GROUPS WITH ISOMORPHIC HOMOLOGY GROUPS AND
NON-CANCELLATION IN HOMOTOPY THEORY

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0. Introduction

When Mislin [4] invented the notion of genus for (isomorphism classes of) finitely
generated nilpotent groups, it was immediately recognized that this constituted a
source of examples of groups with isomorphic homology groups. We say that the
(isomorphism classes of) groups $M$ and $N$ are in the same genus if, for every prime
$p$, the $p$-localizations $M_p$ and $N_p$ are isomorphic. But then $H_*M, H_*N$ are finitely
generated abelian groups in the same genus, and hence isomorphic.

It was noted in [3, 4] that $G(N)$, the genus of $N$, is particularly easy to calculate
if the commutator subgroup of $N$ is finite, since $G(N)$ then may be given the struc-
ture of a finite abelian group which is a quotient of some group $(\mathbb{Z}/e)^*/\{\pm 1\}$, where
$e$ is an integer easily calculable from $N$. In [1, 2] the calculation was carried out for
certain specific groups. Thus, let $n$ be a positive integer, let $u$ be prime to $n$ and let
t be the order of $u$ modulo $n$. We consider the group

$$N = \langle x, y; x^n = 1, yxy^{-1} = x^u \rangle. \quad (0.1)$$

Now let $m$ be a positive integer prime to $t$ and let

$$N(m) = \langle x, y; x^n = 1, yxy^{-1} = x^{u^m} \rangle, \quad (0.2)$$

so that $N = N(1)$. One may then prove [2]:

**Theorem 0.1.** (i) $N$, and hence also $N(m)$, is nilpotent if and only if $p | (u - 1)$ for
every prime divisor $p$ of $n$.

(ii) If $m \not\equiv 1 \mod t$, then $N \not\equiv N(m)$.

(iii) If $N$ is nilpotent, then $N(m) \in G(N)$.

(iv) $N$ embeds as a normal subgroup of $N(m)$ with quotient $C_t$, where
$lm \equiv 1 \mod t$; and $N(m)$ embeds as a normal subgroup of $N$ without quotient $C_n$. 
We now suppose \( N \) in (0.1) to be nilpotent and write
\[
\begin{align*}
n &= p_1^{v_1}p_2^{v_2} \cdots p_j^{v_j}, \quad p_1 < p_2 < \cdots < p_j, \quad v_i \geq 1 \\
u &= 1 + cp_1^{\mu_1}p_2^{\mu_2} \cdots p_j^{\mu_j}, \quad p_i \neq c, \quad \mu_i \geq 1.
\end{align*}
\]
(0.3)

Set
\[
\kappa_i = \begin{cases} 
0 & \text{if } v_i \leq \mu_i, \\
\nu_i - \mu_i & \text{if } v_i > \mu_i.
\end{cases}
\]
(0.4)

Then one may show, by a slight generalization of the argument in [2],

**Theorem 0.2.** (i) The order of \( u \) modulo \( n \) is given by
\[
t = p_1^{\kappa_1}p_2^{\kappa_2} \cdots p_j^{\kappa_j}
\]
(0.5)

unless \( p_1 = 2, \mu_1 = 1, \kappa_i \geq 1 \)

(ii) \( G(N) \cong (\mathbb{Z}/t)^*/\{ \pm 1 \} \), where \( t \) is given by (0.5), unless \( p_i = 2, \mu_1 = 1, \kappa_i > 1 \).

**Remark.** In the exceptional case we write \( u = -1 + c'2^{\mu'} \) where \( c' \) is odd, so that \( \mu' > 2 \). We then replace \( \kappa_1 \) in (0.5) by
\[
\kappa_i' = \begin{cases} 
1 & \text{if } 2 \leq v_i \leq \mu', \\
\nu_i - \mu' & \text{if } v_i > \mu'.
\end{cases}
\]

(0.4)

to obtain the order of \( u \) modulo \( n \). Theorem 0.2(ii) remains valid in the exceptional case, with the value of \( t \) thus modified.

Theorem 0.2 gives us the structure of \( G(N) \). Now it was shown in [3] that if, in the group \( G(N) \), \( N_1 + N_2 + \cdots + N_j = N'_1 + N'_2 + \cdots + N'_j \), then \( \prod_{i=1}^j N_i = \prod_{i=1}^j N'_i \). We conclude that there exists an integer \( k \) such that
\[
N^k \equiv N(m)^k, \quad \text{if } N \text{ is nilpotent},
\]
(0.6)

and that we may take \( k = |G(N)| = \frac{1}{2} \Phi(t) \) if \( t \neq 2 \).

Now Wilkerson and others have pointed to the connection between genus and non-cancellation phenomena. We proved in [2] that, whether or not \( N \) is nilpotent, we have the isomorphism
\[
N \times C \cong N(m) \times C.
\]
(0.7)

One may, in fact, calculate the homology groups of \( N \) and \( N(m) \) using the Lyndon–Hochschild–Serre spectral sequence based on the extensions
\[
C_n \rightarrowtail N \twoheadrightarrow C, \quad C_n \rightarrowtail N(m) \twoheadrightarrow C.
\]
(0.8)

It is then easy, in any case, to see that the homology groups are finitely generated, so that it follows readily from (0.7) that
\[
H_iN \cong H_iN(m),
\]
(0.9)

whether or not \( N \) is nilpotent.
Our principal purpose in this paper is to associate with the groups \( N, N(m) \) certain spaces \( X = X(1), X(m) \) which exhibit phenomena paralleling those described in Theorem 0.1, (0.6), (0.7), (0.9). Our procedure is based on (0.8) which presents \( N \) and \( N(m) \) as semidirect products for actions of \( C \) on \( C_n \) (or \( \mathbb{Z}/n \)). Thus we have two groups-with-operators, both having \( \mathbb{Z}/n \) as underlying groups; and these groups-with-operators are isomorphic if and only if the associated semidirect products are isomorphic. We then proceed to realize these groups with operators by means of spaces \( X(1), X(m) \) such that \( \pi_1 X(1) = \pi_1 X(m) = C \) and \( \pi_2 X(1) = \pi_2 X(m) = \mathbb{Z}/n \) but the actions are those associated with the extensions (0.8), respectively, that is,

\[
\xi \cdot a = ua \quad \text{in } X(1), \quad \xi \cdot a = u^m a \quad \text{in } X(m).
\]

where \( \pi_1 X(1) - \pi_1 X(m) = \langle \xi \rangle \) and \( a \in \pi_2 X(1) - \pi_2 X(m) \).

Note that we distinguish notationally between the multiplicative groups \( C, C_n \) and the additive groups \( \mathbb{Z}, \mathbb{Z}/n \). This proves convenient in describing group actions. We then draw particular attention to the noncancellations phenomenon (Theorem 1.2)

\[
X(1) \neq X(m), \quad X(1) \times S^1 \cong X(m) \times S^1
\]

since this is the first example we have constructed where a circle factor cannot be cancelled.

1. The construction

We assign to \( n, u, t, m, l \) the meanings they had in the Introduction. We now proceed to construct a polyhedron \( X(m) \) corresponding to the group \( N(m) \).

Let \( M \) be a connected polyhedron with \( \pi_1 M = C_t \), \( \pi_2 M = \mathbb{Z}/n \) with \( \pi_1 M \) acting on \( \pi_2 M \) by

\[
\eta \cdot a = ua, \quad a \in \pi_2 M,
\]

where \( \pi_1 M = \langle \eta \rangle \). If the action is nilpotent, that is, if \( p \mid (u - 1) \) for every prime divisor \( p \) of \( n \), then we may assume \( M \) nilpotent. Now

\[
H^2(M; \mathbb{Z}) \ni \text{Ext}(H_1 M, \mathbb{Z}) = \text{Ext}(\mathbb{Z}/t, \mathbb{Z}) = \mathbb{Z}/t = \langle g \rangle,
\]

where \( g \) is a homotopy class, \( g : M \to K(\mathbb{Z}, 2) \). We use \( g \) to induce a circle bundle over \( M \), thus

\[
S^1 \to X(1) \xrightarrow{h} M \xrightarrow{g} K(\mathbb{Z}, 2).
\]

If we apply \( H_1 \) to (1.2) we get precisely the extension class \( \mathbb{Z} \to H_1 X(1) \to \mathbb{Z}/t \) of \( g \). Since \( g \) generates \( \text{Ext}(\mathbb{Z}/t, \mathbb{Z}) \) it follows that \( H_1 X(1) = \mathbb{Z} \). If we apply \( \pi_1 \) to (1.2) we
get a central extension $C \to \pi_1 X(1) \to C$, which shows that $\pi_1 X(1) = C$. It is plain that
\[ h_*: \pi_i X(1) \cong \pi_i M, \quad i \geq 2. \]  
(1.2)

Let $\pi_i X(1) = \langle \xi \rangle$ with $h_* \xi = \eta$. Then $\pi_1 X(1)$ acts on $\pi_2 X(1)$ by the rule
\[ \xi \cdot a = ua, \quad a \in \pi_2 X(1), \]  
(1.3)

which justifies the notation $X(1)$. Notice that $X(1)$ is nilpotent if $M$ is nilpotent.

Now consider the map $I: K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$. This induces the commutative diagram
\[
\begin{array}{ccc}
S^1 & \xrightarrow{t} & X(1) & \xrightarrow{h} & M & \xrightarrow{g} & K(\mathbb{Z}, 2) \\
\downarrow f & & \downarrow & & \downarrow \text{Id} & & \\
S^1 & \xrightarrow{t} & Y & \xrightarrow{h'} & M & \xrightarrow{lg} & K(\mathbb{Z}, 2)
\end{array}
\]  
(1.4)

Noting that $lg$ also generates $\text{Ext}(\mathbb{Z}/l, \mathbb{Z})$, we obtain results for the fibering $h'$ analogous to those for $h$. Thus
\[ f_*: \pi_i X(1) \cong \pi_i Y, \quad i \geq 2 \]  
(1.5)

and we may choose a generator $\xi'$ for the cyclic infinite group $\pi_1 Y$ such that
\[ f_* \xi = \xi'^l. \]  
(1.6)

It follows that the action of $\pi_1 Y$ on $\pi_2 Y$ satisfies $\xi'^l \cdot a = ua$. Now $\xi'^l \cdot a = a$ and $lm \equiv 1 \mod t$. Thus $\xi' \cdot a = \xi'^lm \cdot a = u^m a$, and we may therefore take $Y = X(m)$. Since $N(1) \neq N(m)$, it follows immediately that
\[ X(1) \neq X(m). \]  
(1.7)

However, since $f_*$ satisfies (1.5), (1.6), we immediately infer that $f$ is, homotopically, an $l$-sheeted cyclic regular covering, $f: X(1) \to X(m)$. There is obviously an $m$-sheeted cyclic regular covering $X(m) \to X(1)$.

**Theorem 1.1.** If $M$ is nilpotent of finite type, then $X(1), X(m)$ are nilpotent spaces of finite type in the same genus.

**Proof.** We may assume, by Dirichlet's Theorem, that $l$ is a prime which does not divide $n$. If we localize at a prime $p$ dividing $n$, then $l_p: K(\mathbb{Z}_p, 2) \cong K(\mathbb{Z}_p, 2)$, so $f_*$ is a homotopy equivalence. If we localize at a prime $p$ which does not divide $n$, then $g_\rho = 0$, so that
\[ X(1)_p = (M \times S^1)_p \cong X(m)_p. \]

**Theorem 1.2.** $X(1) \times S^1 = X(m) \times S^1$. 
Proof. Let $A = (l, s')$ be a unimodular matrix. Then we may regard $A$ as a homotopy equivalence,

$$A : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2).$$

Since $tg = 0$, we have $(g, 0)(l, s') = (lg, 0)$. Thus we obtain the diagram

$$\begin{array}{ccc}
S^1 \times S^1 & \to & X(1) \times S^1 \\
\downarrow A & & \downarrow \phi \\
S^1 \times S^1 & \to & X(m) \times S^1 \\
\end{array}$$

from which it immediately follows that $F$ is a homotopy equivalence.

Corollary 1.3. If $M$ has finitely generated homology groups, then

$$H_i X(1) \cong H_i X(m), \quad i \geq 0.$$ 

Proof. If $M$ has finitely generated homology groups, so do $X(1), X(m)$. Moreover $X(1), X(m)$ are connected so that $H_0 X(1) = H_0 X(m) = \mathbb{Z}$. From Theorem 1.2 it follows that

$$H_i X(1) \oplus H_{i+1} X(1) \cong H_i X(m) \oplus H_{i+1} X(m), \quad i \geq 0$$

(1.8)

The corollary now follows by induction on $i$, using the evident cancellation property of finitely generated abelian groups.

Remark. Of course, Corollary 1.3 follows directly from Theorem 1.1 in the nilpotent case.

We observe in the Introduction that there exists $k$ such that $N(1)^k \cong N(m)^k$ if $N(1), N(m)$ are nilpotent; in fact, we may take $k = \frac{1}{2} \Phi(t)$, though, in general, a smaller value of $k$ is available [2]. We now prove

Theorem 1.4. If $N(1)^k \cong N(m)^k$, then $X(1)^k = X(m)^k$.

Proof. Let $\phi : N(1)^k \cong N(m)^k$. Now $N(1)^k$ (and, likewise, $N(m)^k$) has a torsion subgroup $C_n^k$ with quotient $C^k$, inducing an action of $C^k$ on $(\mathbb{Z}/n)^k$ given by

$$\xi \cdot (a_1, a_2, ..., a_k) = (a_1, a_2, ..., ua_i, ..., a_k), \quad \text{in the case of } N(1)^k.$$ 

$$\xi \cdot (a_1, a_2, ..., a_k) = (a_1, a_2, ..., u^m a_i, ..., a_k), \quad \text{in the case of } N(m)^k,
where $C^k = \langle \xi_1, \xi_2, \ldots, \xi_k \rangle$. Thus $\phi$ induces
\[
\begin{array}{ccc}
C^k_n & \xrightarrow{\alpha} & N(1)^k \\
\downarrow \phi & & \downarrow \beta \\
C^k_m & \xrightarrow{\beta} & C^k
\end{array}
\]
(1.9)

where $\phi, \alpha, \beta$ are isomorphisms. Let $\beta$ correspond, in the given basis for $C^k$, to the matrix $B = (b_{ij})$. Then it follows from (1.9) and the given actions that
\[
u_{mb_i} \equiv \begin{cases} 
1 \mod n & \text{if } i \neq j, \\
u \mod n & \text{if } i = j.
\end{cases}
\]
(1.10)

Since the order of $u \mod n$ is $t$ and since $m$ is prime to $t$ with $ml \equiv 1 \mod t$, we infer from (1.10) that
\[
b_{ij} \equiv \begin{cases} 
0 \mod t & \text{if } i \neq j, \\
l \mod t & \text{if } i = j.
\end{cases}
\]
(1.11)

Since $B$ is invertible, we have a homotopy equivalence $B : K(\mathbb{Z}^k, 2) \to K(\mathbb{Z}^k, 2)$ and, since $tg = 0$, (1.11) yields the commutative diagram
\[
\begin{array}{ccc}
(S^1)^k & \xrightarrow{B} & X(1)^k \\
\downarrow & & \downarrow \\
(S^1)^k & \xrightarrow{\Phi} & X(m)^k
\end{array}
\]
(1.12)

from which it follows that $F : X(1)^k \simeq X(m)^k$.

References