}(\tilde{\phi}_k) \exp(2^{-q/2} \max(1, |s|)) \\ &\leq C \exp(\varepsilon \max(1, |s|)), \quad k \in \mathbb{N}, \end{aligned}$$

where $C \in (0, \infty)$ is independent of $k \in \mathbb{N}$. Hence, by (4.9) and the majorized convergence theorem,

$$\int_{\mathbb{R}} \tilde{\phi}_k(s\delta_x) v_{\lambda}(ds) \rightarrow \int_{\mathbb{R}} \tilde{\phi}(s\delta_x) v_{\lambda}(ds) \quad \text{as } k \rightarrow \infty. \tag{4.12}$$

Finally, ∂_x acts continuously on $(\mathcal{D}_{\lambda})_{\tau_2}$, and hence $\|\partial_x \phi_k - \partial_x \phi\|_{\lambda, \tau_2} \rightarrow 0$. Thus, by (4.10), we get $n_{\tau_1, q_1}(\partial_x \phi_k - \partial_x \phi) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, $(\partial_x \phi_k)(\omega) \rightarrow (\partial_x \phi)(\omega)$ as $k \rightarrow \infty$, which, together with (4.12) concludes the proof. \square

Let $(\mathcal{D}_{\lambda})_{-\tau}$ denote the dual space of $(\mathcal{D}_{\lambda})_{\tau}$. Analogous to (2.11), we realize $(\mathcal{D}_{\lambda})_{-\tau}$ as the Hilbert space consisting of sequences $F = (F^{(n)})_{n=0}^{\infty}$, $F^{(n)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{\otimes n}$, such that

$$\|F\|_{\lambda, -\tau}^2 := \sum_{n=0}^{\infty} \|F^{(n)}\|_{-\tau}^2 < \infty,$$

with the scalar product in $(\mathcal{D}_{\lambda})_{-\tau}$ generated by the Hilbertian norm $\|\cdot\|_{\lambda, -\tau}$, and the dual pairing of $F = (F^{(n)})_{n=0}^{\infty}$ with an element

$$\phi = \sum_{n=0}^{\infty} \langle \cdot \otimes^n :_{\lambda}, f^{(n)} \rangle \in (\mathcal{D}_{\lambda})_{\tau} \tag{4.13}$$

is given by

$$\langle\langle F, \phi \rangle\rangle = \sum_{n=0}^{\infty} \langle \overline{F^{(n)}}, f^{(n)} \rangle n!.$$

By (4.1), we get the following representation of the dual space of (\mathcal{D}_{λ}) :

$$(\mathcal{D}_{\lambda})^* = \text{ind lim}_{\tau \in T} (\mathcal{D}_{\lambda})_{-\tau}.$$

Each test space $(\mathcal{D}_\lambda)_\tau$ can be embedded into $(\mathcal{D}_\lambda)_{-\tau}$ by setting, for ϕ of form (4.13), $\iota(\phi) := (f^{(n)})_{n=0}^\infty$. In what follows, we will just write ϕ instead of $\iota(\phi)$ to simplify notations.

For each $x \in X$, we define an operator ∂_x^\dagger on $(\mathcal{D}_\lambda)^*$ by

$$\partial_x^\dagger (F^{(n)})_{n=0}^\infty := (\delta_x \hat{\otimes} F^{(n-1)})_{n=0}^\infty.$$

Evidently, ∂_x^\dagger is the dual operator of ∂_x and acts continuously in each $(\mathcal{D}_\lambda)_{-\tau}$, $\tau \geq \tau_0$ (with τ_0 as in the proof of Lemma 4.1). We define operators $:\omega(x) :_\lambda \cdot$ and $\omega(x) \cdot$, acting continuously from each $(\mathcal{D}_\lambda)_\tau$ into $(\mathcal{D}_\lambda)_{-\tau}$, $\tau \geq \tau_0$, by

$$\begin{aligned} :\omega(x) :_\lambda \cdot &:= \partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x + \partial_x^\dagger \partial_x \partial_x, \\ \omega(x) \cdot &:= \omega(x) :_\lambda \cdot + c_\lambda \text{id}. \end{aligned} \tag{4.14}$$

Analogous to $A_1^-(\xi)$, we define operators $A_2^-(\xi)$, $A^0(\xi)$, and $A^+(\xi)$. Let also $\langle : \omega^{\otimes 1} :_{\lambda, \xi} \cdot$ and $\langle \omega, \xi \rangle \cdot$ denote the operators of multiplication by $\langle : \omega^{\otimes 1} :_{\lambda, \xi} \rangle$ and $\langle \omega, \xi \rangle$, respectively. We then easily get the following integral representation:

$$\begin{aligned} A^+(\xi) &= \int_X \sigma(dx) \xi(x) \partial_x^\dagger, & A^0(\xi) &= \int_X \sigma(dx) \xi(x) \partial_x^\dagger \partial_x, \\ A_2^-(\xi) &= \int_X \sigma(dx) \xi(x) \partial_x^\dagger \partial_x \partial_x, \\ \langle : \omega^{\otimes 1} :_{\lambda, \xi} \cdot &= \int_X \sigma(dx) \xi(x) : \omega(x) :_\lambda \cdot, \\ \langle \omega, \xi \rangle \cdot &= \int_X \sigma(dx) \xi(x) \omega(x) \cdot, \end{aligned} \tag{4.15}$$

where the integrals are understood in the sense that one applies pointwise the integrand operator to a test function from (\mathcal{D}_λ) , then dualizes the result with another test function, and finally integrates the obtained function with respect to the measure σ .

Before formulating the next theorem, we note that, for each $x \in X$, the operator ∇_x introduced in Remark 4.1 acts continuously on (\mathcal{D}_λ) . This can be easily shown using methods as in the proof of Theorem 4.1. Furthermore, we introduce on (\mathcal{D}_λ) continuous operators $\nabla_{\alpha-\beta, x}$ and $\mathcal{U}_{\alpha-\beta, x}$ as follows:

$$\nabla_{\alpha-\beta, x} \phi(\omega) := \frac{\phi(\omega + (\alpha - \beta)\delta_x) - \phi(\omega)}{\alpha - \beta},$$

$$\mathcal{U}_{\alpha-\beta, x} \phi(\omega) := \phi(\omega - (\alpha - \beta)\delta_x).$$

Notice that, in the case $\lambda \in [0, 2)$, we have $\alpha - \beta = \alpha - \bar{\alpha}$, which is a purely imaginary number, so when writing either $\phi(\omega + (\alpha - \beta)\delta_x)$ or $\phi(\omega - (\alpha - \beta)\delta_x)$, we understand under ϕ its entire extension to $\mathcal{D}'_{\mathbb{C}}$.

Theorem 4.2. For each $x \in X$, ∂_x^\dagger considered as an operator from (\mathcal{D}_λ) into $(\mathcal{D}_\lambda)^*$ has the following representation:

$$\partial_x^\dagger = \begin{cases} : \omega(x) :_\lambda \cdot (1 + \alpha \nabla_{\alpha-\beta, x})^2 \mathcal{U}_{\alpha-\beta, x} \\ \quad - (1 + \alpha \nabla_{\alpha-\beta, x}) \nabla_{\alpha-\beta, x} \mathcal{U}_{\alpha-\beta, x}, & \lambda \neq 2, \\ \omega(x) \cdot (\nabla_x - 1)^2 + (\nabla_x - 1), & \lambda = 2. \end{cases} \tag{4.16}$$

Proof. We prove the theorem only in the case $\lambda \neq 2$ and refer to [21, Lemma 7.1] for the case $\lambda = 2$. By (4.14) and (1.4),

$$\begin{aligned} : \omega(x) :_\lambda \cdot &= \partial_x^\dagger (1 + \lambda \partial_x + \partial_x^2) + \partial_x \\ &= \partial_x^\dagger (1 - \alpha \partial_x) (1 - \beta \partial_x) + \partial_x. \end{aligned} \tag{4.17}$$

Fix $\tau \geq \tau_0$ and choose $\tau' \in T$, $\tau' > \tau$, such that the operators $(1 + \alpha \nabla_{\alpha-\beta, x})^2 \mathcal{U}_{\alpha-\beta, x}$ and $(1 + \alpha \nabla_{\alpha-\beta, x}) \nabla_{\alpha-\beta, x} \mathcal{U}_{\alpha-\beta, x}$ act continuously from $(\mathcal{D}_\lambda)_{\tau'}$ into $(\mathcal{D}_\lambda)_\tau$. Take any $\varphi \in \mathcal{D}$ such that $e^{\langle \cdot, \varphi \rangle} \in (\mathcal{D}_\lambda)_{\tau'}$. Then, by (4.3) and (4.17),

$$\begin{aligned} : \omega(x) :_\lambda \cdot e^{\langle \cdot, \varphi \rangle} &= \partial_x^\dagger (1 - \alpha (\Psi_\lambda^{-1}(\varphi))(x)) (1 - \beta (\Psi_\lambda^{-1}(\varphi))(x)) e^{\langle \cdot, \varphi \rangle} \\ &\quad + (\Psi_\lambda^{-1}(\varphi))(x) e^{\langle \cdot, \varphi \rangle}. \end{aligned} \tag{4.18}$$

Denoting

$$\nabla_{\alpha-\beta} \varphi(x) := \frac{e^{\varphi(x)(\alpha-\beta)} - 1}{\alpha - \beta},$$

we get

$$(\Psi_\lambda^{-1}(\varphi))(x) = \frac{\nabla_{\alpha-\beta} \varphi(x)}{1 + \alpha \nabla_{\alpha-\beta} \varphi(x)} \tag{4.19}$$

(compare with [26, formula (7.2)]). By (4.18) and (4.19),

$$: \omega(x) :_\lambda \cdot e^{\langle \cdot, \varphi \rangle} = \partial_x^\dagger \frac{e^{\varphi(x)(\alpha-\beta)}}{(1 + \alpha \nabla_{\alpha-\beta} \varphi(x))^2} e^{\langle \cdot, \varphi \rangle} + \frac{\nabla_{\alpha-\beta} \varphi(x)}{1 + \alpha \nabla_{\alpha-\beta} \varphi(x)} e^{\langle \cdot, \varphi \rangle},$$

which yields

$$\begin{aligned} : \omega(x) :_\lambda \cdot &(1 + \alpha \nabla_{\alpha-\beta} \varphi(x))^2 e^{-\varphi(x)(\alpha-\beta)} e^{\langle \cdot, \varphi \rangle} \\ &= \partial_x^\dagger e^{\langle \cdot, \varphi \rangle} + \nabla_{\alpha-\beta} \varphi(x) (1 + \alpha \nabla_{\alpha-\beta} \varphi(x)) e^{-\varphi(x)(\alpha-\beta)} e^{\langle \cdot, \varphi \rangle}. \end{aligned} \tag{4.20}$$

Since the set of all functions $e^{\langle \cdot, \varphi \rangle}$ with φ as above is total in $(\mathcal{D}_\lambda)_{\tau'}$ and since $: \omega(x) :_\lambda \cdot$ and ∂_x^\dagger act continuously from $(\mathcal{D}_\lambda)_\tau$ into $(\mathcal{D}_\lambda)_{-\tau}$, (4.20) implies (4.16). \square

Corollary 4.1. For any $\phi \in (\mathcal{D}_\lambda)$ and $\xi \in \mathcal{D}$, we have for all $\omega \in \mathcal{D}'$

$$\begin{aligned} & A^+(\xi)\phi(\omega) \\ &= \langle \omega(x) - c_\lambda, \xi(x)(1 + \alpha \nabla_{\alpha-\beta, x})^2 \mathcal{U}_{\alpha-\beta, x} \phi(\omega) \rangle \\ &\quad - \langle 1, \xi(x)(1 + \alpha \nabla_{\alpha-\beta, x}) \nabla_{\alpha-\beta, x} \mathcal{U}_{\alpha-\beta, x} \phi(\omega) \rangle, \quad \lambda \neq 2, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} A^+(\xi)\phi(\omega) &= \langle \omega(x), \xi(x)(\nabla_x - 1)^2 \phi(\omega) \rangle \\ &\quad + \langle 1, \xi(x)(\nabla_x - 1)\phi(\omega) \rangle, \quad \lambda = 2, \end{aligned}$$

where x denotes the variable in which the dualization is carried out.

Proof. Again, we prove only in the case $\lambda \neq 2$ and refer to [21, Theorem 7.1] for the case $\lambda = 2$. Fix any $\tau \geq \tau_0$. Using methods as in the proof of Theorem 4.1, we conclude the existence of $\tau_1 \in T$, $\tau_1 > \tau$, such that, for any fixed $\omega \in \mathcal{H}_{-\tau}$ and $z \in \mathbb{C}$, $|z| \leq 2|\alpha - \beta|$, and for any sequence $\{\phi_k, k \in \mathbb{N}\} \subset (\mathcal{D}_\lambda)_{\tau_1}$ such that $\phi_k \rightarrow \phi$ in $(\mathcal{D}_\lambda)_{\tau_1}$ as $k \rightarrow \infty$, we have $\xi \tilde{\phi}_{k,z} \rightarrow \xi \tilde{\phi}_z$ in $\mathcal{H}_{\tau, \mathbb{C}}$ as $k \rightarrow \infty$. Here,

$$X \ni x \mapsto \tilde{\phi}_{k,z}(x) := \phi_k(\omega + z\delta_x) \in \mathbb{C}, \quad X \ni x \mapsto \tilde{\phi}_z(x) := \phi(\omega + z\delta_x) \in \mathbb{C}.$$

Choose $\tau_2 > \tau_1$ such that $Cn_{\tau_1, 1}(\cdot) \leq \|\cdot\|_{\lambda, \tau_2}$, $C > 0$, and choose $\tau_3 > \tau_2$ such that $A^+(\xi)$ acts continuously from $(\mathcal{D}_\lambda)_{\tau_3}$ into $(\mathcal{D}_\lambda)_{\tau_2}$. Fix any $\varphi \in \mathcal{D}$ such that $e^{\langle \cdot, \varphi \rangle} \in (\mathcal{D}_\lambda)_{\tau_3}$. Then, for any $\phi \in (\mathcal{D}_\lambda)$, we get by (the proof of) Theorem 4.2 and (4.15)

$$\begin{aligned} \langle A^+(\xi)e^{\langle \cdot, \varphi \rangle}, \psi \rangle &= \int_X \xi(x) \langle \partial_x^\dagger e^{\langle \cdot, \varphi \rangle}, \psi \rangle \sigma(dx) \\ &= \int_X \xi(x)(1 + \alpha \nabla_{\alpha-\beta} \varphi(x))^2 e^{-\varphi(x)(\alpha-\beta)} \\ &\quad \times \langle \omega(x) :_\lambda \cdot e^{\langle \cdot, \varphi \rangle}, \psi \rangle \sigma(dx) \\ &\quad - \int_X \xi(x)(1 + \alpha \nabla_{\alpha-\beta} \varphi(x)) \nabla_{\alpha-\beta} \varphi(x) e^{-\varphi(x)(\alpha-\beta)} \\ &\quad \times \langle e^{\langle \cdot, \varphi \rangle}, \psi \rangle \sigma(dx) \\ &= \langle \langle \cdot : \otimes^1 :_\lambda, \xi(1 + \alpha \nabla_{\alpha-\beta} \varphi)^2 e^{-\varphi(\alpha-\beta)} \rangle \cdot e^{\langle \cdot, \varphi \rangle}, \psi \rangle \sigma(dx) \\ &\quad - \langle \langle 1, \xi(1 + \alpha \nabla_{\alpha-\beta} \varphi) \nabla_{\alpha-\beta} \varphi e^{-\varphi(\alpha-\beta)} \rangle e^{\langle \cdot, \varphi \rangle}, \psi \rangle, \end{aligned}$$

which implies (4.21) for $\phi = e^{\langle \cdot, \varphi \rangle}$ and $\omega \in \mathcal{H}_{-\tau}$. Now, approximate an arbitrary $\phi \in (\mathcal{D}_\lambda)$ in the $\|\cdot\|_{\lambda, \tau_3}$ norm by a sequence $\{\phi_k, k \in \mathbb{N}\}$ of linear combinations of

exponents as above. Then, $A^+(\xi)\phi_k \rightarrow A^+(\xi)\phi$ as $k \rightarrow \infty$ in the $\|\cdot\|_{\lambda, \tau_2}$ norm, and hence in the $n_{\tau_1, 1}(\cdot)$ norm. In particular, $A^+(\xi)\phi_k(\omega) \rightarrow A^+(\xi)\phi(\omega)$ for any $\omega \in \mathcal{H}_{-\tau_1}$. Furthermore, for any fixed $\omega \in \mathcal{H}_{-\tau}$, we get

$$\xi(1 + \alpha \nabla_{\alpha-\beta, \bullet})^2 \mathcal{U}_{\alpha-\beta, \bullet} \phi_k(\omega) \rightarrow \xi(1 + \alpha \nabla_{\alpha-\beta, \bullet})^2 \mathcal{U}_{\alpha-\beta, \bullet} \phi(\omega),$$

$$\xi(1 + \alpha \nabla_{\alpha-\beta, \bullet}) \nabla_{\alpha-\beta, \bullet} \mathcal{U}_{\alpha-\beta, \bullet} \phi_k(\omega) \rightarrow \xi(1 + \alpha \nabla_{\alpha-\beta, \bullet}) \nabla_{\alpha-\beta, \bullet} \mathcal{U}_{\alpha-\beta, \bullet} \phi(\omega)$$

in $\mathcal{H}_{-\tau, \mathbb{C}}$. From here, we evidently get (4.21) for an arbitrary $\phi \in (\mathcal{D}_\lambda)$ and an arbitrary $\omega \in \mathcal{H}_{-\tau}$. \square

Remark 4.3. Analogous to Corollary 4.1, one can derive from Theorems 4.1, 4.2 and (4.15) explicit formulas for the action of the operators $A^0(\xi)$ and $A_2^-(\xi)$ (see also [21, Theorems 7.2, 7.3] for the case $\lambda = 2$).

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