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Upper bounds on the bisection width of 3- and 4-regular graphs

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This paper is dedicated to our friend of long years Prof. GIORGIO AUSIELLO with the best wishes

Abstract

We derive new upper bounds on the bisection width of graphs which have a regular vertex degree. We show that the bisection width of sufficiently large 3-regular graphs with $|V|$ vertices is at most $(\frac{1}{6}+\varepsilon)|V|$, $\varepsilon > 0$. For the bisection width of sufficiently large 4-regular graphs we show an upper bound of $(\frac{2}{5}+\varepsilon)|V|$, $\varepsilon > 0$.

Keywords: Graph partitioning; Bisection width; Regular graphs; Local improvement

1. Introduction

There are graph-partitioning problems in a wide range of applications. The task is to divide the set of vertices of a graph equally into a given number of parts while keeping the number of crossing edges between vertices belonging to different parts, called the *cut size* of the partition, as small as possible. The special case of a partition of the graph into 2 parts is called a *bisection*, and the minimal cut size of all balanced bisections of a graph is called its *bisection width*. Its calculation is NP-complete for arbitrary graphs [11] and remains NP-complete for regular graphs [2].

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There are several results on bounds on the bisection width of regular graphs (discussed below). Results for 3- and 4-regular graphs are of special interest because these are the lowest non-trivial degrees.

It is of general theoretical interest to improve previous upper bounds on 3- and 4-regular graphs. Moreover, there are some direct applications of these results. As a motivating example, upper bounds on the bisection width of 4-regular graphs have successfully been applied to the configuration of transputer systems [13].

1.1. Definitions and previous results

Let $G=(V,E)$ be a simple undirected graph with vertex set V of cardinality $n:=|V|$ and edge set E. A graph is d-regular if for all $v\in V$ it is $|\{w\in V; \{v,w\}\in E\}|=d$.

Let $\pi: V\to\{0,1\}$ be a *bisection* of G. It distributes the vertices among parts V_0 and V_1. We focus on *balanced* bisections, i.e. the number of vertices in the parts differ by at most 1. Let $\mathrm{cut}(\pi):=|\{\{v,w\}\in E; \pi(v)\neq\pi(w)\}|$ be the *cut size* of π. The *bisection width* of a graph G is $bw(G):=\min\{\mathrm{cut}(\pi); \pi$ is a balanced bisection of $G\}$.

The bisection width is known for some graph classes with regular degree such as tori, cube-connected-cycles [21] or butterflies [4].

There are several results on bounds on the bisection width of arbitrary regular graphs. Clark and Entringer [6] present an upper bound of $\frac{n+138}{3}$ for the bisection width of 3-regular graphs. Kostochka and Melnikov improve this asymptotically and show an upper bound of $\frac{n}{4}+\mathrm{O}(\sqrt{n}\log n)$ [16]. Recently, an upper bound of $0.198n+\mathrm{O}(\log(n))$ has been proved in [27]. Hromkovic and Monien [13] proved an upper bound of $\frac{n}{2}+1$ for the bisection width of 4-regular graphs with $n\geqslant 350$. A general upper bound of $\frac{n}{2}+5$ for 4-regular graphs with any number of vertices is proven in [27]. The result of [16] for 3-regular graphs above is a corollary of an upper bound of $\frac{d-2}{4}n+\mathrm{O}(d\sqrt{n}\log n)$ for the bisection width of d-regular graphs in the same paper. An upper bound of $\frac{d-2}{4}n+1$ for $n\geqslant n_0(d)$ with some function $n_0(d)$ is shown in [20,24] by generalizing the techniques of [13]. Alon [1] uses probabilistic arguments to show that the bisection width is at most $(\frac{d}{2}-\frac{3\sqrt{d}}{16\sqrt{2}})\frac{n}{2}$ for d-regular graphs with $n>40d^9$.

There are results for lower bounds of almost all d-regular graphs. Bollobas [5] shows that for $d\to\infty$ the bisection width of almost every d-regular graph is at least $(\frac{d}{2}-\sqrt{\ln(2)\cdot d})\frac{n}{2}$. For $d=4$ he shows that almost all 4-regular graphs have a bisection width of at least $\frac{11}{50}n=0.22n$. Furthermore, Kostochka and Melnikov show that almost every 3-regular graph has a bisection width of at least $\frac{1}{9.9}n\approx 0.101n$ [17].

There are some (slightly weaker) results for explicitly constructible infinite graph classes with high bisection width. The *Ramanujan Graphs* (see e.g. [7,18,19,22]) have a regular degree d and are defined by having $\lambda_2\geqslant d-2\sqrt{d-1}$ with λ_2 being the second smallest eigenvalue of the Laplacian of the graph. The well-known spectral lower bound $\frac{\lambda_2\cdot n}{4}$ on the bisection width of a graph (cf. [9]) directly leads to a lower bound of $(\frac{d}{2}-\sqrt{d-1})\frac{n}{2}$ for d-regular Ramanujan graphs. This implies lower bounds of $0.042n$ and $0.133n$ for the bisection widths of 3-regular and 4-regular Ramanujan graphs. The spectral lower bound has been improved in [3] to a lower bound of $0.082|V|$ for the bisection width of large 3-regular Ramanujan graphs and a lower bound of $0.176|V|$ for the bisection width of large 4-regular Ramanujan graphs.

There are many heuristics for graph partitioning which are successfully being used in applications. Furthermore, efficient software implementations of the most relevant methods are available by using software tools like e.g. CHACO [12], JOSTLE [28], METIS [15], SCOTCH [26] or PARTY [25]. These heuristics try to calculate a bisection with a small cut size. However, they do not guarantee an approximation of the bisection width. Recently, it has been shown that the bisection width can be approximated by a polynomial time algorithm within a factor of $O(\log^2(|V|))$ [10].

1.2. New results and outline of the paper

In this paper we improve previous upper bounds on the bisection width of large 3- and 4-regular graphs.

Two main lemmas in this paper are not exclusively connected to the bisection problem and may also be of interest on their own. Although they are used in the bisection Sections 3 and 4, we kept them outside those sections because they are also fairly technical. Therefore, they are described in their own Section 2.

In Section 3 we prove for any $\varepsilon > 0$ an upper bound of $(\frac{1}{6} + \varepsilon)n$ on the bisection width of large 3-regular graphs and in Section 4 we prove an upper bound of $(\frac{2}{5} + \varepsilon)n$ on the bisection width of large 4-regular graphs.

As discussed above, there are large 3-regular graphs with a bisection width of at least $0.101n$ and large 4-regular graphs with a bisection width of at least $0.22n$. Thus, the results are optimal up to constant factors and our results improve these factors.

Parts of Sections 2 and 3 of this paper are published in a preliminary and short version in the proceedings of the Symposium on Mathematical Foundations of Computer Science (MFCS) 2001 [23].

1.3. Iterative local improvement with helpful sets

The proofs in this paper are constructive and follow an iterative local improvement scheme. It starts with an arbitrary balanced bisection. If the cut size of it does not fulfill the stated upper bound, it performs two steps to improve the bisection as illustrated in Fig. 1.

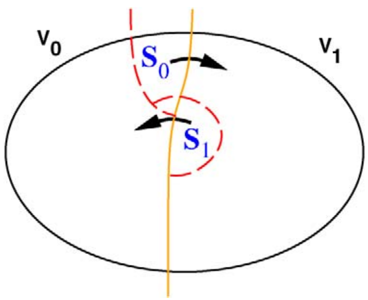

Fig. 1. One iteration of a local improvement.

In the first step, a small set $S_0 \subset V_0$ is moved to V_1. S_0 is chosen such that this move decreases the cut size. In the second step, a set $S_1 \subset V_1 \cup S_0$ with $|S_1| = |S_0|$ is moved to V_0. S_1 is chosen such that the cut size does not increase too much, i.e. such that the increase is less than the decrease in the first step. Thus, the resulting bisection is balanced and has a smaller cut size. These steps are repeated until the cut size drops below the upper bound. The proofs in this paper ensure that there are sets S_0 and S_1 with the desired property as long as the cut size is higher than the stated upper bound.

This local improvement scheme has successfully been used to derive upper bounds on the bisection width of 4-regular graphs [13,20]. Furthermore, it is the basis for the *Helpful-Set* heuristic which is able to calculate bisections with low cut sizes for very large graphs in a short time [8,24]. An implementation of the Helpful-Set heuristic can be found in the software tool PARTY [25].

A move of a set of vertices from one part to the other changes the cut size of the bisection. The helpfulness of the set is the amount of this change.

Definition 1. Let π be a bisection of a graph $G = (V, E)$. For $S \subset V_p(\pi)$, $p \in \{0, 1\}$, let

$$H(S) = \left|\left\{\{v, w\} \in E;\, v \in S,\, w \in V \backslash V_p(\pi)\right\}\right| - \left|\left\{\{v, w\} \in E;\, v \in S,\, w \in V_p(\pi) \backslash S\right\}\right|$$

be the *helpfulness* of S. S is called $H(S)$-*helpful*.

2. Two technical lemmas

In this section we prove two technical lemmas. Although these two lemmas are used to derive bounds on the bisection width, they are not exclusively connected to this topic, i.e. they may be of interest on their own.

Lemma 1 is the main lemma to derive an upper bound on the bisection width of 3-regular graphs and will be used to prove Lemma 3 in Section 3. Lemma 2 will be used in Lemma 6 of Section 4 to show the first step of the local improvement step for deriving an upper bound on the bisection width of 4-regular graphs (although the lemma itself makes a statement on graphs with maximum degree of 3).

The following lemma makes a statement on 3-regular graphs with black and red edges. Such a graph is e.g. constructed in Lemma 3, where one side of a bisection is considered and the helpfulness of a set can be stated as the relation of the red and black edges.

Lemma 1. *Let $G = (V, E)$, $E = B \uplus R$, be a 3-regular graph with black edges B and red edges R. Let each vertex be adjacent to at least one black edge. Let $|R| > (\frac{1}{2} + \varepsilon)|V|$ for an $\varepsilon > 0$. Then there is a set $S \subset V$ of size $\mathrm{O}(\frac{1}{\varepsilon})$ such that the number of red edges between vertices of S is larger than the number of black edges between S and $V \backslash S$.*

Proof. Each vertex is adjacent to at most 2 red edges. Thus, it is $\varepsilon < \frac{1}{2}$.

Let b_i, $i \in \{1, 2, 3\}$, be the number of vertices which are adjacent to i black edges. For a set $S \subset V$ let $b_i(S)$, $i \in \{1, 2, 3\}$, be the number of vertices of black degree i in S. It is

$|V| = b_1 + b_2 + b_3$ and $2|R| = 2b_1 + b_2$ and it follows

$$\frac{|R|}{|V|} = \frac{2b_1 + b_2}{2(b_1 + b_2 + b_3)}. \tag{1}$$

The fact $|V| < 2|R|$ leads us to

$$b_3 = |V| - b_1 - b_2 < 2|R| - b_1 - b_2 = b_1. \tag{2}$$

For $b_3 \approx b_1$ Eq. (1) leads to $\frac{|R|}{|V|} \approx \frac{1}{2}$. The idea of this proof is to show that if we could not find a set fulfilling the lemma, then b_3 is close enough to b_1 such that $\frac{|R|}{|V|} < \frac{1}{2} + \varepsilon$ and we get a contradiction to the condition of the lemma.

Call a set $S \subset V$ *positive* if it has more internal red edges than external black edges, *negative* if it has more external black edges than internal red edges and *neutral* if the numbers are equal.

Consider the graph consisting of black edges only and let F be the family of its connected components. Clearly, the elements of F are neutral or positive. As a simple example, a positive set $I \in F$ of size $\mathrm{O}(\frac{1}{\varepsilon})$ fulfills the lemma.

Let $r(I)$, $I \in F$, be the number of edges which can be removed from I without splitting it into disconnected components. It holds $b_3(I) = b_1(I) + 2r(I) - 2$, because I is a connected component. Let $r = \sum_{I \in F} r(I)$. It is

$$b_3 = b_1 + 2r - 2|F|. \tag{3}$$

Let $\delta > 0$ be a constant. The value of δ will be assigned below. A set $I \in F$ is called *small* if $|I| \leqslant \frac{1}{\delta}$ and *large* otherwise. Denote with $\alpha(S)$, $S \subset V$, the number of red edges between vertices of S and vertices of small sets in F. Denote with $s(S)$, $S \subset V$, the size of the union of S and all small sets of F which are connected to S via a red edge. A set $I \in F$ is called *thin* if $s(I) \leqslant \frac{\alpha(I)+1}{\delta}$ and *thick* otherwise.

The outline of the proof is the following. We will find a positive set of size at most $\frac{20}{\delta} + 1$. Since we will set later that $\delta = \frac{\varepsilon}{1+2\varepsilon}$, we can say that any positive set of size at most $\mathrm{O}(\frac{1}{\delta})$ fulfills the lemma and we are finished. We iteratively remove elements of F and prove that there remains a large and thin set $I \in F$ which has some additional properties (discussed below) and that we can find a set $S \subset I$ which, together with small sets connected to it, fulfills the lemma.

There are some simple cases. If there is a positive small set $I \in F$, then I fulfills the lemma. If there is a red edge between two different small sets $I_a, I_b \in F$, then $I_a \cup I_b$ fulfills the lemma because I_a and I_b are neutral themselves and the union has an additional internal red edge. Both kind of fulfilling sets have a size of at most $\frac{2}{\delta}$. Thus, we may assume that there is no red edge within or between small sets of F.

In the following we remove sets from F iteratively.

Step 1. As long as there is a small set $I \in F$ which has at least one cycle of black edges, remove I from F.

Step 2. As long as there is a large and thick set $I \in F$, remove I from F and remove all small sets $J \in F$ which are connected to I via a red edge.

Notice that the actions in both steps may change the values $\alpha(J)$ and $s(J)$ for the sets $J \in F$ and thin sets of F may change to thick sets.

After these removals, all elements in F are either

(i) large and thin or
(ii) small and the black edges form a tree.

If there is a large and thin set $I \in F$ with $\alpha(I) > 4r(I)$, we will show below how we can find a set $S \subset I$ which, together with small sets connected to it, fulfills the lemma. Before we do so, we show that such a set I exists in F.

Assume that all large and thin sets $I \in F$ have $\alpha(I) \leqslant 4r(I)$.

Let F_1 be the family of (a) the large sets remaining in F, (b) the large sets removed from F in Step 2 and (c) the small sets removed from F in Step 2. We show that the average size of an element in F_1 is at least $\frac{1}{\delta}$. For each set $I \in F_1$ of type (a) it is $|I| \geqslant \frac{1}{\delta}$. For each set $I \in F_1$ of type (b) it is $s(I) \geqslant \frac{\alpha+1}{\delta}$ with α being the value of $\alpha(I)$ at the time I was removed from F in Step 2. This removal operation caused the removal of k, $k \leqslant \alpha$, small sets S_i, $1 \leqslant i \leqslant k$, from F (it is $k < \alpha$ if there is a small set connected to I via more than one red edge). These small sets belong to sets of type (c). Clearly, it is $|I \cup \bigcup_{1 \leqslant i \leqslant k} S_i| \geqslant \frac{\alpha+1}{\delta} \geqslant \frac{k+1}{\delta}$. Overall, the elements of F_1 have an average size of at least $\frac{1}{\delta}$, i.e. it is $|V| = b_1 + b_2 + b_3 \geqslant \frac{|F_1|}{\delta}$. With $b_3 < b_1$ (Eq. (2)) we get

$$|F_1| < \delta(2b_1 + b_2). \tag{4}$$

Let F_2 be the family of (a) the small sets removed from F in Step 1 and (b) the small sets I remaining in F with $|I| \geqslant 2$. Clearly, F_1 and F_2 are disjoint. Let $|F_2(a)|$ be the number of sets of type (a) and $|F_2(b)|$ be the number of sets of type (b). For each set I of type (a) it is $r(I) \geqslant 1$. For each set I of type (b) it is $|I| \geqslant 2$ due to the fact that each vertex is incident to at least 1 black edge. Furthermore, in each set I of type (b) the black edges form a tree and, thus, there are at least 4 red edges connecting vertices of I with vertices of $V \backslash I$. Notice that these vertices may only belong to large and thin sets remaining in F. Otherwise, if one of them would belong to a small set we would have found a fulfilling set as discussed above and if one of them would belong to a large and thick set, I would have been removed in Step 2 above.

There are at least $4|F_2(b)|$ red edges connecting the small sets of F with the large and thin sets of F. However, there are at most $4\sum_{I \in F, I \text{ is large}} r(I)$ red edges of this kind due to the fact that $\alpha(I) \leqslant 4r(I)$ for all large $I \in F$. Thus, it is

$$r \geqslant \left|F_2(a)\right| + \sum_{I \in F, I \text{ is large}} r(I) \geqslant \left|F_2(a)\right| + \left|F_2(b)\right|.$$

This leads us with Eq. (3) to

$$b_3 = b_1 + 2r - 2|F_1| - 2|F_2| \geqslant b_1 - 2|F_1| \geqslant b_1 - 2|F_1|. \tag{5}$$

With Eqs. (4) and (5) we can rewrite Eq. (1) to

$$\frac{|R|}{|V|} = \frac{2b_1 + b_2}{2(b_1 + b_2 + b_3)} < \frac{2b_1 + b_2}{2(2b_1 + b_2 - 2\delta(2b_1 + b_2))} = \frac{1}{2 - 4\delta}.$$

However, it is $\frac{|R|}{|V|} > \frac{1}{2} + \varepsilon$ due to the condition of the lemma. This leads to contradiction for $\delta \leqslant \frac{\varepsilon}{1+2\varepsilon}$ and we set $\delta = \frac{\varepsilon}{1+2\varepsilon}$ as mentioned above.

This shows that after the removal operations there remains a large and thin set $I \in F$ with $\alpha(I) > 4r(I)$. In the remainder we show how we can find a set $S \subset I$ which, together with small sets connected to it, fulfills the lemma.

Let $G_I = (I, E_I)$ be the subgraph of black edges induces by I. It is $|E_I| = |I| - 1 + r(I)$. Construct a reduced graph $\hat{G} = (\hat{V}, \hat{E})$ from G_I in the following way. In a first step successively delete all vertices of degree one until no such vertices remain. In a second step delete all paths of vertices of degree 2 (they connect vertices of degree 3) and replace each path with a single edge. For each new edge $e \in \hat{E}$ denote with $T(e)$ the deleted tree consisting of the replaced path and all (recursively) adjacent vertices deleted in step one, i.e. $I = \hat{V} \cup \bigcup_{e \in \hat{E}} T(e)$.

It is $|\hat{E}| = |\hat{V}| - 1 + r(I)$. $\hat{G}$ is regular of degree 3 and, thus, it is $|\hat{E}| = \frac{3}{2}|\hat{V}|$. It follows that $|\hat{V}| = 2(r(I) - 1)$ and $|\hat{E}| = 3(r(I) - 1)$.

Call an edge $e \in \hat{E}$ *fat* if it is $s(T(e)) > \frac{4}{\delta}\alpha(T(e))$. Let $x = \sum_{e \in \hat{E},\ e \text{ not fat}} \alpha(T(e))$. For a $v \in \hat{V}$ let $e(v)$ be the set of non-fat edges incident to v and let $n(v) = \sum_{e \in e(v)} \alpha(T(e))$. Clearly, it is $\sum_{v \in \hat{V}} n(v) = 2x$.

Assume that $x \leqslant 3r(I) - 1$. Then, at least $\frac{1}{4}\alpha(I) + 1$ of the red connections belong to fat edges. Thus, it is

$$s(I) \geqslant \sum_{e \in \hat{E},\ e \text{ is fat}} s\big(T(e)\big) > \frac{4}{\delta} \sum_{e \in \hat{E},\ e \text{ is fat}} \alpha\big(T(e)\big) \geqslant \frac{\alpha(I) + 1}{\delta}.$$

This is a contradiction to I being thin. Therefore, it is $x > 3r(I) - 1$.

There exists a vertex $v \in \hat{V}$ with $n(v) \geqslant 4$ because otherwise it would be

$$x = \frac{1}{2} \sum_{v \in \hat{V}} n(v) \leqslant \frac{3}{2}|\hat{V}| = 3\big(r(I) - 1\big).$$

Distinguish between the following cases.

(1) If $n(v) = 4$, then the union of $v \cup \bigcup_{e \in e(v)} T(e)$ and the 4 adjacent small sets has a size of at most $\frac{16}{\delta} + 1$ and has at least 4 internal red edges and at most 3 external black edges, fulfilling the lemma.
(2) If $n(v) \geqslant 5$ and $v \in \hat{V}$ is incident to two non-fat edges $e_1, e_2 \in e(v)$ with $\alpha(T(e_1)) = \alpha(T(e_2)) = 2$, then the union of $v \cup T(e_1) \cup T(e_2)$ and the 4 adjacent small sets has a size at most $\frac{16}{\delta} + 1$ and, again, has at least 4 internal red edges and at most 3 external black edges, fulfilling the lemma.

Thus, if we remain unsuccessful there is a non-fat edge $e \in e(v)$ with $\alpha(T(e)) \geqslant 3$. In the following we show how to find a set in $T(e)$ which, together with small sets connected to it, fulfills the lemma.

$T(e)$ has two external black edges with respect to G_I. Mark both vertices of $I \backslash T(e)$ which are connected to $T(e)$ via a black edge. Let K be the union of $T(e)$ and both marked vertices. K is a tree and both marked vertices are leaves. Remember that it is $\alpha(T(e)) \geqslant 3$

and $s(T(e)) \leqslant \frac{4}{\delta}\alpha(T(e))$. Take one of the marked leaves as the root of a tree with directed edges pointing from the root to the leaves. For each $v \in K$ let $T(v)$ be the subtree with root v.

Now designate the vertices v in the tree which have the following properties: (i) $T(v)$ does not contain a marked vertex, (ii) $\alpha(T(v)) = 2$ and (iii) $T(v)$ does not contain any vertex which has the properties (i) and (ii), i.e. v is as low as possible in the tree. If it is $s(T(v)) \leqslant \frac{20}{\delta}$ for a designated vertex v, then the union of $T(v)$ with the two adjacent small sets has two internal red edges and only one external black edge (the one connecting v to the rest of the tree) and the lemma is fulfilled. Now let $s(T(v)) > \frac{20}{\delta}$ for all designated vertices v.

Construct a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{V} \subset K$ being the union of the designated vertices, the two marked vertices and all paths between them. Let y be the number of designated vertices. Thus, $\tilde{G}$ has a root with degree 1, $y + 1$ leaves and y vertices of degree 3.

Let $\tilde{P}$ be the set of paths of vertices of degree 2 in $\tilde{G}$. It is $|\tilde{P}| = 2y + 1$. For each vertex v on these paths let w be its (possible) neighbor in $K \backslash \tilde{V}$. Clearly, $T(w) \cap \tilde{V} = \emptyset$ because $\alpha(T(w)) \leqslant 1$ due to the construction above. Let $U(v) = \{v\} \cup T(w)$ if w exists and $U(v) = \{v\}$ otherwise.

For a path $P \in \tilde{P}$ let $U(P) = \bigcup_{v\in P} U(v)$, $\alpha(P) = \alpha(U(P))$, and $s(P) = s(U(P))$. It is

$$\alpha\bigl(T(e)\bigr) = 2y + \sum_{P\in\tilde{P}} \alpha(P).$$

Furthermore, it is

$$s\bigl(T(e)\bigr) > \frac{20}{\delta} y + \sum_{P\in\tilde{P}} s(P).$$

The fact that $s(T(e)) \leqslant \frac{4\alpha(T(e))}{\delta}$ leads us to

$$0 \leqslant \frac{4}{\delta}\alpha\bigl(T(e)\bigr) - s\bigl(T(e)\bigr) < \sum_{P\in\tilde{P}} \left(\frac{4}{\delta}\alpha(P) - s(P)\right) - \frac{12}{\delta} y. \tag{6}$$

Let $\tilde{P}^{+} = \{P \in \tilde{P};\ \frac{4}{\delta}\alpha(P) \geqslant s(P)\}$. From Eq. (6) it follows

$$\sum_{P\in\tilde{P}^{+}} \left(\frac{4}{\delta}\alpha(P) - s(P)\right) > \frac{12}{\delta} y. \tag{7}$$

For each vertex $v \in T(e)$ of degree 3 let $P(v) \subset \tilde{P}^{+}$ be the set of paths incident to v and let $n(v) = \sum_{P\in P(v)} \alpha(P)$.

We first show that there is one such vertex v with $n(v) \geqslant 4$. Assume that no such vertex exists, i.e. it is $n(v) \leqslant 3$ for all vertices of degree 3. Notice that there are y such vertices. This implies

$$\sum_{P\in\tilde{P}^{+}} \alpha(P) \leqslant \sum_{v \text{ has degree 3 in } T(e)} n(v) \leqslant 3y \tag{8}$$

which is a contradiction due to Eq. (7).

Thus, there exists a vertex $v \in T(e)$ of degree 3 with $n(v) \geqslant 4$. A path $P \in \tilde{P}^+$ corresponds to a non-fat edge as described above and, similar to above, we distinguish between the following cases.

(1) If $n(v) = 4$, then the union of $v \cup \bigcup_{P \in P(v)} U(e)$ and the 4 adjacent small sets has a size of at most $\frac{16}{\delta} + 1$ and has at least 4 internal red edges and at most 3 external black edges, fulfilling the lemma.
(2) If $n(v) \geqslant 5$ and v is incident to two paths $P_1, P_2 \in P(v)$ with $\alpha(P_1) = \alpha(P_2) = 2$, then the union of $v \cup U(P_1) \cup U(P_2)$ and the 4 adjacent small sets has a size of at most $\frac{16}{\delta} + 1$ and, again, has at least 4 internal red edges and at most 3 external black edges, fulfilling the lemma.

We conclude that we either found a set of size at most $\frac{16}{\delta} + 1$ fulfilling the lemma or there is a path $P \in \tilde{P}^+$ with $\alpha(P) \geqslant 3$ and $\alpha(U(v)) \leqslant 1$ for each $v \in P$.

We will take 3 small sets which are connected to $\bigcup_{v \in P} U(v)$ via a red edge and unify them with a subpath of P such that the subpath connects the 3 small sets. This union is a positive set. We will show that there are 3 small sets such that the connecting subpath is not too long, i.e. such that the size of the union is at most $\frac{20}{\delta}$.

For each $v \in P$ we denote a weight of $w(v)$ to v which is the size of the union of $U(v)$ and the (possible) small set connected to $U(v)$ via a red edge. It is $s(P) = \sum_{v \in P} w(v)$. Furthermore, v is called a red vertex if there is a red edge connecting a small set with any vertex of $U(v)$. Due to the construction above, at most one vertex of $U(v)$ may be connected to a small set. Thus, there are $\alpha(P)$ red vertices r_i, $1 \leqslant i \leqslant \alpha(P)$, and there are $\alpha(P) + 1$ (possible empty) paths P_i, $1 \leqslant i \leqslant \alpha(P) + 1$, of non-red vertices in P. Let $w(P_i) = \sum_{v \in P_i} w(v)$ be the weight of a subpath and let $y_i = w(r_i) + w(P_{i+1})$, $1 \leqslant i \leqslant \alpha(P)$, be the sum of the weight of a red vertex and the path on one side of this red vertex. It is $\sum_{j=1}^{\alpha(P)} y_j \leqslant s(P)$.

Let $\triangle = \min\{y_i + y_{i+1} + y_{i+2};\ 1 \leqslant i \leqslant \alpha(P) - 2\}$. It is

$$\bigl(\alpha(P) - 2\bigr)\triangle \leqslant 3 \sum_{j=1}^{\alpha(P)} y_j \leqslant 3s(P)$$

and, because of $s(P) \leqslant \frac{4}{\delta}\alpha(P)$, it is

$$\triangle \leqslant \frac{\alpha(P)}{\alpha(P) - 2} \cdot \frac{12}{\delta}.$$

If $\alpha(P) \leqslant 4$, then the union of P and the adjacent small sets is of size at most $\frac{20}{\delta}$ and has 2 external black edges and at least 3 internal red edges, fulfilling the lemma. If $\alpha(P) \geqslant 5$, it is $\frac{\alpha(P)}{\alpha(P)-2} \leqslant \frac{5}{3}$ and the definition of $\triangle$ ensures that there is an $1 \leqslant i \leqslant \alpha(P) - 2$ such that the union of $U(v_i) \cup U(P_{i+1}) \cup U(v_{i+1}) \cup U(P_{i+2}) \cup U(v_{v+2})$ and the 3 small sets connected to v_i, v_{i+1} and v_{i+2} is a positive set with a size of at most $\frac{20}{\delta}$. □

Lemma 2 will be used in Lemma 6 to show the first step of the local improvement step for 4-regular graphs.

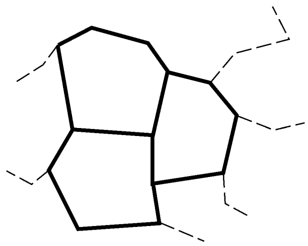

Fig. 2. Repeatedly delete all vertices of degree 1 (doted paths) until there are only vertices of degree 2 and 3 (cycles) remaining.

Lemma 2. *Let $G = (V, E)$ be a graph with a maximum degree of 3, $e = |E| \geqslant (1+\beta)|V|$, $\beta > 0$ and $|V| \geqslant \frac{8}{\beta}$. There is a set $S \subset V$ of size $|S| = \mathrm{O}(\frac{\log(|V|)}{\beta})$ with at least $|S| + 1$ internal edges. Furthermore, S can be computed in time $\mathrm{O}(|V|)$.*

Proof. Let x, y and z be the number of vertices of degree 3, 2 and 1. Then $x + y + z = n$ and $3x + 2y + z = 2e$. It holds

$$x - z = 3x + 2y + z - 2n = 2e - 2n \geqslant 2\beta n \quad \text{and}$$
$$y = n - x - z \leqslant (1 - 2\beta)n - 2z.$$

In a first step, repeatedly delete all vertices of degree 1. Adjacent vertices of degree 2 become vertices of degree 1. Adjacent vertices of degree 3 become vertices of degree 2 and, possibly, vertices of degree 1 later on. These new vertices of degree 1 are also deleted until only vertices of degree 2 and 3 are left as shown in Fig. 2 (right). Thus, all connected components without any cycles and all induced subtrees will be deleted.

For each initial vertex of degree 1, the vertices along the path to the closest vertex of degree 3 are deleted and that vertex of degree 3 becomes a vertex of degree 2. Therefore, for each initial vertex of degree 1 the number of vertices of degree 3 decreases by at most 1. Although many vertices of degree 2 may be deleted, in the worst case a vertex of degree 1 is directly connected to a vertex of degree 3 and, thus, for each initial vertex of degree 1 the number of vertices of degree 2 may increase by at most 1. Let x_1 and y_1 be the number of vertices with degree 3 and 2 after step one. Now it holds

$$x_1 \geqslant x - z \geqslant 2\beta n \quad \text{and} \quad y_1 \leqslant y + z \leqslant (1 - 2\beta)n - 2z + z \leqslant (1 - 2\beta)n.$$

In the second step delete long paths of vertices of degree 2. The average length of the paths is

$$\frac{y_1}{\frac{3}{2}x_1} \leqslant \frac{1 - 2\beta}{3\beta} =: \lambda.$$

Now delete all paths of a length of at least $4\lambda + 2$ one after another. Let r be the number of such paths. The number of deleted vertices of degree 2 is at least $r(4\lambda + 2)$. Each deletion of a path transforms two vertices of degree 3 to vertices of degree 2. The original number of vertices of degree 2 is y_1. However, this number can increase by 2 for each deleted path.

In general, at most $y_1 + 2r$ vertices of degree 2 can be deleted, thus

$$4r\frac{1-2\beta}{3\beta} + 2r = r(4\lambda + 2) \leqslant (1-2\beta)n + 2r.$$

This leads to $r \leqslant \frac{3}{4}\beta n$. The above implies that the number x_2 of vertices of degree 3 after step two fulfills

$$x_2 \geqslant x_1 - 2r \geqslant \left(2 - \frac{3}{2}\right)\beta n = \frac{1}{2}\beta n \geqslant 4.$$

Now consider a graph that consists of the x_2 vertices with degree 3 and edges between them if they are connected via a path along vertices of degree 2. The length g of a shortest cycle in a graph is called the *girth* of a graph. If the girth is odd, it is $g \leqslant 2\log(\frac{n+2}{3}) + 1$ and if the girth is even it is $g \leqslant 2\log(\frac{n+2}{2})$ (see e.g. [14]). Thus, in a 3-regular graph with x_2 vertices there is a cycle with at most $O(\log(x_2))$ vertices. Now perform a breath-first search which starts from the cycle vertices until either a new path between any two cycle vertices or a cycle in one of the search trees is found. It is easy to see that such a new path as well as such a new cycle consists of at most $O(\log(x_2))$ vertices. The set S that consists of the first cycle and either the path or the second cycle has $|S| = O(\log(x_2))$ and $|S| + 1$ internal edges.

After transforming S back to the original graph, the paths between the vertices of degree 3 consist of at most $O(\lambda \cdot \log(n))$ vertices of degree 2. Thus, $|S| = O(\frac{\log(n)}{\beta})$ and S has $|S| + 1$ internal edges.

The construction of S requires several steps, each of which can be performed in time $O(|V|)$. Thus, the set S can be constructed in time $O(|V|)$. □

3. Upper bound on the bisection width of 3-regular graphs

In this section we derive a new upper bound on the bisection width of 3-regular graphs. The proof is based on the iterative local improvement scheme described in the previous section. We will use Lemma 3 for the first step of the improvement scheme and Lemma 4 for the second step as described in Section 1.3. These lemmas will be used to prove the theorem at the end of this section. Lemma 1 will be used to prove Lemma 3.

In the following we state the Lemmas 3 and 4 which are used for the two steps of the local improvement scheme. Before we do so, we classify the vertices.

Definition 2. The vertices of V_0 (or V_1) are classified according to their distance to the cut. It is $V_0 = C \uplus D \uplus E$ with C vertices being at a distance of 1 to the cut, i.e. they are incident to a cut edge. D vertices are at a distance of 2 and E vertices at a distance of at least 3. D vertices are further classified with respect to the number of adjacent C vertices. I.e. $D = D_3 \uplus D_2 \uplus D_1$ and each D_x vertex is adjacent to x vertices in C. Overall, it is $V_0 = C \uplus D_3 \uplus D_2 \uplus D_1 \uplus E$.

The values c, d_3, d_2, d_1 and e specify the number of vertices of each type and the values $c(X)$, $d_3(X)$, $d_2(X)$, $d_1(X)$ and $e(X)$ denote the number of the according vertices in a set $X \subset V$.

Lemma 3. *Let π be a bisection of a 3-regular graph $G = (V, E)$ with $V = V_0 \uplus V_1$. If* $\mathrm{cut}(\pi) > (\frac{1}{3} + 2\varepsilon)|V_0|$, $\varepsilon > 0$, *then there is an at least* 1*-helpful set of size* $\mathrm{O}(\frac{1}{\varepsilon})$ *in* V_0.

Proof. We focus on part V_0 of the bisection only. Let $m := |V_0|$.

There are some structures which would directly lead to small 1-helpful sets.

(i) If a C vertex is incident to two or three cut edges, it is an at least 1-helpful set by itself.
(ii) A set of three connected C vertices is 1-helpful.
(iii) If there are two adjacent C vertices and one of them is adjacent to a D_3 vertex, then the union of the adjacent C vertices, the D_3 vertex and its two other adjacent C vertices form a 1-helpful set of size 5.
(iv) Another 1-helpful set can be formed if two D_3 vertices are adjacent to a common C vertex. Then, the union of both D_3 vertices and their adjacent C vertices is a 1-helpful set of size 7.
(v) Let v be a vertex which is adjacent to a C vertex which itself is adjacent to another C vertex or a D_3 vertex. If both other neighbors of v are from $C \cup D_2 \cup D_3$, the union of the mentioned vertices and their adjacent C vertices forms an at least 1-helpful set of size at most 11.
Not that case (v) describes a quite general type of helpful sets. (v) includes the cases (iii) and (iv).

In the remainder we can assume that these types of structures do not exist. Especially, since (i) is excluded, it is $c = \mathrm{cut}(\pi)$.

We derive the helpful set in three steps.

1. Perform several transformations on the edges between vertices of V_0 to avoid certain structures in the graph. Let $\bar{G} = (V, \bar{E})$ be the transformed graph.
2. Derive a 1-helpful set $\bar{H}$ of size $\mathrm{O}(\frac{1}{\varepsilon})$ in $\bar{G}$.
3. Inverse the transformations in the reverse order. The helpfulness of the set might decrease. In this case, we include some further vertices to the set during the inverse transformations and derive a 1-helpful set H of G with $|H| = \mathrm{O}(|\bar{H}|)$.

We manipulate the edges between vertices of V_0 and transform G into a graph $\bar{G} = (V, \bar{E})$ which has no adjacent C vertices and no D_3 vertices.

We perform two types of transformations. The transformation type 1 is performed until there are no adjacent C vertices. Then, transformation type 2 is performed until there are no D_3 vertices. The transformations are illustrated in Fig. 3. Each transformation deletes two edges $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ and includes two edges $\{\alpha, \gamma\}$ and $\{\beta, \delta\}$. Thus, the graph remains 3-regular.

Transformation type 1. Avoid adjacent C vertices (illustrated in Fig. 3, left). It is $\alpha, \beta \in C$, $\gamma \in D_1 \cup D_2$ (due to (ii) and (iii) above) and there is a $\delta \in D_1 \cup E$ (due to (ii) and (v) above).

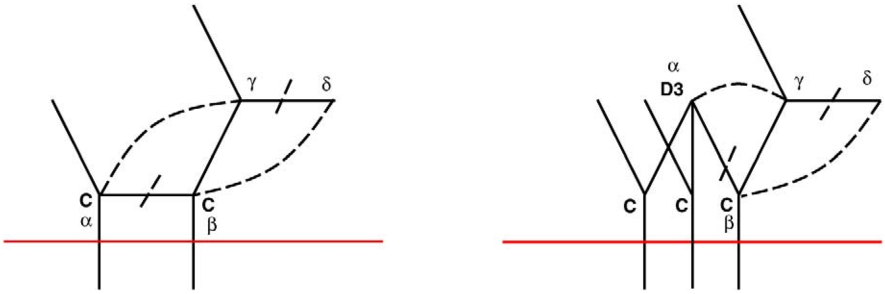

Fig. 3. The transformations of the edges. Left: Adjacent C vertices. Right: A D_3 vertex.

Transformation type 2. Avoid D_3 vertices (illustrated in Fig. 3, right). It is $\alpha \in D_3$, $\beta \in C$, $\gamma \in D_1 \cup D_2$ (due to (iii) and (iv) above) and there is a $\delta \in D_1 \cup E$ (due to (v) above).

Notice that the transformations of type 1 do not generate a new pair of adjacent C vertices and that the transformations of type 2 neither generate a new pair of adjacent C vertices nor a new D_3 vertex.

The graph $\bar{G} = (V, \bar{E})$ is the result after all transformations.

We now show how a 1-helpful set in $\bar{G}$ can be transformed to a 1-helpful set in G. We inverse all previous transformations and do these inversions in the reverse order. Consider a transformation of the graph and assume there is a 1-helpful set $\bar{H}$ of the graph after this transformation.

Clearly, if $\{\alpha, \beta, \gamma, \delta\} \cap \bar{H} = \emptyset$, the inverse transformation does not change the helpfulness of $\bar{H}$. Furthermore, the helpfulness of $\bar{H}$ does not change if the size of $|\{\alpha, \beta, \gamma, \delta\} \cap \bar{H}|$ is 1, 3 or 4. To be more precise, the helpfulness of $\bar{H}$ may only decrease if $\alpha, \gamma \in \bar{H}$ and $\beta, \delta \notin \bar{H}$ or if $\beta, \delta \in \bar{H}$ and $\alpha, \gamma \notin \bar{H}$.

For these two cases we describe the following enlargements of $\bar{H}$ in order to keep it at least 1-helpful.

Inverse of transformation type 2.
- (i) If $\alpha, \gamma \in \bar{H}$, $\beta, \delta \notin \bar{H}$: Let $H = \bar{H} \cup \{\beta\}$.
- (ii) If $\beta, \delta \in \bar{H}$, $\alpha, \gamma \notin \bar{H}$: Let $H = \bar{H} \cup \{\alpha, \gamma\} \cup \{v \in C; \{v, \alpha\} \in E\}$.

Inverse of transformation type 1.
- (i) If $\alpha, \gamma \in \bar{H}$, $\beta, \delta \notin \bar{H}$: Let $H = \bar{H} \cup \{\beta\}$.
- (ii) If $\beta, \delta \in \bar{H}$, $\alpha, \gamma \notin \bar{H}$: Let $H = \bar{H} \cup \{\alpha, \gamma\}$.

We have to ensure that an inverse transformation does not increase the size of the set too much. Each inverse transformation enlarges the set by at most a constant number of vertices. We show that all inverse transformations together increase the size of the set by not more than a constant factor.

Consider the enlargement of case (ii) of the inverse transformation type 2. We count the number of C vertices of the set which are not adjacent to a D_3 vertex which is also in the set. At the beginning, all C vertices of the set fulfill this condition. Each time the inverse transformation of type 2 increases the set, this value decreases by at least one because of the vertex β. This value cannot increase, because each new C vertex in the set is already adjacent to a D_3 vertex of the set. Thus, the number of enlarging transformations of case (ii) of type 2 is bounded by the initial number of C vertices in the set. The case (i) only includes new C vertices. This number is at most 3 times the number of D vertices in the final set.

For the inverse of type 1 we count the number of C vertices which are not adjacent to another C vertex which is also in the set. At the beginning, all C vertices of the set fulfill this condition. Each time the inverse transformation of type 1 increases the set, this value decreases by at least one. This value cannot increase, because each new C vertex in the set is already adjacent to another C vertex in the set. Thus, the number of enlarging transformations of type 1 is bounded by the initial number of C vertices in the set.

Overall, all inverse transformations together enlarge the size of the set by at most a constant factor.

It is left to show that we can find a small helpful set in $\bar{G}$. It is $d_3 = 0$ (due to transformation type 2) and $2c = 2d_2 + d_1$ (due to transformation type 1). Furthermore, because of $|V_0| < 3c$ (due to the condition in the lemma) it is (similar to Eq. (2))

$$e = |V_0| - c - d_2 - d_1 < 2c - d_2 - d_1 = 2d_2 + d_1 - d_2 - d_1 = d_2\,. \tag{9}$$

In the following we construct a new graph and apply Lemma 1. The new graph K consists of the D and E vertices of $\bar{G}$, i.e. $K = (U, F)$ with $U = D \uplus E$. Let $F = B \uplus R$ with black edges B and red edges R. The black edges are the edges between the D and E vertices as in $\bar{G}$. Furthermore, there is a red edge between two vertices if they are adjacent to a common C vertex in $\bar{G}$, i.e. $|R| = c$. Thus, K is 3-regular with a maximum red degree of 2, due to the fact that there are no D_3 vertices. It is

$$\begin{aligned} |R| = c &> \left(\frac{1}{3} + 2\varepsilon\right)(c + d_2 + d_1 + e) \\ &> \left(\frac{1}{3} + 2\varepsilon\right)\frac{3}{2}(d_2 + d_1 + e) = \left(\frac{1}{2} + 3\varepsilon\right)|U|. \end{aligned}$$

Thus, K fulfills the requirements of Lemma 1 for $\bar{\varepsilon} = 3\varepsilon$. We use Lemma 1 to derive a set S of D and E vertices with size $\mathrm{O}(\frac{1}{\bar{\varepsilon}}) = \mathrm{O}(\frac{1}{\varepsilon})$.

The number b_{ext} of external black edges of S with respect to K is equal to the number of edges between S and other D and E vertices in V_0. The number r_{int} of internal red edges of S with respect to K is equal to the number of C vertices in V_0 which are connected to two vertices in S. Lemma 1 ensures $r_{\text{int}} > b_{\text{ext}}$.

Let $\hat{S}$ be the union of S with all adjacent C-vertices. It is $|\hat{S}| = \mathrm{O}(\frac{1}{\varepsilon})$. Each external black edge connects $\hat{S}$ with $V_0 \backslash \hat{S}$. The external red edges are neutral, because they connect S via a C vertex to $V_0 \backslash \hat{S}$. Thus, such a C vertex has one edge to V_1 and one edge to $V_0 \backslash \hat{S}$. Each internal red edge is a C vertex which is connected to two vertices in S. Thus, such a vertex has one edge to V_1 and no edges to $V_0 \backslash \hat{S}$. Overall, there are $r_{\text{int}} + r_{\text{ext}}$ edges between $\hat{S}$ and V_1 and $b_{\text{ext}} + r_{\text{ext}}$ edges between $\hat{S}$ and $V_0 \backslash \hat{S}$. This leads to $H(\hat{S}) = r_{\text{int}} - b_{\text{ext}} > 0$ and $\hat{S}$ fulfills the lemma. □

The following lemma is used for the second step of the local improvement strategy. We denote with '$\log(x)$' the logarithm of x to the basis 2.

Lemma 4. *Let $G = (V, E)$ be a connected 3-regular graph and let π be a bisection of G. If $|V_1(\pi)| < 3 \cdot \mathrm{cut}(\pi)$ and $0 < x < |V_1(\pi)|$, then there is a set $S \subset V_1(\pi)$ with $|S| = x$ and $H(S) \geqslant -1 - \lfloor \log(|S|) \rfloor$.*

Proof. We first discuss the following cases.

(i) If $x \leqslant 2$, any set S of x vertices which are incident to a cut edge has the desired property $H(S) \geqslant -1 - \log(|S|)$.

(ii) If we find a set $Z \subset V_1(\pi)$ with $|Z| \leqslant x$ and $H(Z) \geqslant 0$, we can move Z from V_1 to V_0 without increasing the cut size. It remains to apply the lemma again with $\bar{x} = x - |Z|$. Notice that in the case $H(Z) > 0$ the move may result in $|V_1| \geqslant 3 \cdot \mathrm{cut}$. In this case vertices which are incident to a cut edge can be moved from V_1 to V_0 until we either moved a total of x vertices or until it holds $|V_1| < 3 \cdot \mathrm{cut}$. In the latter case we apply the lemma again.

(iii) If we find a set $Z \subset V_1(\pi)$ with $\frac{x}{2} \leqslant |Z| \leqslant x$ and $H(Z) \geqslant -1$, we can move Z from V_1 to V_0 with increasing the cut size by at most 1. It remains to apply the lemma again with $\bar{x} = x - |Z| < \frac{x}{2}$. This will construct a set $\bar{S}$ with $|\bar{S}| = x - |Z|$ and $H(\bar{S}) \geqslant -1 - \log(|\bar{S}|)$, and a unified set $S = Z \cup \bar{S}$ with $|S| = x$ and $H(S) \geqslant -1 - 1 - \log(|\bar{S}|) \geqslant -1 - \log(|S|)$.

In the following we can exclude the existence of certain small 0-helpful sets as illustrated in Fig. 4. One example are C vertices incident to two or three cut edges and any set of two adjacent C vertices. A D_3 vertex, together with its adjacent C vertices, also forms a 0-helpful set.

In the remainder there are no such sets, i.e. it is $\mathrm{cut} = c$, $d_3 = 0$ and $2c = 2d_2 + d_1$. Because of $|V_1| < 3c$ it holds Eq. (9).

Consider the graph induced by the vertex set $D \cup E$ and its connected components. Let F be the family of these components. For a set $I \subset V_1(\pi)$ define the enlarged set $Z(I) = I \cup \{v \in C; \exists w \in I \text{ with } \{v, w\} \in E\}$ which includes the adjacent C-vertices. Clearly, each

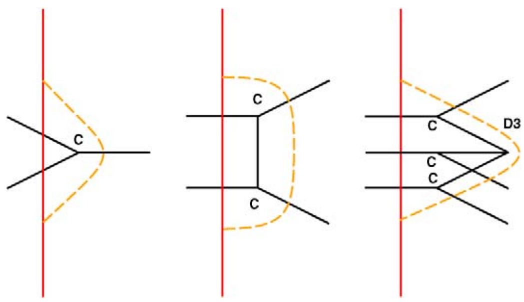

Fig. 4. Sets Z with $H(Z) \geqslant 0$.

set $Z(I)$ for an $I \in F$ is at least 0-helpful. If there is a set $Z(I)$, $I \in F$, with $|Z(I)| \leqslant x$, we proceed as discussed in case (ii).

Consider a connected component $I \in F$ and let $K = (I, J)$ be the subgraph of G induced by I. The E vertices in K have degree 3, D_1 vertices have degree 2 and D_2 vertices have degree 1. It is easy to see that $e(I) \geqslant d_2(I)$ iff K contains a cycle and $e(I) = d_2(I) - 2$ otherwise. Because of Eq. (9) there is an $I \in F$ for which the induced subgraph is a tree.

Let $I \in F$ be a connected component with the induced subgraph $T = (I, J)$ being a tree. Assign a weight $w(v)$ to each vertex v in the tree with $w(v) = |Z(\{v\})|$. For each vertex v this is one higher than the number of C vertices adjacent to v. Thus, each leaf has a weight of 3, each vertex of degree 2 has a weight of 2 and each vertex with a degree of 3 has a weight of 1. It is $\sum_{v \in L} w(v) = |Z(L)|$ for an $L \subset I$ if there are no C vertices which are connected to two vertices of L. It is $\sum_{v \in L} w(v) > |Z(L)|$ for an $L \subset I$ if there is at least one such vertex.

With $|Z(I)| > x$ it is $\sum_{v \in T} w(v) > x$. Clearly, for this type of weight distribution there is an edge in T which separates T into T_1 and T_2 with $\frac{x}{2} \leqslant \sum_{v \in T_1} w(v) \leqslant x$.

If $|Z(T_1)| < \sum_{v \in T_1} w(v)$, it is $|Z(T_1)| < x$ and $H(Z(T_1)) \geqslant 0$ and we proceed with case (ii) above. If $|Z(T_1)| = \sum_{v \in T_1} w(v)$, it is $\frac{x}{2} \leqslant |Z(T_1)| \leqslant x$ and $H(Z(T_1)) \geqslant -1$. We proceed with case (iii) above. □

Theorem 1. *For any $\varepsilon > 0$ there is a value $n(\varepsilon)$ such that the bisection width of any 3-regular graph $G = (V, E)$ with $|V| > n(\varepsilon)$ is at most $(\frac{1}{6} + \varepsilon)|V|$.*

Proof. We start with an arbitrary bisection and follow the iterative local improvement scheme described in Section 1.3. As long as the cut is above the bound, we repeatedly use Lemmas 3 and 4 to calculate a new bisection with a lower cut. Thus, we can limit our focus on one iteration of the two lemmas. Let π_0 be a balanced bisection at the start of the iteration with $\mathrm{cut}(\pi_0) > (\frac{1}{6} + \varepsilon)|V|$.

Step 1. We construct a small helpful set $S \subset V_0$. Set $k = 3 \cdot \lceil \log(\frac{1}{\varepsilon}) \rceil$. The value of k is discussed below. We apply Lemma 3 several times. Each time we find an at least 1-helpful set. We proceed until we reach a total helpfulness of at least k, i.e. we apply the lemma k' times with $k' \leqslant k$. Let $S_i \subset V_0$, $1 \leqslant i \leqslant k'$, with $|S_i| = \mathrm{O}(\frac{1}{\varepsilon})$ be the sets constructed with Lemma 3. After a 1-helpful set S_i is constructed, it is moved from V_0 to V_1 and the next set S_{i+1} is constructed. Let $S = \biguplus_{1 \leqslant i \leqslant k'} S_i$. It is $|S| = k' \cdot \mathrm{O}(\frac{1}{\varepsilon}) = k \cdot \mathrm{O}(\frac{1}{\varepsilon})$ and $H(S) \geqslant k$.
It remains to show that the requirement of Lemma 3 is fulfilled before each construction of a helpful set. Let $\bar{\varepsilon} = \frac{\varepsilon}{2}$. It is $|V| \geqslant 2|V_0| - 1$ and $\mathrm{cut}(\pi_0) > (\frac{1}{3} + 2\bar{\varepsilon})|V_0(\pi_0)| - (\frac{1}{6} + \bar{\varepsilon}) + \bar{\varepsilon}|V|$ at the beginning. Let $n(\varepsilon)$ be large enough such that $\bar{\varepsilon}|V| \geqslant k + (\frac{1}{6} + \bar{\varepsilon})$ for all $|V| > n(\varepsilon)$. Thus, it is $\mathrm{cut}(\pi_0) > (\frac{1}{3} + 2\bar{\varepsilon})|V_0(\pi_0)| + k$. Each application of Lemma 3 decreases the size of the cut. We perform the lemma as long as $\mathrm{cut}(\pi) > \mathrm{cut}(\pi_0) - k > (\frac{1}{3} + 2\bar{\varepsilon})|V_0(\pi_0)| \geqslant (\frac{1}{3} + 2\bar{\varepsilon})|V_0(\pi)|$ with π being the current bisection. Thus, the condition $\mathrm{cut}(\pi) > (\frac{1}{3} + 2\bar{\varepsilon})|V_0(\pi)|$ is true before each application.
Let π_1 be the new bisection with $\mathrm{cut}(\pi_1) = \mathrm{cut}(\pi_0) - H(S)$.

Step 2. If $H(S) = k$, it is $\mathrm{cut}(\pi_1) = \mathrm{cut}(\pi_0) - k$. If $H(S) > k$, it is $\mathrm{cut}(\pi_1) < \mathrm{cut}(\pi_0) - k$ and we change π_1 by iteratively moving border vertices from V_1 to V_0 until we either get to $\mathrm{cut}(\pi_1) = \mathrm{cut}(\pi_0) - k$ or to a balanced bisection (in this case we are already finished). Each move of a border vertex decreases the imbalance of the bisection and increases the cut by at most one.
Let $i := |V_1(\pi_1)| - \frac{n}{2}$ be the imbalance of π_1. It is $i \leqslant k \cdot \mathrm{O}(\frac{1}{\varepsilon})$. We use Lemma 4 to find a balancing set $\bar{S} \subset V_1(\pi_1)$ with $|\bar{S}| = i$.
Lemma 4 can only be applied if $|V_1(\pi_1)| < 3 \cdot \mathrm{cut}(\pi_1)$. The fact $\mathrm{cut}(\pi_0) > (\frac{1}{6} + \varepsilon)n$ implies $|V_1(\pi_1)| = \frac{n}{2} + i < 3\,\mathrm{cut}(\pi_0) - 3\varepsilon \cdot n + i = 3\,\mathrm{cut}(\pi_1) + 3k - 3\varepsilon \cdot n + i \leqslant 3\,\mathrm{cut}(\pi_1)$ if $3k + i \leqslant 3\varepsilon \cdot n$. Clearly, there is a value $n(\varepsilon)$ such that this equation holds for all graphs with $n > n(\varepsilon)$.
We use Lemma 4 to get a set $\bar{S} \subset V_1(\pi_1)$ with $|\bar{S}| = i$ and $H(\bar{S}) \geqslant -1 - \log(i)$. The move of $\bar{S}$ from V_1 to V_0 results in a balanced bisection π_2 with $\mathrm{cut}(\pi_2) \leqslant \mathrm{cut}(\pi_1) + 1 + \log(i)$.

We need to ensure $\mathrm{cut}(\pi_2) < \mathrm{cut}(\pi_0)$ in order to show a decrease of the cut size. It is $\mathrm{cut}(\pi_2) \leqslant \mathrm{cut}(\pi_0) - k + 1 + \log(i)$ and $i \leqslant k \cdot x\frac{1}{\varepsilon}$. for some constant x. Choosing $k = 3 \cdot \log(\frac{1}{\varepsilon})$ fulfills $k > 1 + \log(k \cdot x\frac{1}{\varepsilon})$ for $\frac{1}{\varepsilon} \geqslant 2^6$ and $\frac{1}{\varepsilon} \geqslant x$. $\square$

4. Upper bound on the bisection width of 4-regular graphs

In this section we are presenting a new upper bound on the bisection width of 4-regular graphs. The proof is based on the iterative local improvement scheme described in Section 1.3.

We categorize the vertices of a 4-regular graph according to their number of external edges in the bisection as shown in Fig. 5. Notice that these categories differ from the ones in Section 3.

Definition 3. Let $G = (V, E)$ be a 4-regular graph and let $\pi(G)$ be a bisection of V. A-vertices are incident to 3 or 4 cut edges, B-vertices are incident to 2 cut edges, C-vertices are incident to 1 cut edge and D-vertices are not incident to any cut edge.

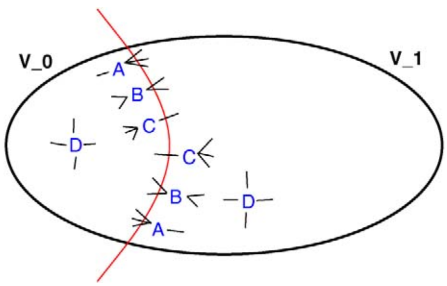

Fig. 5. A, B, C and D vertices.

Whenever this categorization is used, only one part of the bisection is considered and the values $a = |A|$, $b = |B|$, $c = |C|$ and $d = |D|$ denote the cardinalities of the sets in the according part.

Lemmas 5 and 2 are used in Lemma 6 to show the first step of the local improvement step. Lemma 7 shows the second step of the local improvement. These two steps are combined in Theorem 2 to improve the upper bound for large 4-regular graphs. It is an extension and improvement of the local improvement Helpful-Set approach of [13]. At the end of this section we will list an algorithm for calculating a bisection of 4-regular graphs which corresponds to the results of this section.

Lemma 5. *Let π be a bisection of a 4-regular graph $G = (V, E)$. Let B be the set of B vertices and D be the set of D vertices in $V_0(\pi)$. Then there is an at least 2-helpful set $S_0 \subset V_0(\pi)$ of size at most $3\lceil \log(n) \rceil + 1$ in $V_0(\pi)$ or there is an injective function $f : B \to D$ with the property: for each $u \in B$ there is a path of a length of at most $\lceil \log(n) \rceil$ from u to $f(u)$ along C vertices in $V_0(\pi)$. The set S_0, or the function f with the paths from u to $f(u)$ for each $u \in B$, can be computed in time $O(|V|)$.*

Proof. An A vertex in $V_0(\pi)$ is a 2-helpful set by itself and we are finished if such a vertex exists. Thus, in the remainder we may assume that no such vertex exist.

In the following we are trying to find small 2-helpful sets. Clearly, a B vertex is a 0-helpful set by itself. A cycle of C vertices is a 0-helpful set, too. We will search for a short path of C vertices which connects either two B vertices or a B vertex with a short cycle of C vertices. Both cases are 2-helpful sets.

Perform a breath-first search which starts from all B vertices simultaneously and proceeds along C vertices only. Thus, there is a search tree for each B vertex. Let $t = \lceil \log(n) \rceil$ be the maximal depth of the search trees. Proceed until either (a) a depth of t is reached, (b) a certain 2-helpful set is found in any search tree or (c) in each search tree 2 D vertices are found, i.e. each such D vertex is connected to the according B vertex via a path of C vertices and the length of the path is at most t. A 2-helpful set occurs if there are either (i) a C vertex in two different search trees of two B vertices B_1 and B_2 or (ii) a cycle-edge within one search tree. In case (i) the two B vertices B_1 and B_2 and the path of C vertices between them is a 2-helpful set of size at most $2t + 1$. In case (ii), the cycle of C vertices, the B vertex and the path between the cycle and the B vertex is a 2-helpful set of size at most $2t + 1$, too.

If there is no such 2-helpful set to be found, we can show that there are at least 2 D vertices in each search tree. Otherwise, we show that the size of a single search tree already exceeds the total number of vertices. Consider one search tree. The B vertex of this search tree is adjacent to the root of a binary tree of C vertices of depth $t - 1$ as shown in Fig. 6.

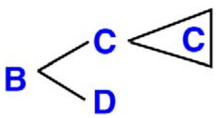

Fig. 6. If at most one D vertex can be found, the search tree consists of at least one complete binary tree of C vertices of depth $t - 1$, i.e. there would be at least $2 + 2^t - 1 \geqslant n + 1$ vertices in the search tree.

The other branch adjacent to B may be blocked by one D vertex. Thus, the size of this search tree is at least $2 + 2^t - 1 \geqslant n + 1$. This contradiction shows that at least 2 D vertices were found in each search tree.

Now construct a bipartite graph which consists of the B and D vertices, such that a vertex B will be connected to a vertex D if it was reached in the search. B vertices have a degree of 2 in this graph. If a D-vertex has a degree of at least 3, the union of this D vertex and 3 paths to the B vertices is a 2-helpful set S with $|S| \leqslant 3t + 1$. Otherwise, the D vertices have a degree of at most 2. Clearly, we are left with a bipartite graph with the B vertices having a degree of 2 and the D vertices having a maximum degree of 2. A maximum matching in such a graph leads us to an injective function $f : B \to D$. Due to the maximal depth of search during the construction of the trees we can guarantee that for each $u \in B$ there is a path of length at most $\lceil \log(n) \rceil$ from u to $f(u)$ along C vertices.

The breath-first search performed to find S can be done in linear time. If such a set S is not found then the search trees for different B vertices have disjoint sets of edges. We stop when two D vertices have been found per search tree. Therefore each edge belongs to at most 2 paths from B vertices to D vertices. □

Lemma 6. *Let π be a bisection of a 4-regular graph $G = (V, E)$ with $V = V_0 \uplus V_1$, $|V_0| \leqslant |V_1|$. If $\mathrm{cut}(\pi) > (0.8 + \varepsilon) \cdot |V_0|$ and $|V_0| \geqslant \frac{80}{25\varepsilon}$ for some $\varepsilon > 0$, then there is an at least 2-helpful set of size $\mathrm{O}(\frac{1}{\varepsilon}(\log(n))^2)$ in $V_0(\pi)$. This set can be calculated in time $\mathrm{O}(|V|)$.*

Proof. Consider only the vertices of part $V_0(\pi)$ throughout this proof. Lemma 5 shows that there is either a 2-helpful set S of size $\mathrm{O}(\log(n))$ or there is an injective function $f : B \to D$ which assigns each vertex of B to a different vertex of D, connected with a path of length $\mathrm{O}(\log(n))$ along B or C vertices. The proof is finished if the first case holds true.

Now construct a new graph $\bar{G} = (\bar{V}, \bar{E})$, which is a copy of G with some transformations. The vertices $u \in B$ and $f(u) \in D$ are transformed to two C vertices as shown in Fig. 7. Choose any neighbor x of u in part $V_1(\pi)$ and any neighbor w of $f(u)$ with $w \neq u$ and $\{u, w\} \notin E$. There is such a vertex w, because $f(u)$ has got 4 and u has only got 2 internal neighbors. Now delete the edges $\{u, x\}$ and $\{f(u), w\}$ and include the edges $\{u, w\}$ and $\{x, f(u)\}$. This is performed for any $u \in B$. The transformations can be performed all in one step, due to the function being injective. The involved B and D vertices are becoming C vertices and the result is a graph $\bar{G}$ with only C and D vertices.

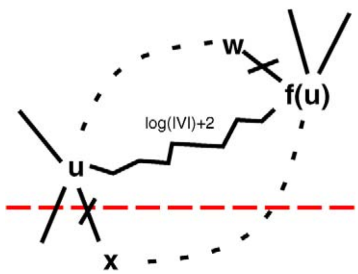

Fig. 7. Transform $u \in B$ and $f(u) \in D$ to C-vertices.

Only C vertices are incident to cut edges, i.e. $c = \mathrm{cut}(\pi)$. The number of edges in the new graph $\bar{G}$ between C and D vertices cannot exceed $4d$ and, therefore, the number of edges between C vertices is at least $\frac{3c-4d}{2} = \frac{3c-4(r-c)}{2} = \frac{7}{2}c - 2r$ where $r = |V_0|$.

Furthermore, consider the graph $H = (U, F)$ which consists of all C vertices and edges between C vertices. It holds $|U| = c = \mathrm{cut}(\pi)$ and $|F| \geqslant \frac{7}{2}|U| - 2r > \frac{7}{2}|U| - \frac{2\cdot\mathrm{cut}(\pi)}{0\cdot 8+\varepsilon} = \frac{7}{2}|U| - \frac{2|U|}{0.8+\varepsilon} = (1+\beta)|U|$ with $\beta := \frac{25\varepsilon}{8+10\varepsilon}$. It follows from $|V_0| \geqslant \frac{80}{25\varepsilon}$ that $|U| = \mathrm{cut}(\pi) \geqslant (0.8+\varepsilon)|V_0| \geqslant \frac{8}{\beta}$. Lemma 2 ensures that there is a set $S \subset U$ of size $\mathrm{O}(\frac{1}{\varepsilon}\log(|U|)) = \mathrm{O}(\frac{1}{\varepsilon} \cdot \log n)$ with at least $|S|+1$ internal edges. With respect to graph $\bar{G}$, the set S of C vertices with $|S|+1$ edges is at least 2-helpful.

S is at least 2-helpful with respect to $\bar{G}$, but we need a 2-helpful set with respect to G. We perform the transformations from G to $\bar{G}$ in the reverse direction. Again, the function being injective allows the performance of all reverse transformations in one step. In each reverse transformation we possibly enlarge the set S such that it remains 2-helpful. The result is a 2-helpful set S with respect to G. In each reverse transformation, a pair of vertices u and $f(u)$ is transformed back to B and D vertices and the edges are set back to the old constellation.

It is easy to see that only in the case $f(u) \in S$ and $u, w \notin S$ the helpfulness may decrease by the reverse transformation. In this case the helpfulness decreases by two and we include the path from u to $f(u)$ into S. The enlargement of S increases the helpfulness by at least 2, resulting in an at least 2-helpful set S. The length of each path is $\mathrm{O}(\log(n))$. Thus, it is $|S| = \mathrm{O}(\frac{1}{\varepsilon}(\log(n))^2)$.

The transformations of the graph and the used lemmas run in time $\mathrm{O}(|V|)$ each. Thus, the 2-helpful set can be calculated in time $\mathrm{O}(|V|)$. □

Hromkovic and Monien [13] proved the following lemma for re-balancing a bisection of a 4-regular graph without increasing the cut size too much.

Lemma 7. [13] *Let π be a bisection of a 4-regular graph $G = (V, E)$ with $|V_0(\pi)| < |V_1(\pi)| \leqslant \frac{5}{4}\mathrm{cut}(\pi)$ and $\mathrm{cut}(\pi) \geqslant 4$. Then a bisection $\bar{\pi}$ with $|V_0(\bar{\pi})|, |V_1(\bar{\pi})| \leqslant \lceil\frac{|V|}{2}\rceil$ and $\mathrm{cut}(\bar{\pi}) \leqslant \mathrm{cut}(\pi) + 2$ can be constructed.*

The algorithmic idea of Lemma 7 is to move A and B vertices from the larger to the smaller side (the cut size will not increase), followed by the move of cycles of C-vertices (the cut size will not increase), finished by a move of a connected component of C-vertices of a size equal to the remaining imbalance (this might increase the cut size by 2, but it is applied at most once). These steps can be performed by a breath-search approach and, thus, Lemma 7 needs a time of $\mathrm{O}(|V|)$.

We use Lemma 6 for the first step of the improvement scheme and Lemma 7 for the second step as described in Section 1.3. These lemmas are used to prove the following theorem for the bisection width of 4-regular graphs.

Theorem 2. *For any $\varepsilon > 0$ there is a value $n(\varepsilon)$ for which the bisection width of any 4-regular graph $G = (V, E)$ with $|V| > n(\varepsilon)$ is at most $(0.4+\varepsilon)|V|$. Such a bisection can be calculated in time $\mathrm{O}(|V|^2)$.*

Proof. Lemma 6 constructs a set of size $\mathrm{O}(\frac{1}{\varepsilon}(\log(n))^2)$. Let α be a constant such that the size of these sets is at most $\alpha\frac{1}{\varepsilon}(\log(n))^2$. Furthermore, we choose $n(\varepsilon)$ large enough such that it holds (i) $\frac{n(\varepsilon)}{2} - 1 \geqslant \frac{80}{25\varepsilon}$, (ii) $n(\varepsilon) \geqslant \frac{4}{\varepsilon} + 2$, (iii) $\frac{n(\varepsilon)}{2} - 1 - 2\alpha\frac{1}{\varepsilon}(\log(n))^2 \geqslant \frac{80}{25\varepsilon}$, and (iv) $n(\varepsilon) \geqslant \frac{4}{5\varepsilon}2\alpha\frac{1}{\varepsilon}(\log(n))^2 + 5$.

We start with an arbitrary bisection. We follow the iterative local improvement scheme as described in Section 1.3. As long as the cut is above the bound, we repeatedly use Lemmas 6 and 7 to calculate a new bisection with a lower cut. Thus, we can limit our focus on one iteration of the two lemmas. Let π_0 be a balanced bisection at the start of the iteration with $\mathrm{cut}(\pi_0) > (0.4 + \varepsilon)|V|$.

Step 1. W.l.o.g. let $|V_0| \leqslant |V_1|$. We construct a small helpful set $S \subset V_0$ by applying Lemma 6 one or two times. The first application of Lemma 6 leads us to an at least 2-helpful set $S_1 \subset V_0$ and we move S_1 to V_1. If the cut size decreased by at least 4, we do not apply Lemma 6 again and set $S = S_1$. If the cut size decreased by exactly 2 (note that a 4-regular graph can only have an even cut size), we apply Lemma 6 again to get another at least 2-helpful set $S_2 \subset V_0$. We also move S_2 to V_1 and set $S = S_1 \uplus S_2$. Overall, it is $|S| \leqslant 2\alpha\frac{1}{\varepsilon}(\log(n))^2$ and $H(S) \geqslant 4$.
It remains to show that the requirements of Lemma 6 are fulfilled before each construction of a helpful set. From $|V| \geqslant 2|V_0|$ and from the bound (i) on $n(\varepsilon)$ it follows that the requirement of Lemma 6 is fulfilled for the first application. We apply Lemma 6 a second time only if $\mathrm{cut}(\pi) = \mathrm{cut}(\pi_0) - 2$ with π being the current bisection. Then from the bound (ii) on $n(\varepsilon)$ it follows $\mathrm{cut}(\pi) = \mathrm{cut}(\pi_0) - 2 > (0,8 + 2\varepsilon) \cdot |V_0(\pi_0)| - 2 \geqslant (0,8 + \varepsilon) \cdot |V_0(\pi_0)| > (0,8 + \varepsilon) \cdot |V_0(\pi)|$ and together with (iii) the condition for Lemma 6 holds true again. Let π_1 be the new bisection. It is $\mathrm{cut}(\pi_1) = \mathrm{cut}(\pi_0) - H(S) \leqslant \mathrm{cut}(\pi_0) - 4$.

Step 2. If $H(S) = 4$, it is $\mathrm{cut}(\pi_1) = \mathrm{cut}(\pi_0) - 4$. If $H(S) > 4$, it is $\mathrm{cut}(\pi_1) < \mathrm{cut}(\pi_0) - 4$ and we change π_1 by iteratively moving border vertices from V_1 to V_0 until we either get to $\mathrm{cut}(\pi_1) = \mathrm{cut}(\pi_0) - 4$ or to a balanced bisection (in this case we are already finished). Each move of a border vertex decreases the imbalance of the bisection and increases the cut by at most two.
After these operations it is cut $(\pi_1) = \mathrm{cut}(\pi_0) - 4$ and $|V_1(\pi_1)| = \lceil\frac{n}{2}\rceil + i$, $i \leqslant 2\alpha\frac{(\log(n))^2}{\varepsilon}$, where i is the size of imbalance which is at most the size of the set S moved from V_0 to V_1 in Step 1. Thus, it is

$$\frac{5}{4}\mathrm{cut}(\pi_1) = \frac{5}{4}\mathrm{cut}(\pi_0) - 5 > \left(\frac{1}{2} + \frac{5}{4}\varepsilon\right) \cdot n - 5 \geqslant \left|V_1(\pi_1)\right| = \frac{n}{2} + i$$

by the use of the bound (iv) on $n(\varepsilon)$.
We use Lemma 7 to get a balanced bisection π_2 with $\mathrm{cut}(\pi_2) \leqslant \mathrm{cut}(\pi_1) + 2$.

Thus, it is $\mathrm{cut}(\pi_2) \leqslant \mathrm{cut}(\pi_1) + 2 \leqslant \mathrm{cut}(\pi_0) - 2$. We repeat the local improvement iteration until we arrive at a cut size fulfilling the bound.

This proof is an iteration of the Lemmas 6 and 7. Thus, each iteration takes time $\mathrm{O}(|V|)$. Furthermore, the cut size is reduced by at least 2 in each iteration. Thus, the total time requirement is $\mathrm{O}(|V|^2)$. □

Table 1
Algorithm for 4-regular graphs fulfilling the bound of Theorem 2

```
Input: G = (V, E), ε > 0. Output: π.

generate an arbitrary bisection π of V;

WHILE cut(π) > (0.4 + ε)|V|

  /* Step 1: construct and move a 4-helpful set (Lemma 6) */
  IF there is an A vertex v ∈ V_0        /* (Lemma 5) */
    S_1 = {v}:
  ELSE
    perform a breath-first-search from each B vertex along C vertices until either:
    (i) a path of C vertices between two B vertices is found:
      in this case let S_1 be the path of C vertices and the two B vertices;
    (ii) a cycle of C vertices is found:
      in this case let S_1 be the cycle of C vertices together with the path to the B vertex;
    (iii) a depth of ⌈log(|V|)⌉ is reached:
      in this case we know that each tree is connected to at least two D vertices;
      IF there is a D vertex marked by at least three B vertices
        let S_1 be this D vertex together with the paths to three B vertices.
      ELSE
        construct a graph of C vertices as described in Lemma 6;
        let S_1 be a connected component with more edges than vertices; /* (Lemma 2) */
        possibly enlarge S_1 by (short) paths as described in Lemma 6;
      END IF;
  END IF;
  move S_1 to V_1;        /* S_1 is at least 2-helpful */
  IF S_1 was only 2-helpful
    calculate another 2-helpful set S_2 ⊂ V_0 and move it to V_1;        /* as before */
  END IF;

  /* Step 2: construct and move an at least -2-helpful balancing set (Lemma 7) */
  WHILE π is not balanced
    IF there is an A or B vertex in V_1
      move this vertex to V_0;
    ELSE IF there is a cycle of C vertices in V_1
      move it (or a part of it if π will become balanced) to V_0;
    ELSE
      move a connected component of C vertices (of size equal to the imbalance) to V_0;
    END IF;
  END WHILE;

END WHILE;
```

Table 1 shows the algorithm of Theorem 2 for 4-regular graphs. Although the asymptotic runtime of the algorithm is $O(|V|^2)$, it is left to say that this asymptotic analysis is very pessimistic and may possibly be improved. Our experiences from an implementation for 4-regular graphs [8] based on a weaker bound on the bisection [13] tell us that this algorithm will most likely exhibit a nearly linear runtime behavior.

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