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# Graphs associated with vector spaces of even dimension: A link with differential geometry

Luis Boza [a,1], Alfonso Carriazo [b,2], Luis M. Fernández [b,*]

[a] Departamento de Matemática Aplicada I, Escuela Técnica Superior de Arquitectura, Universidad de Sevilla, Av./Reina Mercedes, n. 2, 41012 Sevilla, Spain
[b] Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad de Sevilla, Apartado de Correos 1160, 41080 Sevilla, Spain

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ABSTRACT

We define a new association between graphs and orthonormal bases of even-dimensional Euclidean vector spaces endowed with an special isomorphism motivated by the recently introduced theory of submanifolds associated with graphs. We provide several interesting examples and we analyze the shape of such graphs by proving some general results.



## 1. Introduction

In this paper, we introduce a new method to associate vector spaces endowed with an inner product and graphs, through orthonormal bases, motivated by the theory of submanifolds associated with graphs developed by the last two authors in [3,4] (jointly with A. Rodríguez-Hidalgo), even if the first paper in which graphs were used to visualize the behavior of submanifolds was really [2]. In all

* Corresponding author.
*E-mail addresses:* boza@us.es (L. Boza), carriazo@us.es (A. Carriazo), lmfer@us.es (L.M. Fernández).

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those papers, submanifolds of almost Hermitian manifolds are considered. Let us recall that an almost Hermitian manifold $(\widetilde{M}, J, g)$ is a triple made up by a differentiable manifold $\widetilde{M}$ of even dimension $2n$, a tensor field $J$ of type $(1,1)$ on $\widetilde{M}$ such that $J^2 = -Id$, and a Riemannian metric $g$ on $\widetilde{M}$ such that $g(JX, JY) = g(X, Y)$, for any vector fields $X, Y$ on $\widetilde{M}$ (for more background on this theory, we recommend, for example, [15]).

Given an $m$-dimensional Riemannian manifold $M$ isometrically immersed in $(\widetilde{M}, J, g)$, let $\mathcal{B} = \{X_1, \ldots, X_{2n}\}$ be a local orthonormal frame of $\widetilde{M}$ defined on a neighborhood $U$ of a point $p \in M$. Then, for any $q \in U$, we define the weighted graph $G_{\mathcal{B},q}$ given by the set of vertices $\{1, \ldots, 2n\}$ such that the edge $\{i, j\}$ exists if and only if $g_q(J_qX_{iq}, X_{jq}) \neq 0$, with weight $g_q^2(J_qX_{iq}, X_{jq})$.

As this procedure produces labeled and weighted graphs, it was necessary to take into account a slightly different notion of isomorphism in [3,4], by imposing that isomorphisms preserve labels and, sometimes, weights. Hence, for the purpose of those papers, an isomorphism (resp. weak isomorphism) between two such graphs with $2n$ vertices is just the identity map from $\{1, \ldots, 2n\}$ into itself, preserving adjacency and edge weights (resp. adjacency). As usual, we say that two graphs are isomorphic (resp. weakly isomorphic) if there exists an isomorphism (resp. a weak isomorphism) between them.

Now we can define the association between submanifolds and graphs. Let $G$ be a weighted graph with vertices $\{1, \ldots, 2n\}$. Then, we say that $M$ is *associated* (resp. *weakly associated*) with $G$ if for any $p \in M$ there exists a neighborhood $U(p)$ and a local orthonormal frame $\mathcal{B} = \{X_1, \ldots, X_{2n}\}$ on $U$ satisfying the following conditions:

($a$) $\{X_1, \ldots, X_m\}$ are tangent to $M$ and $\{X_{m+1}, \ldots, X_{2n}\}$ are normal to $M$.
($b$) For any $q \in U$, the graph $G_{\mathcal{B},q}$ is isomorphic (resp. weakly isomorphic) to $G$.

Obviously, every submanifold associated with a graph is also weakly associated with it, since graph isomorphisms are in particular weak isomorphisms. On the other hand, it is not necessary for $G$ to be a weighted graph to define the weak association. Moreover, as it was pointed out in [4], the weak association of submanifolds with graphs is not so strange at all. Indeed, it was proved there that it is a natural local fact: given any submanifold $M$ of an almost Hermitian manifold, there always exists an open submanifold of $M$ which is weakly associated with a graph [4, Theorem 3.1].

Let us look at this association pointwise: if a submanifold $M$ is weakly associated with a graph $G$ through a local orthonormal frame

$$\mathcal{B} = \{X_1, \ldots, X_{2n}\},$$

then, for any point $p \in M$, the graph $G_{\mathcal{B},p}$ is weakly isomorphic to $G$ and hence, $G$ is associated with the orthonormal basis $\mathcal{B}_p = \{X_1(p), \ldots, X_{2n}(p)\}$ of the $2n$-dimensional Euclidean vector space $T_p(\widetilde{M})$, endowed with the inner product $g_p$ and the isometry $J_p$.

From this starting point, in this paper we shall associate graphs with Euclidean even dimensional vector spaces (due to the fact that almost Hermitian manifolds are of even dimension too), where the inner product plays the role of the almost Hermitian metric. We shall show how, on these spaces, it is always possible to consider an isometry $F$ such that $F^2 = -Id$ (which corresponds to the almost complex structure from the Theory of Submanifolds setting).

It is necessary to notice here that to associate vector spaces and graphs is not a new subject. For instance, see Chapter 6 of [6]. Moreover, vector representations of graphs (see [12] for definition) are of interest because they allow one to use linear algebra to study properties of graphs being represented. In particular, orthogonal representations in the real field $\mathbb{R}$ were used by Lovász [9] in his solution of the Shannon capacity of the pentagon. A related construction was considered by Erdös and Simonovits [5]. The smallest dimension where these vectors can be constructed is quite interesting and it is called the *geometrical dimension* of the graph, but other problems concerning graphs have been also studied (see [7,10,11] for more details). We recommend [8] for more background on this subject.

The importance of the new association introduced in this paper resides first in the fact that it naturally arises from the one of papers [3,4] and thus, it can reinforce the established link between two traditionally remote areas in Mathematics like Discrete Mathematics and Differential Geometry, hence becoming a very useful tool for the general problem of the classification of submanifolds. However, this new association is also important by itself because it sets up a general framework with many interesting problems to solve: the shape of the graphs admitting such an association, their behavior under possible changes of orthonormal bases, the characterization of such graphs belonging to relevant families, etc.

The paper is organized as follows: after a preliminaries section in which we present the basic concepts and results of Graph Theory for later use, in Section 3 we define the association between graphs and orthonormal bases which corresponds to the weak association between graphs and submanifolds introduced in [3,4]. We provide several examples in dimensions 2, 4, 6 and 8 and we study some appropriate changes of bases producing very useful ones. Next, Section 4 is dedicated to investigate the shape of these graphs. We show some general conditions for them and we prove that a graph is associated with an orthonormal basis if and only if each of its components also is. We use this fact to completely determine the graphs of maximum degree at most 3 admitting such an association. All the results presented in this section are applicable to the Theory of Submanifolds setting. In particular, Proposition 4.1 and Corollaries 4.5 to 4.8 generalize Lemmas 3.5, 3.6, 3.10 and 3.11, Propositions 3.7, 3.12 and 3.13, Theorem 5.5 and Corollary 5.6 of [4]. Furthermore, from Theorems 4.11, 4.15 and 4.16 we can easily deduce similar results for submanifolds weakly associated with graphs. Finally, we can close a question left open in [4] concerning submanifolds weakly associated with graphs in dimension 6. There, it was proved that the only graphs with 6 vertices which can be weakly associated with submanifolds are those ones of Table 1 and explicit examples of such associations were given for all of them except $G_{207}$. Now, we can adapt the orthonormal basis included in the table to produce the corresponding orthonormal frame establishing the weak association of this graph with a submanifold.

## 2. Preliminaries

A *graph* $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set of vertices and $E(G)$ is a prescribed set of unordered pairs of distinct vertices of $V(G)$, called edges. Given a pair $\{i, j\}$ in $E(G)$, $i$ and $j$ are said to be *adjacent* vertices and $\{i, j\}$ is said to be *incident* to both $i$ and $j$. The *degree* of a vertex is the number of edges incident to it. A vertex of degree 0 is called an *isolated vertex*. A vertex subset is an *independent set* and its vertices are said to be *independent* if no two of them are adjacent. A graph is said to be *regular of degree d* is all its vertices are of degree $d$. In particular, a regular graph of degree 3 is called a *cubic* graph.

The *complement* of a graph $G = (V(G), E(G))$ is the graph $\overline{G}$ such that $V(\overline{G}) = V(G)$ and an edge $e \in E(\overline{G})$ if and only if $e \notin E(G)$.

If $G$ is a graph and $e$ is one of its edges, the graph $G - e$ is that one obtained from $G$ by deleting $e$.

Given two graphs $G_1$ and $G_2$, their *union* $G = G_1 \cup G_2$ is the graph such that $V(G) = V(G_1) \bigsqcup V(G_2)$ and $E(G) = E(G_1) \bigsqcup E(G_2)$ (that is, the graph made by putting together $G_1$ and $G_2$ with no additional edges between them). We use the notation $kG$ ($k \in \mathbb{N}$) for the graph $\underbrace{G \cup \cdots \cup G}_{k}$.

The graph consisting of $n$ vertices, all of them of maximum degree $n-1$ is denoted by $K_n$ and called *the complete graph of n vertices*. A *path* is a graph determined by an alternating sequence of distinct vertices and edges in which each edge is incident with the two vertices immediately preceding and following it. The path of $n \geqslant 2$ vertices is denoted by $P_n$. A *cycle* is a closed path, that is, a path beginning and ending with the same vertex. The cycle of $n \geqslant 3$ vertices is denoted by $C_n$. The *d-cube graph* $Q_d$ ($d \geqslant 1$) is the 1-skeleton of the $d$-dimensional hypercube $\{(x_1, \ldots, x_d) : 0 \leqslant x_j \leqslant 1,\ j = 1, \ldots, d\}$ (that is, the graph consisting of the vertices and edges of the hypercube). Obviously, $Q_d$ is a regular graph with $2^d$ vertices of degree $d$. In particular, $Q_3$ is a cubic graph.

A graph is said to be *bipartite* if its vertices can be partitioned into two sets, called *partite sets*, in such a way that no edge joins two vertices in the same set. A *complete bipartite graph* is a bipartite graph in which each vertex in one partite set is adjacent to all the vertices in the other partite set. In this case, if the two partite sets have cardinal numbers $n$ and $m$, respectively, then this graph is denoted by $K_{n,m}$.

Throughout this paper, we are labeling graphs by distinguishing their vertices from one another by consecutive natural numbers. Therefore, we identify the vertex set of a graph with $n$ vertices with the set $\{1, \ldots, n\}$.

In Graph Theory, an isomorphism between two graphs is a one-to-one correspondence between their vertex sets which preserves adjacency. We say that two graphs are isomorphic if there exists an isomorphism between them. For more background on Graph Theory, we refer to [6].

## 3. Associating graphs with orthonormal bases

Let $V$ be a Euclidean vector space of dimension $2n$ endowed with an inner product (denoted by $\cdot$) and let $F$ be an isometry on $V$ (that is, $Fv \cdot Fw = v \cdot w$, for any $v, w \in V$) such that $F^2 = -Id$. Observe that, for any $v \in V$, it is known that $v$ is orthogonal to $Fv$ because $v \cdot Fv = -F^2 v \cdot Fv = -Fv \cdot v$ and thus $v \cdot Fv = 0$.

In these conditions it is always possible to construct special orthonormal bases of $V$ as follows: let $w_1$ be any unit vector of $V$. Then, $Fw_1$ is a unit vector too and, moreover, orthogonal to $w_1$. Next, if $n > 1$, let $w_2$ be any unit vector of $V$ orthogonal to both $w_1$ and $Fw_1$. It is easy to show that $Fw_2$ is another unit vector orthogonal to $w_1$, $Fw_1$ and $w_2$. Continuing this procedure, we get an orthonormal basis $\{w_1, \ldots, w_{2n}\}$ of $V$, where we are denoting $w_{n+k} = Fw_k$, $k = 1, \ldots, n$. Furthermore, we observe that $Fw_{n+k} = -w_k$, for any $k = 1, \ldots, n$. The orthonormal bases obtained this way are called *F-bases*. Conversely, it is also easy to prove that if $\{w_1, \ldots, w_{2n}\}$ is an orthonormal basis of $V$, then there exists a unique isometry $F$ on $V$ such that $F^2 = -Id$ and $\{w_1, \ldots, w_{2n}\}$ is an $F$-basis. In fact, $F$ is defined as $Fw_k = w_{n+k}$ and $Fw_{n+k} = -w_k$, for any $k = 1, \ldots, n$.

Now, let $\mathcal{B} = \{v_1, \ldots, v_{2n}\}$ be an arbitrary orthonormal basis of $V$. We can define a graph $G_{\mathcal{B}}$ by following these steps:

1. We consider a vertex for every vector in the basis, labeled with its corresponding natural index. Actually, we will sometimes identify vectors and vertices by using the same notation.
2. We say that the $\{v_i, v_j\}$ edge exists if and only if $Fv_i \cdot v_j \neq 0$. Notice that there are no loops in $G_{\mathcal{B}}$, since $Fv_i \cdot v_i = 0$, for any $i = 1, \ldots, 2n$, as we have already pointed out above.

We say that a labeled graph $G$ and the basis $\mathcal{B}$ are *associated* if $G$ is isomorphic to $G_{\mathcal{B}}$.

Now, we are going to present some examples.

**Example 3.1.** Let $V$ be a 2-dimensional Euclidean vector space and $F$ an isometry of $V$ such that $F^2 = -Id$. If we consider an $F$-basis, $\{w_1, w_2\}$, then it is associated with the graph $K_2$ because $Fw_1 \cdot w_2 = Fw_1 \cdot Fw_1 = w_1 \cdot w_1 = 1$.

**Example 3.2.** Let $V$ be a 4-dimensional Euclidean vector space and $F$ an isometry of $V$ such that $F^2 = -Id$ and let us consider an $F$-basis $\{w_1, w_2, w_3, w_4\}$. Then, it is associated with the graph $K_2 \cup K_2$.

Now, let $\theta \in (0, \pi/2)$. If we choose

$$v_1 = \cos\theta w_1 + \sin\theta w_2, \quad v_2 = w_3, \quad v_3 = w_4, \ v_4 = -\sin\theta w_1 + \cos\theta w_2,$$

it is easy to see that $\{v_1, v_2, v_3, v_4\}$ is an orthonormal basis of $V$ associated with the graph $C_4$.

Finally, if we choose

$$\widetilde{v}_1 = \cos\theta v_1 + \sin\theta v_2, \quad \widetilde{v}_2 = \sin\theta v_1 - \cos\theta v_2, \quad \widetilde{v}_3 = v_3, \quad \widetilde{v}_4 = v_4,$$

then, $\{\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3, \widetilde{v}_4\}$ is an orthonormal basis of $V$ associated with the graph $K_4$.

Actually, the three graphs from Example 3.2 are the only non-isomorphic ones which can be associated with an orthonormal basis in a 4-dimensional vector space, as we will show in Section 4.

**Example 3.3.** Let $V$ be a 6-dimensional Euclidean vector space, $\alpha_1, \alpha_2, \alpha_3 \in (0, \pi/2)$ and $F$ an isometry of $V$ such that $F^2 = -Id$. Let us choose an $F$-basis

$$\{w_1, w_2, w_3, w_4 = Fw_1, w_5 = Fw_2, w_6 = Fw_3\}.$$

If we consider the orthonormal bases given in Table 1, then they are associated with the indicated graphs, numbered as in [13, pp. 9–11].

**Example 3.4.** Let $V$ be a 8-dimensional Euclidean vector space, $\alpha_1, \alpha_2 \in (0, \pi/2)$ and $F$ an isometry of $V$ such that $F^2 = -Id$. Let us consider an $F$-basis

$$\{w_1, w_2, w_3, w_4, w_5 = Fw_1, w_6 = Fw_2, w_7 = Fw_3, w_8 = Fw_4\}.$$

Then, if we put

$$\begin{aligned}
v_1 &= \cos\alpha_1 \cos\alpha_2 w_1 + \sin\alpha_1 \cos\alpha_2 w_2 + \cos\alpha_1 \sin\alpha_2 w_3 + \sin\alpha_1 \sin\alpha_2 w_4,\\
v_2 &= w_5,\\
v_3 &= w_7,\\
v_4 &= -\cos\alpha_1 \sin\alpha_2 w_1 - \sin\alpha_1 \sin\alpha_2 w_2 + \cos\alpha_1 \cos\alpha_2 w_3 + \sin\alpha_1 \cos\alpha_2 w_4,\\
v_5 &= \cos\alpha_2 w_6 + \sin\alpha_2 w_8,\\
v_6 &= -\sin\alpha_1 w_1 + \cos\alpha_1 w_2,\\
v_7 &= -\sin\alpha_1 w_3 + \cos\alpha_1 w_4,\\
v_8 &= -\sin\alpha_2 w_6 + \cos\alpha_2 w_8,
\end{aligned}$$

it is easy to see that $\{v_1, \ldots, v_8\}$ is an orthonormal basis of $V$ associated with the cubic graph $Q_3$ shown in Fig. 1.

A first question concerning this association can be its dependence on the chosen orthonormal basis. For example, let us consider $n = 2$ and the basis $\mathcal{B} = \{w_1, \ldots, w_4\}$ given in Example 3.2 which is associated with Graph 1 ($K_2 \cup K_2$) shown in Fig. 2. If we now put $\widetilde{\mathcal{B}} = \{\sqrt{2}/2(w_1 + w_2), \sqrt{2}/2(w_1 - w_2), w_3, w_4\}$, then $\widetilde{\mathcal{B}}$ is also an orthonormal basis of $V$ which is associated with Graph 2 ($K_{2,2}$) in Fig. 2. Obviously, these two graphs are far from being isomorphic.

Therefore, it is interesting to look for appropriate changes of bases preserving association with graphs. In this sense, we will introduce some operators, which will allow us to simplify the structure of the orthonormal bases associated with a given graph. Then, if $\mathcal{B} = \{w_1, \ldots, w_{2n}\}$ is an $F$-basis of V and $v \in V$ is written as $v = \sum_{k=1}^{n} a_k w_k + \sum_{k=1}^{n} b_k w_{n+k}$, we define the following operators:

(i) For any integer $p$ such that $1 \leqslant p \leqslant n$,

$$o_p(v) = \sum_{k=1, k\neq p}^{n} a_k w_k + \sum_{k=1, k\neq p}^{n} b_k w_{n+k} - a_p w_p - b_p w_{n+p}.$$

(ii) For any integer $p$ such that $1 \leqslant p \leqslant n$ and any $\alpha \in [0, 2\pi]$,

$$\begin{aligned}
\phi_{p,\alpha}(v) = &\sum_{k=1, k\neq p}^{n} a_k w_k + \sum_{k=1, k\neq p}^{n} b_k w_{n+k}\\
&+ (a_p \cos\alpha - b_p \sin\alpha) w_p + (b_p \cos\alpha + a_p \sin\alpha) w_{n+p}.
\end{aligned}$$

**Table 1**
Graphs and orthonormal bases of Example 3.3.

| Graph | Corresponding orthonormal basis |
|---|---|
| $G_{61} = 3K_2$ | $v_1 = w_1;\ v_2 = w_4;$ <br> $v_3 = w_2;\ v_4 = w_5;$ <br> $v_5 = w_3;\ v_6 = w_6.$ |
| $G_{85} = C_4 \cup K_2$ | $v_1 = w_1;\ v_2 = w_4;\ v_3 = w_3;$ <br> $v_4 = \sin\alpha_1 w_2 - \cos\alpha_1 w_6;\ v_5 = \cos\alpha_1 w_2 + \sin\alpha_1 w_6;$ <br> $v_6 = w_5.$ |
| $G_{166} = K_4 \cup K_2$ | $v_1 = w_1;\ v_2 = w_4;$ <br> $v_3 = \frac{1}{\sqrt{2}} w_3 - \frac{1}{2}(w_2 + w_6);\ v_4 = \frac{1}{\sqrt{2}} w_3 + \frac{1}{2}(w_2 + w_6);$ <br> $v_5 = w_5;\ v_6 = \frac{1}{\sqrt{2}}(w_6 - w_2).$ |
| $G_{154} = K_{3,3} - e$ | $v_1 = w_1;\ v_2 = \frac{1}{\sqrt{2}}(w_4 + w_5);$ <br> $v_3 = \frac{1}{\sqrt{2}}(w_5 - w_4);\ v_4 = \frac{1}{\sqrt{2}}(w_2 + w_6);$ <br> $v_5 = \frac{1}{\sqrt{2}}(w_6 - w_2);\ v_6 = w_3.$ |
| $G_{174} = K_{3,3}$ | $v_1 = \frac{1}{\sqrt{2}}(\sin\alpha_1 w_1 + \cos\alpha_1 w_2 + w_3);\ v_2 = \frac{1}{\sqrt{2}}(w_4 + w_6);$ <br> $v_3 = \cos\alpha_1 w_1 - \sin\alpha_1 w_2;\ v_4 = w_5;$ <br> $v_5 = \frac{1}{\sqrt{2}}(\sin\alpha_1 w_1 + \cos\alpha_1 w_2 - w_3);\ v_6 = \frac{1}{\sqrt{2}}(w_6 - w_4).$ |
| $G_{194} = \overline{K_{1,3} \cup K_2}$ | $v_1 = \frac{1}{\sqrt{2}}(w_2 + w_3);\ v_2 = \frac{1}{\sqrt{2}} w_3 - \frac{1}{2}(w_6 - w_2);$ <br> $v_3 = \frac{1}{\sqrt{2}} w_3 + \frac{1}{2}(w_6 - w_2);\ v_4 = \frac{1}{\sqrt{2}}(w_4 + w_5);$ <br> $v_5 = \frac{1}{\sqrt{2}}(w_5 - w_4);\ v_6 = w_1.$ |
| $G_{202} = \overline{P_4 \cup 2K_1}$ | $v_1 = \frac{1}{\sqrt{2}}(w_1 - w_2);\ v_2 = \frac{1}{\sqrt{2}}(w_4 + w_6);$ <br> $v_3 = \frac{1}{2}(w_1 + w_2 + w_3 + w_5);\ v_4 = \frac{1}{2}(w_1 + w_2 - w_3 - w_5);$ <br> $v_5 = \frac{1}{\sqrt{2}}(w_4 - w_6);\ v_6 = \frac{1}{\sqrt{2}}(w_3 - w_5).$ |
| $G_{204} = \overline{3K_2}$ | $v_1 = \sin\alpha_1 w_1 + \cos\alpha_1 w_2;\ v_2 = \sin\alpha_3 w_4 + \cos\alpha_3 w_6;$ <br> $v_3 = \sin\alpha_2 w_3 + \cos\alpha_2 w_5;\ v_4 = \sin\alpha_1 w_2 - \cos\alpha_1 w_1;$ <br> $v_5 = \sin\alpha_2 w_5 - \cos\alpha_2 w_3;\ v_6 = \sin\alpha_3 w_6 - \cos\alpha_3 w_4.$ |
| $G_{205} = \overline{P_3 \cup 3K_1}$ | $v_1 = w_6;\ v_2 = \frac{1}{\sqrt{2}}(w_6 - w_5);$ <br> $v_3 = \frac{1}{2}(w_1 - w_2) - \frac{1}{\sqrt{2}} w_4;\ v_4 = \frac{1}{2}(w_1 - w_2) + \frac{1}{\sqrt{2}} w_4;$ <br> $v_5 = \frac{1}{2}(w_1 + w_2 + w_3 + w_5);\ v_6 = \frac{1}{2}(w_1 + w_2 - w_3 - w_5).$ |
| $G_{206} = \overline{2K_2 \cup 2K_1}$ | $v_1 = \frac{1}{2}(w_1 + w_2 + w_3 + w_5);\ v_2 = \frac{1}{\sqrt{2}}(w_1 - w_2);$ <br> $v_3 = \frac{1}{2}(w_3 - w_5) + \frac{1}{\sqrt{2}} w_4;\ v_4 = \frac{1}{2}(w_1 + w_2 - w_3 - w_5);$ <br> $v_5 = \frac{1}{2}(w_3 - w_5) - \frac{1}{\sqrt{2}} w_4;\ v_6 = w_6.$ |
| $G_{207} = K_6 - e$ | $v_1 = w_1;\ v_2 = w_2;$ <br> $v_3 = \frac{1}{\sqrt{3}}(w_3 + w_4 + w_5);\ v_4 = \frac{1}{\sqrt{15}}(-3w_3 + w_4 + 2w_5 + w_6);$ <br> $v_5 = \frac{1}{\sqrt{3}}(w_4 - w_5 + w_6);\ v_6 = \frac{1}{\sqrt{15}}(-w_3 + 2w_4 - w_5 - 3w_6).$ |
| $G_{208} = K_6$ | $v_1 = \frac{1}{2}(w_1 + w_2 + w_3 + w_5);$ <br> $v_2 = \frac{1}{\sqrt{2}} w_4 + \frac{1}{2}(w_1 - w_2);\ v_3 = \frac{1}{\sqrt{2}} w_6 - \frac{1}{2}(w_3 - w_5);$ <br> $v_4 = \frac{1}{2}(w_1 - w_2) - \frac{1}{\sqrt{2}} w_4;$ <br> $v_5 = \frac{1}{2}(w_3 - w_5) + \frac{1}{\sqrt{2}} w_6;$ <br> $v_6 = \frac{1}{2}(w_1 + w_2 - w_3 - w_5).$ |

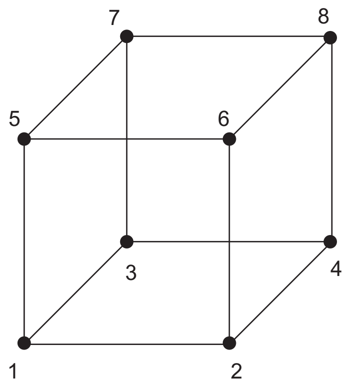


**Fig. 1**. The graph $Q_3$.

(iii) For any integers $p \neq q$ such that $1 \leqslant p, q \leqslant n$ and any $\alpha \in [0, 2\pi]$,

$$\begin{aligned}\varphi_{p,q,\alpha}(v) = &\sum_{k=1,k\neq p,q}^{n} a_k w_k + \sum_{k=1,k\neq p,q}^{n} b_k w_{n+k}\\ &+ (a_p \cos\alpha - a_q \sin\alpha) w_p + (a_q \cos\alpha + a_p \sin\alpha) w_q\\ &+ (b_p \cos\alpha - b_q \sin\alpha) w_{n+p} + (b_q \cos\alpha + b_p \sin\alpha) w_{n+q}.\end{aligned}$$

Thus, we have the following immediate results.

**Lemma 3.5.** *For any integers $p \neq q$ such that $1 \leqslant p, q \leqslant n$ and any $\alpha \in [0, 2\pi]$, the above operators are isometries, they commute with F and, therefore, the following properties are satisfied, for any $v, v' \in V$:*

*(i)* $F(o_p(v)) \cdot o_p(v') = Fv \cdot v'$.
*(ii)* $F(\phi_{p,\alpha}(v)) \cdot \phi_{p,\alpha}(v') = Fv \cdot v'$.
*(iii)* $F(\varphi_{p,q,\alpha}(v)) \cdot \varphi_{p,q,\alpha}(v') = Fv \cdot v'$.

**Corollary 3.6.** *For any integers $p \neq q$ such that $1 \leqslant p, q \leqslant n$ and any $\alpha \in [0, 2\pi]$, an orthonormal basis associated with a graph is transformed by the operators $o_p$, $\phi_{p,\alpha}$, $\varphi_{p,q,\alpha}$ in another orthonormal basis associated with the same graph.*

Now, we can prove:

**Theorem 3.7.** *Let $\mathcal{B} = \{w_1, \ldots, w_{2n}\}$ be an F-basis and let G be a graph associated with a certain orthonormal basis. Then, there exists another orthonormal basis $\mathcal{B}' = \{v_1, \ldots, v_{2n}\}$ associated with G such that, for any $j = 1, \ldots, 2n$, we have:*

$$v_j = \begin{cases} w_1 & \text{if } j = 1,\\ \displaystyle\sum_{k=2}^{j} a_{j,k} w_k + \sum_{k=1}^{j-1} b_{j,k} w_{n+k} & \text{if } 2 \leq j \leq n,\\ \displaystyle\sum_{k=2}^{n} a_{j,k} w_k + \sum_{k=1}^{n} b_{j,k} w_{n+k} & \text{if } n+1 \leq j \leq 2n.\end{cases}$$

*Moreover, we can suppose that $a_{k,k} \geqslant 0$, for any $k = 2, \ldots, n$.*

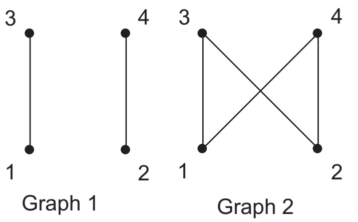


**Fig. 2**. Changing bases.

**Proof.** Let $\{v_1, \ldots, v_{2n}\}$ be an orthonormal basis associated with $G$. For any $j = 1, \ldots, 2n$, we identify $v_j = \sum_{k=1}^{n} a_{j,k} w_k + \sum_{k=1}^{n} b_{j,k} w_{n+k}$ with the vector $\overrightarrow{v}_j = (a_{j,1}, \ldots, a_{j,n}, b_{j,1}, \ldots, b_{j,n})$.

If we successively apply to any $\overrightarrow{v}_j$ the operators $\phi_{h,-\arctan(b_{1,h}/a_{1,h})}$, with $1 \leqslant h \leqslant n$, then, the vector $\overrightarrow{v}_1$ changes to one like

$$(a'_{1,1}, \ldots, a'_{1,n}, \underbrace{0, \ldots, 0}_{n}).$$

Thus, we can suppose without loss of generality that $b_{1,h} = 0$, for any $h = 1, \ldots, n$. Now, if we successively apply to any $\overrightarrow{v}_j$ the operators

$$\varphi_{1,h,-\arctan(a_{1,h}/a_{1,1})}, \quad 2 \leqslant h \leqslant n,$$

the vector $\overrightarrow{v}_1$ changes to

$$(a'_{1,1}, \underbrace{0, \ldots, 0}_{2n-1}).$$

Hence, we can also suppose without loss of generality that $a_{1,h} = 0$, for any $h = 2, \ldots, n$. Finally, applying $o_1$, if necessary, to any $\overrightarrow{v}_j$, we get that the vector $\overrightarrow{v}_1$ is transformed into

$$(1, \underbrace{0, \ldots, 0}_{2n-1}).$$

Given that $\overrightarrow{v}_j$, with $2 \leqslant j \leqslant 2n$, is orthogonal to $\overrightarrow{v}_1$, we have that $a_{j,1} = 0$.

Next, we denote again by $a_{j,k}$ and $b_{j,k}$ the coefficients of $\overrightarrow{v}_j$. If we successively apply to any $\overrightarrow{v}_j$ the operators $\phi_{h,-\arctan(b_{2,h}/a_{2,h})}$, with $2 \leqslant h \leqslant n$, the vector $\overrightarrow{v}_2$ changes to one like

$$(0, a'_{2,2}, \ldots, a'_{2,n}, b'_{2,1}, \underbrace{0, \ldots, 0}_{n-1})$$

and we can suppose without loss of generality that $b_{2,h} = 0$, for any $h = 2, \ldots, n$. Now, if we successively apply to any $\overrightarrow{v}_j$ the operators

$$\varphi_{2,h,-\arctan(a_{2,h}/a_{2,2})}, \quad 3 \leqslant h \leqslant n,$$

the vector $\overrightarrow{v}_2$ is transformed into

$$(0, a'_{2,2}, \underbrace{0, \ldots, 0}_{n-2}, b'_{2,1}, \underbrace{0, \ldots, 0}_{n-1}).$$

Then, we can suppose without loss of generality that $a_{2,h} = 0$, for any $h = 3, \ldots, n$. If $a'_{2,2} < 0$, we can apply $o_2$ to obtain that the second coefficient of $\overrightarrow{v}_2$ is non-negative.

As before, we denote again by $a_{j,k}$ and $b_{j,k}$ the coefficients of $\overrightarrow{v}_j$. By using an inductive process, let us suppose that

$$\overrightarrow{v}_p = (0, a_{p,2}, \ldots, a_{p,p}, \underbrace{0, \ldots, 0}_{n-p}, b_{p,1}, \ldots, b_{p,p-1}, \underbrace{0, \ldots, 0}_{n-p+1})$$

for a certain value of $p \in \{2, \ldots, n-1\}$ and that $a_{p,p} \geqslant 0$. Next, let us successively apply to any $\overrightarrow{v}_j$ the operators $\phi_{h,-\arctan(b_{p+1,h}/a_{p+1,h})}$, with $p+1 \leqslant h \leqslant n$, in order for the vector $\overrightarrow{v}_{p+1}$ to be transformed into one like

$$(0, a'_{p+1,2}, \ldots, a'_{p+1,n}, b'_{p+1,1}, \ldots, b'_{p+1,p}, \underbrace{0, \ldots, 0}_{n-p}).$$

Therefore we can suppose without loss of generality that $b_{p+1,h} = 0$, for any $h = p+1, \ldots, n$. Now, if we successively apply to any $\overrightarrow{v}_j$ the operators

$$\varphi_{p+1,h,-\arctan(a_{p+1,h}/a_{p+1,p+1})},$$

with $p+2 \leqslant h \leqslant n$, the vector $\overrightarrow{v}_{p+1}$ changes to

$$(0, a'_{p+1,2}, \ldots, a'_{p+1,p+1}, \underbrace{0, \ldots, 0}_{n-p-1}, b'_{p+1,1}, \ldots, b'_{p+1,p}, \underbrace{0, \ldots, 0}_{n-p}),$$

so we can suppose without loss of generality that $a_{p+1,h} = 0$, for any $h = p+2, \ldots, n$. Finally, if $a'_{p+1,p+1} < 0$, we just have to apply $o_{p+1}$ in order to get a non-negative $(p+1)$-th coefficient in $\overrightarrow{v}_{p+1}$, which completes the inductive proof of this result. □

**Theorem 3.8.** *Let $\mathcal{B} = \{w_1, \ldots, w_{2n}\}$ be an F-basis and let G be a graph associated with a certain orthonormal basis such that its first r vertices are independent. Then, there exists another orthonormal basis $\mathcal{B}' = \{v_1, \ldots, v_{2n}\}$ associated with G and such that, for any $j = 1, \ldots, 2n$, we have:*

$$v_j = \begin{cases} w_j & \text{if } j \leqslant r, \\ \sum_{k=r+1}^{j} a_{j,k} w_k + \sum_{k=1}^{j-1} b_{j,k} w_{n+k} & \text{if } r+1 \leqslant j \leqslant n, \\ \sum_{k=r+1}^{n} a_{j,k} w_k + \sum_{k=1}^{n} b_{j,k} w_{n+k} & \text{if } n+1 \leqslant j \leqslant 2n, \end{cases}$$

*with $a_{k,k} \geqslant 0$, for any $k = r+1, \ldots, n$.*

**Proof.** Let $\{v_1, \ldots, v_{2n}\}$ be an orthonormal basis associated with $G$ given by Theorem 3.7. Then, $v_1 = w_1$. Next, by using an inductive process, we suppose that there exists an orthonormal basis $\{v_1, \ldots, v_{2n}\}$ associated with $G$ such that $v_j = w_j$ if $j \leqslant p \leqslant r-1$ and

$$v_{p+1} = \sum_{k=2}^{p+1} a_{p+1,k} w_k + \sum_{k=1}^{p} b_{p+1,k} w_{n+k}.$$

Since $v_j \cdot v_{p+1} = 0$, we have that $a_{p+1,j} = 0$. On the other hand, as we shall shown in Theorem 4.9 it follows that $r \leqslant n$ and so, $p+1 \leqslant n$. Therefore, $v_{n+j} \cdot v_{p+1} = 0$ and we get that $b_{p+1,j} = 0$. Thus, $v_{p+1} = a_{p+1,p+1} w_{p+1}$.

Now, since $v_{p+1} \cdot v_{p+1} = 1$ and $a_{p+1,p+1} \geqslant 0$, we have that $a_{p+1,p+1} = 1$ which completes the inductive proof of $v_j = w_j$ if $j \leqslant r$.

Finally, if $k \leqslant r$ and $j \geqslant r+1$, since $v_k \cdot v_j = 0$, it follows that $a_{j,k} = 0$ and the result holds. □

**Example 3.9.** In Example 3.4, we gave an orthonormal basis $\{v_1, \dots, v_8\}$ associated with the cube $Q_3$. Now, if we reorder it by defining

$$\widetilde{v}_1 = v_6;\ \widetilde{v}_2 = v_7;\ \widetilde{v}_3 = v_1;\ \widetilde{v}_4 = v_4;\ \widetilde{v}_5 = v_3;\ \widetilde{v}_6 = v_2;\ \widetilde{v}_7 = v_8;\ \widetilde{v}_8 = v_5$$

and we consider the operator

$$\Phi = \phi_{4,\pi} \circ \varphi_{3,4,\alpha_2+\pi} \circ \varphi_{2,4,\alpha_1-\frac{\pi}{2}} \circ \varphi_{2,3,\frac{\pi}{2}} \circ \varphi_{1,2,-\alpha_1+\frac{\pi}{2}},$$

then we obtain:

$$\begin{aligned}
&\Phi(\widetilde{v}_i) = w_i,\ i = 1, \dots, 4,\\
&\Phi(\widetilde{v}_5) = -\sin\alpha_1 w_6 + \cos\alpha_1 \sin\alpha_2 w_7 + \cos\alpha_1 \cos\alpha_2 w_8,\\
&\Phi(\widetilde{v}_6) = -\sin\alpha_1 w_5 + \cos\alpha_1 \cos\alpha_2 w_7 - \cos\alpha_1 \sin\alpha_2 w_8,\\
&\Phi(\widetilde{v}_7) = -\cos\alpha_1 \sin\alpha_2 w_5 + \cos\alpha_1 \cos\alpha_2 w_6 + \sin\alpha_1 w_8,\\
&\Phi(\widetilde{v}_8) = \cos\alpha_1 \cos\alpha_2 w_5 + \cos\alpha_1 \sin\alpha_2 w_6 + \sin\alpha_1 w_7.
\end{aligned}$$

Let us notice that $\{\Phi(\widetilde{v}_1), \dots, \Phi(\widetilde{v}_8)\}$ is an orthonormal basis associated with the cube $Q_3$, having the form described in Theorem 3.8 with $r = 4$.

## 4. The shape of graphs associated with orthonormal bases

In this section, we study how a graph associated with an orthonormal basis can be. From now on, all considered graphs will be labeled ones.

**Proposition 4.1.** *A graph associated with an orthonormal basis has not isolated vertices.*

**Proof.** Let us suppose that a graph $G$ with an isolated vertex is associated with an orthonormal basis $\{v_1, \dots, v_{2n}\}$. Let $v_i$ be the vector corresponding to the isolated vertex. Then, $Fv_i \cdot v_j = 0$ for any $j$, which implies that $Fv_i = 0$. But this is a contradiction with the fact that $F$ is an isometry. □

Now, we present two general results:

**Theorem 4.2.** *Let $G$ be a graph with $2n$ vertices $v_1, \dots, v_{2n}$ such that:*

*(1) $v_1, \dots, v_{n-1}$ are independent vertices.*
*(2) $v_n$ is not adjacent to either $v_{n+1}$ or $v_{n+2}$.*
*(3) $v_n$ is adjacent to some of $v_{n+3}, \dots, v_{2n}$.*
*(4) $v_{n+1}$ and $v_{n+2}$ are adjacent.*

*Then, $G$ is not associated with any orthonormal basis.*

**Proof.** Suppose that $G$ is associated with an orthonormal basis. Then, from Theorem 3.8, this basis can be written as

$$v_j = \begin{cases} w_j & \textit{if } j \leqslant n-1,\\ a_{n,n}w_n + \displaystyle\sum_{k=1}^{n-1} b_{n,k}w_{n+k} & \textit{if } j = n,\\ a_{j,n}w_n + \displaystyle\sum_{k=1}^{n} b_{j,k}w_{n+k} & \textit{if } n+1 \leqslant j \leqslant 2n, \end{cases}$$

where $\{w_1, \dots, w_{2n}\}$ is an $F$-basis. Thus,

$$Fv_n = a_{n,n}w_{2n} - \sum_{k=1}^{n-1} b_{n,k}w_k, \tag{4.1}$$

$$Fv_{n+1} = a_{n+1,n}w_{2n} - \sum_{k=1}^{n} b_{n+1,k}w_k. \tag{4.2}$$

By using the fact that $v_n$ is not adjacent to either $v_{n+1}$ or $v_{n+2}$, from (4.1) and (4.2) it follows that $0 = Fv_n \cdot v_{n+1} = a_{n,n}b_{n+1,n}$ and $0 = Fv_n \cdot v_{n+2} = a_{n,n}b_{n+2,n}$. If $a_{n,n} \neq 0$, then $b_{n+1,n} = b_{n+2,n} = 0$ and so, $Fv_{n+1} \cdot v_{n+2} = -a_{n+2,n}b_{n+1,n} + a_{n+1,n}b_{n+2,n} = 0$, which is a contradiction with $v_{n+1}$ and $v_{n+2}$ being adjacent. Therefore, $a_{n,n} = 0$. Now, if $j \geqslant n+3$, $Fv_n \cdot v_j = a_{n,n}b_{j,n} = 0$, which contradicts the fact that $v_n$ is adjacent to one of $v_{n+3}, \ldots, w_{2n}$ and the results holds. □

**Theorem 4.3.** *Given a graph, if it has two subsets of vertices $W_1$ and $W_2$, non-necessarily disjoint, such that*

*(i) $W_1$ has $m_1$ vertices ($m_1 \geqslant 2$) and $W_2$ has $m_2$ vertices with $m_2 < m_1$,*
*(ii) any vertex of $W_1$ is adjacent to, at least, one vertex of $W_2$, and*
*(iii) the common neighbors of any pair of vertices of $W_1$ always lay in $W_2$,*

*then the graph cannot be associated with any orthonormal basis.*

**Proof.** Suppose that the graph is associated with an orthonormal basis, say $\{v_1, \ldots, v_{2n}\}$. Let us choose $v_h, v_k \in W_1$, $h \neq k$ and let us denote by $w'_1, \ldots, w'_{m_2}$ the vectors of the reference corresponding to the vertices of $W_2$. Then,

$$0 = \sum_{i=1}^{2n}(Fv_h \cdot v_i)(Fv_k \cdot v_i) = \sum_{i=1}^{m_2}(Fv_h \cdot w'_i)(Fv_k \cdot w'_i).$$

Consequently, the vectors of $\mathbf{R}^{m_2}$,

$$(Fv_h \cdot w'_1, \ldots, Fv_h \cdot w'_{m_2}) \text{ and } (Fv_k \cdot w'_1, \ldots, Fv_k \cdot w'_{m_2})$$

are orthogonal. Therefore, we have $m_1$ non-null and mutually orthogonal vectors in $\mathbf{R}^{m_2}$, which is a contradiction. □

The above two theorems provide us with some conditions for a graph not to be associated with an orthonormal basis. However, there are graphs that do not satisfy them nor are associated, as we show in the following example:

**Example 4.4.** Let $G$ be the graph with 8 vertices $\{v_1, \ldots, v_8\}$ and edges:

$$\{\{v_1, v_4\}, \{v_1, v_7\}, \{v_1, v_8\}, \{v_2, v_4\}, \{v_2, v_7\}, \{v_2, v_8\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_3, v_8\}, \{v_4, v_6\}, \{v_4, v_7\}, \{v_4, v_8\}, \{v_5, v_6\}, \{v_5, v_7\}, \{v_5, v_8\}, \{v_6, v_7\}, \{v_7, v_8\}\}.$$

Let us suppose that $G$ is associated with an orthonormal basis denoted by $\{v_1, \ldots, v_8\}$ too. Since $v_1, v_2$ and $v_3$ are independent vertices, by using Theorem 3.8, this basis can be written as

$$v_j = \begin{cases} w_j & \text{if } j \leqslant 3, \\ a_{4,4}w_4 + \sum_{k=1}^{3} b_{4,k}w_{4+k} & \text{if } j = 4, \\ a_{j,4}w_4 + \sum_{k=1}^{4} b_{j,k}w_{4+k} & \text{if } 5 \leqslant j \leqslant 8, \end{cases}$$

where $\{w_1, \dots, w_8\}$ is an $F$-basis. Given that the edges $\{v_1, v_5\}$, $\{v_1, v_6\}$, $\{v_2, v_5\}$, $\{v_2, v_6\}$ and $\{v_3, v_7\}$ are not in the graph, we deduce that $b_{5,1} = Fv_1 \cdot v_5 = 0$, $b_{5,2} = Fv_2 \cdot v_5 = 0$, $b_{6,1} = Fv_1 \cdot v_6 = 0$, $b_{6,2} = Fv_2 \cdot v_6 = 0$ and $b_{7,3} = Fv_3 \cdot v_7 = 0$.

Now, since $\{v_4, v_6\}$ is an edge of $G$, $0 \neq Fv_4 \cdot v_6 = a_{4,4}b_{6,4}$ and so, $a_{4,4} \neq 0$ and $b_{6,4} \neq 0$. Moreover, as $\{v_3, v_5\}$ and $\{v_3, v_6\}$ are also edges in the graph, $0 \neq Fv_3 \cdot v_5 = b_{5,3}$ and $0 \neq Fv_3 \cdot v_6 = b_{6,3}$, that is $b_{5,3}, b_{6,3} \neq 0$. Finally, since $\{v_4, v_5\}$ is not an edge in $G$, $0 = Fv_4 \cdot v_5 = a_{4,4}b_{5,4}$, so $b_{5,4} = 0$.

Next, applying that the basis is orthonormal, $0 = v_4 \cdot v_5 = a_{4,4}a_{5,4} + b_{4,3}b_{5,3}$, which gives

$$a_{5,4} = -\frac{b_{4,3}b_{5,3}}{a_{4,4}}.$$

Furthermore, $0 = v_4 \cdot v_6 = a_{4,4}a_{6,4} + b_{4,3}b_{6,3}$, which implies

$$a_{6,4} = -\frac{b_{4,3}b_{6,3}}{a_{4,4}}$$

and

$$0 = v_5 \cdot v_6 = a_{5,4}a_{6,4} + b_{5,3}b_{6,3} = \frac{(a_{4,4}^2 + b_{4,3}^2)b_{5,3}b_{6,3}}{a_{4,4}^2},$$

which contradicts the fact that $a_{4,4} \neq 0$, $b_{5,3} \neq 0$ and $b_{6,3} \neq 0$. Consequently, the graph $G$ cannot be associated with any orthonormal basis.

Nevertheless, Theorem 4.3 is very useful in order to determine some properties about the shape of graphs associated with orthonormal bases. For instance, in the case $m_1 = 2$, we get the following immediate result:

**Corollary 4.5.** *If two vertices of a graph $G$ associated with an orthonormal basis have a common neighbor, then they have another one. Consequently:*

*(i) The connected component of any vertex of degree 1 is $K_2$.*
*(ii) The only possible isolated cycle contained in $G$ is $C_4$. In particular, $G$ has no isolated triangles.*

It is well known (see [13]) that there are 11 non-isomorphic graphs with 4 vertices. From Proposition 4.1 and the above corollary, it is easy to check that only the graphs $K_2 \cup K_2$, $C_4$ and $K_4$ can be associated with an orthonormal basis. In fact, they are, as we showed in Example 3.2.

We have more interesting consequences of Theorem 4.3:

**Corollary 4.6.** *Let $G$ be a graph associated with an orthonormal basis and $v_i$ be a vertex of $G$ with degree $t \geqslant 2$ and $v_{j_1}, \dots, v_{j_t}$ its adjacent vertices. If there is another vertex $v_h$, different from them, which is adjacent to any of the vertices $v_{j_r}$, $1 \leqslant r \leqslant t$, then $v_h$ is adjacent to, at least, another of the vertices $v_{j_k}$, $1 \leqslant k \leqslant t$ and $k \neq r$ (see Fig. 3).*

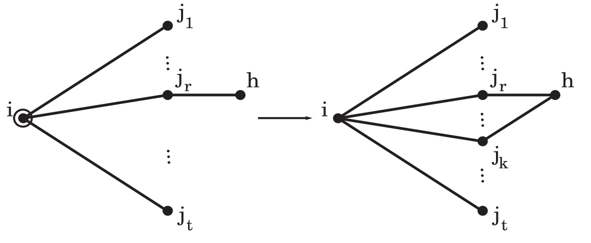


**Fig. 3**. Graphic representation of Corollary 4.6.

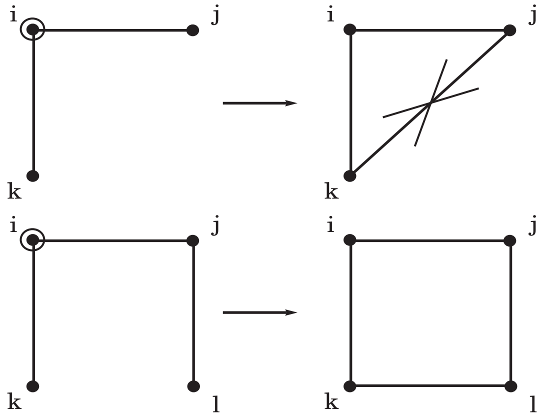


**Fig. 4**. Graphic representation of Corollary 4.7.

**Proof.** If we suppose that $v_h$ is not adjacent to any $v_{j_k}, k \neq r$, then, we apply Theorem 4.3 to the subsets $W_1 = \{v_i, v_h\}$ and $W_2 = \{v_{j_r}\}$. $\square$

**Corollary 4.7.** *Let $G$ be a graph associated with an orthonormal basis and let $v_i$ be a vertex of $G$ with degree* 2*. Denote by $v_j$, $v_k$ its adjacent vertices. Then, $v_j$ and $v_k$ cannot be adjacent vertices. Moreover, if there is another vertex $v_l$, different from $v_i$ and $v_k$, which is adjacent to $v_j$, then $v_l$ is also adjacent to $v_k$ (see Fig. 4).*

**Proof.** Let us suppose that $v_j$ and $v_k$ are adjacent vertices. Then, it is enough to apply Theorem 4.3 to the subsets $W_1 = \{v_i, v_j\}$ and $W_2 = \{v_k\}$. The second part of the statement is a particular case of Corollary 4.6. $\square$

Another particular case of Corollary 4.6 is the following:

**Corollary 4.8.** *Let $G$ be a graph associated with an orthonormal basis and $v_i$ be a vertex of $G$ with degree* 3*. Let $v_j$, $v_k$, $v_l$ denote the adjacent vertices to $v_i$. The following properties are satisfied:*

*(i) If $v_j$ and $v_k$ are adjacent, then $v_l$ is adjacent to both $v_j$ and $v_k$.*
*(ii) If there is another vertex $v_h$, different from $v_i$, $v_k$, $v_l$ which is adjacent to $v_j$, then, $v_h$ is also adjacent to either $v_k$ or $v_l$ (see Fig. 5).*

We observe that, actually, statement (*i*) of this corollary can be obtained from statement (*ii*). In fact, it implies that if there is a triangle in a graph associated with an orthonormal basis and one of its vertices has degree 3, then the triangle lies in a tetrahedron.

By using the above general results, we can now analyze the 6-dimensional case. There are 156 different (in the sense of being non-isomorphic) graphs of 6 vertices; they can be found in pages 9–11 of [13]. Having a look at these graphs, we realize that it is possible to reject all of them, except possibly 15, for being associated with an orthonormal basis. In Example 3.3 we have shown that 12 of them are, in fact, associated with orthonormal bases, by giving specifically such bases in Table 1. The other three are, numbered as in [13], $G_{146} = K_{2,4}$, $G_{195} = \overline{C_4 \cup 2K_1}$ and $G_{201} = \overline{K_3 \cup 3K_1}$. It is easy to check, by using Theorem 4.3, that $G_{146}$ and $G_{195}$ are not associated with any orthonormal basis, taking, in both cases, $W_1$ as the set whose elements are three of the vertices of minimum degree of the corresponding graph and $W_2$ the set whose elements are the two vertices of maximum degree. What about $G_{201}$? To give an answer, we need a new result.

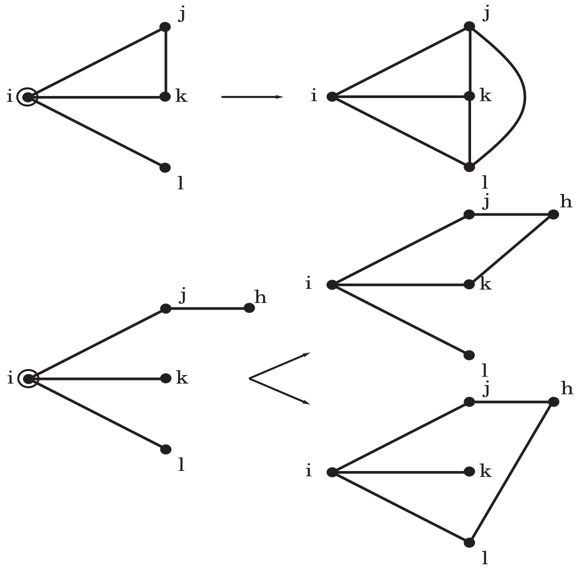


**Fig. 5**. Graphic representation of Corollary 4.8.

**Theorem 4.9.** *The number of independent vertices of a graph associated with an orthonormal basis is, at most, half of its total number of vertices. Moreover, if in such a graph with* $2n$ *vertices there are* $n$ *of them independent, then the other* $n$ *vertices also are.*

**Proof.** Let $G$ be a graph with $2n$ vertices associated with an orthonormal basis $\{v_1, \ldots, v_{2n}\}$ and suppose that there exist $n + k$ $(0 < k < n)$ independent vertices in $G$. Let $W_1$ be the set of such independent vertices and $W_2$ a set of $n + k - 1$ vertices of $G$ containing the $n - k$ vertices of the complementary set of $W_1$. By using the fact that $G$ has no isolated vertices, it is easy to show that $W_1$ and $W_2$ satisfy the conditions of the statement of Theorem 4.3 and, consequently, $G$ cannot be associated with an orthonormal basis, which is a contradiction. Therefore, $G$ has, at most, $n$ independent vertices.

Moreover, suppose that $v_{n+1}, \ldots, v_{2n}$ are independent vertices, that is, $Fv_h \cdot v_k = 0$ when $n + 1 \leqslant h, k \leqslant 2n$. Since $\sum_{j=1}^{2n}(Fv_p \cdot v_j)^2 = 1$, for any $p = 1, \ldots, 2n$, then:

$$n = \sum_{p=n+1}^{2n}\sum_{j=1}^{2n}(Fv_p \cdot v_j)^2 = \sum_{p=n+1}^{2n}\sum_{j=1}^{n}(Fv_p \cdot v_j)^2 = \sum_{j=1}^{n}\sum_{p=n+1}^{2n}(Fv_p \cdot v_j)^2$$
$$= \sum_{j=1}^{n}\sum_{p=n+1}^{2n}(Fv_j \cdot v_p)^2 = \sum_{p=1}^{n}\sum_{j=n+1}^{2n}(Fv_p \cdot v_j)^2.$$

On the other hand,

$$n = \sum_{p=1}^{n}\sum_{j=1}^{2n}(Fv_p \cdot v_j)^2 = \sum_{p=1}^{n}\sum_{j=1}^{n}(Fv_p \cdot v_j)^2 + \sum_{p=1}^{n}\sum_{j=n+1}^{2n}(Fv_p \cdot v_j)^2$$
$$= \sum_{p=1}^{n}\sum_{j=1}^{n}(Fv_p \cdot v_j)^2 + n.$$

Consequently, $Fv_p \cdot v_j = 0$ if $1 \leqslant p, j \leqslant n$ and the first $n$ vertices are also independent. $\square$

From this theorem, we can check that the graph $G_{201}$ cannot be associated with an orthonormal basis because the three vertices of degree 3 are independent vertices and the other ones are not. This completes the study of the 6-dimensional case.

Another immediate consequence of the above theorem is given in the following corollary.

**Corollary 4.10.** *If a bipartite graph $G$ of $2n$ vertices is associated with an orthonormal basis, then $G$ is a subgraph of $K_{n,n}$.*

Next, a natural idea to continue the study of the shape of graphs associated with orthonormal bases could be to increase the number of vertices of the graphs (equivalently, the dimension of the vector spaces) to 8, 10, 12, and so on. But, actually, this is not such a good idea because the number of those graphs grows too much. Nevertheless, we can offer more general results.

First, we recall that a *matching* in a graph $G$ is a subset of edges no two of which have a common vertex and a *perfect matching* or *1-factor* is that one in which each vertex of $V(G)$ is incident on exactly one edge of the matching. Actually, if $|V(G)| = 2n$, a 1-factor in $G$ is just a subgraph isomorphic to $nK_2$. Hence, we can use a well-known theorem of Tutte (see [1,14]), which establishes that a nontrivial graph $G$ contains a 1-factor if and only if, for any proper $S \subseteq V(G)$, the number of odd-components (that is, components with an odd number of vertices) in the graph $G - S$, obtained by removing from $G$ all the vertices of $S$ and all the edges incident to them, is lower or equal than the cardinal of $S$. Therefore, we can prove:

**Theorem 4.11.** *Let $G$ be a graph with $2n$ vertices associated with an orthonormal basis. Then, $G$ admits $nK_2$ as a subgraph.*

**Proof.** Let us suppose that $G$ does not admit $nK_2$ as a subgraph. From Tutte's Theorem, there exists a proper $S \subseteq V(G)$ such that the number of odd-components $H_a$ in the graph $G - S$ is greater than the cardinal of $S$. For any $H_a$, we consider one vertex $v_a \in V(H_a)$ adjacent to one vertex of $S$. Let $W_1$ be the set of all of these $v_a$ and $W_2 = S$. Applying Theorem 4.3, $G$ cannot be associated with an orthonormal basis. □

The converse of the above theorem in not true. For example, consider the graph of four vertices $P_4$. We know that it is not associated with an orthonormal basis, but it contains $2K_2$ as a subgraph. We should like to point out that, adapting the proof of Theorem 4.11 and denoting by $k(G)$ the number of components in a graph $G$, we obtain the following result which is, in some sense, an extension of Tutte's Theorem for graphs associated with orthonormal bases:

**Proposition 4.12.** *Let $G$ be a connected graph associated with an orthonormal basis. Then, $k(G-S) \leqslant |S|$, for any nonempty subset $S$ of $V(G)$.*

Again, the converse is not true. This can be shown by considering Example 4.4: the above inequality holds for this graph (because it does not satisfy the conditions of Theorem 4.3) but it is not associated with any orthonormal basis.

Next, a vectorial subspace $W$ of $V$ is said to be *invariant by $F$* if $Fw \in W$, for any $w \in W$. It is easy to show, by using the same reasoning as above, that any invariant subspace of $V$ is of even dimension and admits an $F$-basis.

**Lemma 4.13.** *Let $G$ be a graph associated with an orthonormal basis. Then, the vertices of a component of $G$ span an invariant subspace of $V$. Moreover, two of such subspaces are mutually orthogonal.*

**Proof.** Let us denote by $\mathcal{B}' = \{v_1, \ldots, v_{2n}\}$ the orthonormal basis associated with $G$ and by $W_i$ the subspace spanned by the vertices of a component $G_i$ of $G$. We can suppose that $\{v_1, \ldots, v_m\}$ are the vectors of $\mathcal{B}'$ spanning $W_i$. Hence, for any $w = \sum_{k=1}^{m} a_k v_k \in W_i$ and any $h \in \{m+1, \ldots, 2n\}$, we have that $Fw \cdot v_h = \sum_{k=1}^{m} a_k F v_k \cdot v_h = 0$ and so $Fw$ must be in $W_i$.

Moreover, if $i \neq j$, then $W_i$ and $W_j$ are mutually orthogonal because their corresponding components $G_i$ and $G_j$ have no adjacent vertices. $\square$

**Theorem 4.14.** *A graph is associated with an orthonormal basis if and only if any of its components also is.*

**Proof.** Let $G$ be a graph associated with an orthonormal basis $\{v_1, \ldots, v_{2n}\}$ and let $G_i$ be a component of $G$. We denote by $W_i$ the subspace of $V$ spanned by the vectors $\{v_{i_1}, \ldots, v_{i_m}\}$ corresponding to the vertices of $G_i$. We consider in $W_i$ the inner product induced by the one in $V$ and the linear application $F_i = F|_{W_i}$, which is an endomorphism of $V_i$ because of Lemma 4.13. It is clear that $F_i$ is an isometry of $W_i$ such that $F_i^2 = -Id$ and that $G_i$ is associated with the orthonormal basis $\{v_{i_1}, \ldots, v_{i_m}\}$.

Conversely, if any component $G_i$ $(i = 1, \ldots, r)$ of $G$ is associated with an orthonormal basis, then there exist a vectorial space $V_i$ (endowed with an inner product and an isometry $F_i$ of $V_i$ such that $F_i^2 = -Id$) and an orthonormal basis $\{v_1^i, \ldots, v_{2n_i}^i\}$ of $V_i$ associated with $G_i$. We consider the direct sum $V = \oplus_i V_i$ with its natural component-to-component inner product and the isometry $F$ of $V$ given by $Fv = \sum_i F_i v_i$, for each $v = \sum_i v_i \in V$. It is clear that $F^2 = -Id$ and that $G$ is associated with the orthonormal basis

$$\{v_1^1, \ldots, v_{2n_1}^1, \ldots, v_1^r, \ldots, v_{2n_r}^r\}. \quad \square$$

Finally, using the above results, we are now going to study the shape of graphs whose vertices have degree less than or equal to 3 and are associated with orthonormal bases.

**Theorem 4.15.** *A cubic graph is associated with an orthonormal basis if and only if its connected components are $K_4$, $K_{3,3}$ or $Q_3$.*

**Proof.** Let $G$ be a cubic graph associated with an orthonormal basis and let $C$ be one of its connected components. We have the following cases:

**Case 1:** *$K_{2,3}$ is a subgraph of $C$.* We denote by $v_1$ and $v_5$ the vertices of degree 3 in the subgraph $K_{2,3}$ and by $v_2$, $v_3$ and $v_4$ their common neighbors. Let $v_6$ be the neighbor of $v_2$ which is neither $v_1$ nor $v_5$. By using Corollary 4.8 (ii) $v_6$ is adjacent to either $v_3$ or $v_4$. We can suppose, without loss of generality, that $v_6$ is adjacent to $v_3$. If it is also adjacent to $v_4$, then $C$ must be $K_{3,3}$. On the other hand, if $v_6$ is not adjacent to $v_4$, let $v_7$ be the neighbor of $v_4$ which is neither $v_1$ nor $v_5$. From Corollary 4.8 (ii) again $v_7$ should be adjacent to either $v_2$ or $v_3$, but this is impossible because $G$ is a cubic graph.

**Case 2:** *$K_3$ is a subgraph of $C$.* Let $v_1$, $v_2$ and $v_3$ be the vertices of $K_3$ and $v_4$ be the other neighbor of $v_1$. By using Corollary 4.8 (i), $v_4$ is adjacent to both $v_2$ and $v_3$. Then, $C$ must be $K_4$.

**Case 3:** *Neither $K_{2,3}$ nor $K_3$ is a subgraph of $C$.* Let $v_1$ be any vertex of $C$ and $v_2$, $v_4$, $v_5$ be its neighbors. Since $K_3$ is not a subgraph of $C$, $v_2$, $v_4$ and $v_5$ are not adjacent two by two. Now, let $v_3$ be a neighbor of $v_2$ different from $v_1$. By virtue of Corollary 4.8 (ii), $v_3$ is adjacent to either $v_4$ or $v_5$. We can suppose, without loss of generality, that $v_3$ is adjacent to $v_4$. Therefore, it cannot be also adjacent to $v_5$ because $K_{2,3}$ is not a subgraph of $C$. Let $v_6$, $v_7$ and $v_8$ be the neighbors of $v_2$, $v_3$ and $v_4$, respectively, not belonging to $\{v_1, v_2, v_3, v_4\}$. Since neither $K_{2,3}$ nor $K_3$ is a subgraph of $C$, it is easy to see that $v_5$, $v_6$, $v_7$ and $v_8$ are different two by two. From Corollary 4.8 (ii), $v_5$ and $v_7$ are adjacent to both $v_6$ and $v_8$ and $C$ is $Q_3$.

Conversely, if the components of a cubic graph $G$ are $K_4$, $K_{3,3}$ and $Q_3$, then it is enough to apply Theorem 4.14 and the existence of orthonormal basis associated with $K_4$, $K_{3,3}$ and $Q_3$ (see Example 3.2, $G_{174}$ from Table 1 and Example 3.4, respectively). $\square$

**Theorem 4.16.** *The only graphs of maximum degree at most 3 which are associated with an orthonormal basis are those whose connected components are $K_2$, $C_4$, $K_4$, $K_{3,3} - e$, $K_{3,3}$ or $Q_3$.*

**Proof.** We have already shown in Examples 3.1–3.4 that $K_2$, $C_4$, $K_4$, $K_{3,3} - e = G_{154}$, $K_{3,3}$ and $Q_3$ are associated with orthonormal bases. Thus, from Theorem 4.14, any graph having some of them as its connected components also is.

To prove the uniqueness, let $G$ be a graph of maximum degree at most 3 associated with an orthonormal basis and let us consider any connected component $C$ of $G$. Then, if $G$ is a cubic graph, by using Theorem 4.15, $C$ is $K_4$, $K_{3,3}$ or $Q_3$. On the other hand, if there is a vertex of degree 1 in $C$, from Corollary 4.5, $C$ is $K_2$. Therefore, we just have to consider the case in which all vertices of $C$ have degree either 2 or 3 with al least one of them of degree 2. Denote this one by $v_1$. Let $v_2$ and $v_3$ be its neighbors. From Corollary 4.5, there exists another common neighbor $v_4$ of $v_2$ and $v_3$. There are two cases:

**Case 1:** $v_2$ *is of degree 2.* Suppose that $v_3$ is of degree 3 and let $v_5$ be its neighbor which is neither $v_1$ nor $v_4$. Then, from Corollary 4.5, $v_1$ and $v_5$ have a common neighbor different from $v_3$. However, the only neighbor of $v_1$ different from $v_3$ is $v_2$, but, since $v_2$ is of degree 2, it cannot be adjacent to $v_5$, which is a contradiction and so, $v_3$ is of degree 2.

Next, if $v_4$ has a neighbor, say $v_5$, different from $v_2$ and $v_3$, from Corollary 4.5, $v_2$ and $v_5$ have another common neighbor, which is not $v_4$. But the only neighbor of $v_2$ different from $v_4$, is $v_1$, which is not adjacent to $v_5$ and we deduce that $v_4$ cannot be of degree 3. Consequently, $C$ is $C_4$.

**Case 2:** $v_2$ *is of degree 3.* Let $v_5$ be the neighbor of $v_2$ which is neither $v_1$ nor $v_5$. From Corollary 4.5, $v_1$ and $v_5$ have another common neighbor, but since $v_1$ is of degree 2, this common neighbor has to be $v_3$. Now, by using Theorem 4.3 for $m = 3$ and since $v_1$ is of degree 2, $v_4$ and $v_5$ have a common neighbor, say $v_6$, different from $v_2$ and $v_3$. Then, $v_6$ cannot be of degree 2 because, if there exists a vertex adjacent to $v_6$, denoted by $v_7$, which is neither $v_4$ nor $v_5$, from Corollary 4.5, $v_4$ and $v_7$ should have another common neighbor, different from $v_6$, but $v_2$ and $v_3$, the other neighbors of $v_4$, cannot be adjacent to $v_7$. Consequently, $v_6$ is of degree 2 and $C$ is $K_{3,3} - e$. $\square$

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