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# TRANSFORMING TIME-VARYING MULTIVARIABLE SYSTEMS INTO BLOCK COMPANION CANONICAL FORMS

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**Abstract**—The problem of transforming a class of linear time-varying continuous time systems into controllable and observable block companion canonical forms is considered. In terms of system block controllability (observability) matrix, this paper generalizes the results of Shieh *et al.* [3] and provides systematic and straightforward algorithms for obtaining block companion canonical forms. An example is provided to illustrate this transformation technique.

## 1. INTRODUCTION

Research on canonical transformations for the class of linear time-varying systems and their applications have drawn extensive attention in the last two decades. Various canonical transformation techniques have been proposed for systems which are uniformly controllable (observable) and have time-invariant controllability (observability) indices [1–16]. Among these contributions, Brunovsky [1] led to the determination of controllable canonical transformations for this class of systems. Ramar and Ramaswami [2] obtained necessary and sufficient conditions for the existence of controllable block companion canonical form. The transformation of time-varying systems into block companion canonical forms was also studied by Shieh *et al.* [3] recently. They showed that if a time-varying system is uniformly controllable (observable) with its controllability (observability) index $q\{= n/p$, (dimension of system state)/(number of system input or output)$\}$ being an integer, the time-varying system can be transformed into the block companion canonical forms in terms of system controllability (observability) matrices. An advantage of the technique provided by [3] is easy to compute.

Although some other canonical transformation techniques are available for the case of non-integer $q$ (see for example, [2, 4–6]), the transformation procedures are rather lengthy when the system order is high due to their computational complexity. Therefore, it is of interest to further develop other canonical techniques which can retain the computational simplicity of [3] for the case of non-integer $q$. In this paper a technique for block companion canonical transformation of time-varying system is presented which is an extension of [3] to the non-integer $q$ cases. Here, the transformation matrices are derived in terms of block controllability (observability) matrices. In the case when $q$ is an integer, the technique proposed here reduces to that of [3].

The paper is organized as follows. The mathematic preliminaries and definitions are stated in Section 2. A method for controllable block companion canonical transformation in terms of block controllability matrix is given in Section 3. An observable block companion canonical transformation in terms of block observability matrix is extended in Section 4. An example is presented in Section 5 to illustrate the proposed transformation technique.

## 2. PRELIMINARIES AND DEFINITIONS

In this section we introduce some notations and definitions which will be used in the later development. All the matrices considered in this paper are real. The set of $p \times q$ matrices will be denoted as $\mathbf{R}^{p\times q}$, and the set of column vectors with $p$ components as $\mathbf{R}^p$.

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The linear time-varying continuous system to be considered in this paper is described by the following equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1a}$$
$$y(t) = C(t)x(t), \tag{1b}$$

where $t \in [0,\infty)$ is the continuous time variable, $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^r$ is the input vector, $y(t) \in \mathbf{R}^m$ is the output vector, and $A(t) \in \mathbf{R}^{n\times n}$, $B(t) \in \mathbf{R}^{n\times r}$ and $C(t) \in \mathbf{R}^{m\times n}$ are time-varying matrices.

Let the matrix differential operators $\mathbf{J}[\cdot]$ and $\mathbf{L}[\cdot]$ be defined as

$$\mathbf{J}[M(t)] = A(t)M(t) - \frac{d}{dt}M(t), \tag{2a}$$
$$\mathbf{L}[P(t)] = P(t)A(t) + \frac{d}{dt}P(t), \tag{2b}$$

for any matrices $M(t) \in \mathbf{R}^{n\times p}$ and $P(t) \in \mathbf{R}^{q\times n}$ differentiable up to sufficiently high order, and

$$\begin{aligned}\mathbf{J}^j[M(t)] &= \mathbf{J}(\mathbf{J}^{j-1}[M(t)]),\\ &= \underbrace{\mathbf{J}\cdot\mathbf{J}\cdots\mathbf{J}}_{j \text{ times}}[M(t)],\end{aligned} \tag{3a}$$
$$\begin{aligned}\mathbf{L}^j[M(t)] &= \mathbf{L}(\mathbf{L}^{j-1}[P(t)]),\\ &= \underbrace{\mathbf{L}\cdot\mathbf{L}\cdots\mathbf{L}}_{j \text{ times}}[P(t)],\end{aligned} \tag{3b}$$

for $j = 1, 2, \ldots$. Also define

$$\mathbf{J}^0[M(t)] = M(t),$$
$$\mathbf{L}^0[P(t)] = P(t).$$

In terms of matrix operators $\mathbf{J}[\cdot]$ and $\mathbf{L}[\cdot]$, the controllability matrix $W^c(t)$ and observability matrix $W^o(t)$ of system (1) can then be defined as

$$W^c(t) = (\mathbf{J}^0[B(t)]\ \mathbf{J}^1[B(t)]\ \ldots\ \mathbf{J}^{n-1}[B(t)]), \tag{4a}$$
$$W^o(t) = (\mathbf{L}^0[C(t)]^T\ \mathbf{L}^1[C(t)]^T\ \ldots\ \mathbf{L}^{n-1}[C(t)]^T)^T, \tag{4b}$$

respectively.

It is well known [17] that system (1) is uniformly controllable (observable) if and only if

$$Rank(W^c(t))(Rank(W^o(t))) = n \quad \text{for all } t \in [0,\infty).$$

Unlike the time-invariant case, the definition of controllability (observability) for time-varying systems are much more complicated, and readers are referred to Chapter 5 of Chen [17] for various definitions. Here, in order to develop our canonical transformation technique, the following definitions are introduced.

DEFINITION 1. Under the assumption that system (1) is uniformly controllable (observable), if there exists a unique $(n \times n)$ submatrix in $W^c(t)$ ($W^o(t)$) that is non-singular for all $t \in [0,\infty)$, then $W^c(t)$ ($W^o(t)$) is said to have a time-invariant controllability (observability) basis. A uniformly controllable (observable) continuous system whose controllability (observability) matrix has a time-invariant basis is said to have time-invariant controllability (observability) indices on $[0,\infty)$.

For a uniformly controllable (observable) system which has time-invariant controllability (observability) indices, the concept of block controllability (observability) is important in developing the block companion canonical transformation technique and is defined below.

DEFINITION 2. Let $\alpha^c$ be the largest non-negative integer which is less than or equal to $n/r$, and

$$\begin{aligned} \beta^c &= n - \alpha^c r, \qquad & 0 \le \beta^c < r, \\ \gamma^c &= r - \beta^c, \qquad & 0 < \gamma^c \le r, \end{aligned}$$

partition $B(t)$ in two blocks as

$$B(t) = (B_1(t) \quad B_2(t)), \tag{5a}$$

where $B_1(t) \in \mathbf{R}^{n\times\gamma^c}$, $B_2(t) \in \mathbf{R}^{n\times\beta^c}$. Then system (1) is said to be block controllable on $[0,\infty)$ if

$$W^c(t,\alpha^c) = (\mathbf{J}^0[B_1(t)]\ \mathbf{J}^0[B_2(t)]\ \dots\ \mathbf{J}^{\alpha^c-1}[B_1(t)]\ \mathbf{J}^{\alpha^c-1}[B_2(t)]\ \mathbf{J}^{\alpha^c}[B_2(t)]), \tag{5b}$$

has full rank, and

$$span(\mathbf{J}^0[B_1(t)],\ \mathbf{J}^0[B_2(t)],\ \dots,\ \mathbf{J}^{\alpha^c-1}[B_1(t)],\ \mathbf{J}^{\alpha^c-1}[B_2(t)]) \supseteq span\{\mathbf{J}^{\alpha^c}[B_1(t)]\}, \tag{5c}$$

for all $t \in [0,\infty)$.

Parallel to Definition 2, we also have

DEFINITION 3. Let $C(t)$ be partitioned as

$$C(t) = \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}, \tag{6a}$$

then system (1) is said to be block observable on $[0,\infty)$ if

$$W^o(t,\alpha^o) = \begin{pmatrix} \mathbf{L}^0[C_1(t)] \\ \mathbf{L}^0[C_2(t)] \\ \vdots \\ \mathbf{L}^{\alpha^o-0}[C_1(t)] \\ \mathbf{L}^{\alpha^0-1}[C_2(t)] \\ \mathbf{L}^{\alpha^o}[C_2(t)] \end{pmatrix}, \tag{6b}$$

has full rank, and

$$row\ span \begin{pmatrix} \mathbf{L}^0[C_1(t)] \\ \mathbf{L}^0[C_2(t)] \\ \vdots \\ \mathbf{L}^{\alpha^0-1}[C_1(t)] \\ \mathbf{L}^{\alpha^0-1}[C_2(t)] \end{pmatrix} \supseteq row\ span\{\mathbf{L}^{\alpha^o}[C_1(t)]\}, \tag{6c}$$

for all $t \in [0,\infty)$, where $\alpha^o$ is the largest non-negative integer which is less than or equal to $n/m$,

$$\begin{aligned} \beta^o &= n - \alpha^o m, \qquad & 0 \le \beta^o < m, \\ \gamma^o &= m - \beta^o, \qquad & 0 < \gamma^o \le m, \end{aligned}$$

and $C_1(t) \in \mathbf{R}^{\gamma^o\times n}$, $C_2(t) \in \mathbf{R}^{\beta^o\times n}$.

In our technique the following assumption for system (1) is required.

**Assumption 1.** System (1) is block controllable (observable) and satisfies the following conditions:

(i) $\frac{d^i}{dt^i}A(t)$, $\frac{d^i}{dt^i}B(t)$ and $\frac{d^i}{dt^i}C(t)$ for $0 \le i \le n-1$ are defined and bounded for all $t \in [0,\infty)$;
(ii) $Rank(B(t)) = r$ and $Rank(C(t)) = m$ for all $t \in [0,\infty)$.

Without confusion, we use $\alpha$, $\beta$, $\gamma$ and $\rho$ for $\alpha^c$, $\beta^c$, $\gamma^c$ and $\rho^c$ in Section 3, and for $\alpha^o$, $\beta^o$, $\gamma^o$ and $\rho^o$ in Section 4, respectively. Sometimes we also drop the parameters in the parentheses for simplicity.

## 3. CONTROLLABLE BLOCK COMPANION FORMS

The technique for controllable block companion canonical transformation is first presented in this section in terms of block controllability matrix (5a). The transformed system is described by

$$\dot{x}^c(t) = A^c(t)x^c(t) + B^c(t)u(t), \tag{7a}$$
$$y^c(t) = C^c(t)x^c(t), \tag{7b}$$

here $A^c(t)$, $B^c(t)$ have the following canonical structure

$$A^c(t) = \begin{pmatrix} 0_{\beta,\beta} & 0_{\beta,\gamma} & I_\beta & 0_{\beta,\gamma} & 0_{\beta,\beta} & \dots & 0_{\beta,\gamma} & 0_{\beta,\beta} \\ 0_{\gamma,\beta} & 0_{\gamma,\gamma} & 0_{\gamma,\beta} & I_\gamma & 0_{\gamma,\beta} & \dots & 0_{\gamma,\gamma} & 0_{\gamma,\beta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0_{\beta,\beta} & 0_{\beta,\gamma} & 0_{\beta,\beta} & 0_{\beta,\gamma} & 0_{\beta,\beta} & \dots & 0_{\beta,\gamma} & I_\beta \\ A^c_{1,1} & A^c_{1,2} & A^c_{1,3} & A^c_{1,4} & A^c_{1,5} & \dots & A^c_{1,\rho} & A^c_{1,\rho+1} \\ A^c_{2,1} & A^c_{2,2} & A^c_{2,3} & A^c_{2,4} & A^c_{2,5} & \dots & A^c_{2,\rho} & A^c_{2,\rho+1} \end{pmatrix}, \tag{7c}$$

$$B^c = \begin{pmatrix} 0_{\beta,\gamma} & 0_{\beta,\beta} \\ 0_{\gamma,\gamma} & 0_{\gamma,\beta} \\ \vdots & \vdots \\ 0_{\beta,\gamma} & 0_{\beta,\beta} \\ I_\gamma & 0_{\gamma,\beta} \\ 0_{\beta,\gamma} & I_\beta \end{pmatrix}, \tag{7d}$$

where $\rho = 2\alpha$, $0_{p,q}$ is the $p$ by $q$ zero matrix, $I_p$ is the $p$th order identity matrix, and

$$A^c_{1,i} \in \mathbf{R}^{\gamma\times\beta}, \qquad A^c_{2,i} \in \mathbf{R}^{\beta\times\beta}, \qquad \text{for } i \le \rho+1 \text{ odd},$$
$$A^c_{1,i} \in \mathbf{R}^{\gamma\times\gamma}, \qquad A^c_{2,i} \in \mathbf{R}^{\beta\times\gamma}, \qquad \text{for } i \le \rho+1 \text{ even},$$

are time-varying submatrices. The transformation of $x(t)$ into $x^c(t)$ is defined by a non-singular matrix

$$x^c(t) = N^c(t)x(t). \tag{8}$$

Substituting (8) into (1) and comparing with (7), results in the following matrix equations

$$A^c(t) = N^c(t)A(t)(N^c(t))^{-1} + \dot{N}^c(t)(N^c(t))^{-1}, \tag{9a}$$
$$B^c(t) = N^c(t)B(t), \tag{9b}$$

and

$$C^c(t) = C(t)(N^c(t))^{-1}. \tag{9c}$$

Then the determination of $N^c(t)$ can be stated as the following theorem.

**THEOREM 1.** If system (1) is block controllable, then there exists a non-singular transformation matrix $N^c(t)$

$$N^c = ((N^c_1)^T\ (N^c_2)^T\ \dots\ (N^c_{\rho-1})^T\ (N^c_\rho)^T\ (N^c_{\rho+1})^T)^T \tag{10}$$

where

$$N^c_i \in \mathbf{R}^{\beta\times n}, \qquad \text{for } i \le \rho+1 \text{ odd},$$
$$N^c_i \in \mathbf{R}^{\gamma\times n}, \qquad \text{for } i < \rho+1 \text{ even}.$$

Determined by

$$N^c_1 = (0_{\beta,\gamma}\ 0_{\beta,\beta}\ \dots\ 0_{\beta,\beta}\ 0_{\beta,\gamma}\ I_\beta)(W^c)^{-1}, \tag{11a}$$

$$N_2^c = (0_{\gamma,\gamma}\ 0_{\gamma,\beta}\ \dots\ I_\gamma\ 0_{\gamma,\beta}\ A_{1,\rho+1}^c)(W^c)^{-1}, \tag{11b}$$

$$N_i^c = \mathbf{L}[N_{i-2}^c] \qquad \text{for } 3 \le i \le \rho+1, \tag{11c}$$

which transforms system (1) into equivalent controllable block companion canonical structure (7).

PROOF. The proof is divided into two parts. First we need to show that matrix $N$ expressed by (11) is the matrix which transforms (1) into canonical structure (7). In accordance with Definition 2, the block controllability matrix can be written as

$$W_1^c = B_1, \qquad W_2^c = B_2, \tag{12a}$$

and

$$W_i^c = W_{i-2}^c A - \dot{W}_{i-2}^c; \qquad \text{for } 3 \le i \le \rho+1 \text{ odd}, \tag{12b}$$
$$W_i^c = W_{i-2}^c A - \dot{W}_{i-2}^c; \qquad \text{for } 4 \le i \le \rho+2 \text{ even}. \tag{12c}$$

Postmultiply $N^c$ on both sides of (9a), and then substitute (10) and (7c)–(7d) into (9a)–(9b), respectively, which yield (11c)–(11d) and

$$N_\rho^c A + \dot{N}_\rho^c = \sum_{i=1}^{\rho+1} A_{1,i} N_i^c, \tag{13a}$$

$$N_{\rho+1}^c A + \dot{N}_{\rho+1}^c = \sum_{i=1}^{\rho+1} A_{2,i} N_i^c, \tag{13b}$$

$$\begin{aligned}
&N_1^c B_1 = 0_{\beta,\gamma}; && N_1^c B_2 = 0_{\beta,\beta},\\
&N_2^c B_1 = 0_{\gamma,\gamma}; && N_2^c B_2 = 0_{\gamma,\beta},\\
&N_3^c B_1 = 0_{\beta,\gamma}; && N_3^c B_2 = 0_{\beta,\beta},\\
&N_4^c B_1 = 0_{\gamma,\gamma}; && N_4^c B_2 = 0_{\gamma,\beta},\\
&\vdots && \vdots\\
&N_{\rho-1}^c B_1 = 0_{\beta,\gamma}; && N_{\rho-1}^c B_2 = 0_{\beta,\beta},\\
&N_\rho^c B_1 = I_\gamma; && N_\rho^c B_2 = 0_{\gamma,\beta},\\
&N_{\rho+1}^c B_1 = 0_{\beta,\gamma}; && N_{\rho+1}^c B_2 = I_\beta.
\end{aligned} \tag{13c}$$

Postmultiply (13a) by $B_1$ and $B_2$, respectively, and consider (13c) we get

$$\begin{aligned}
A_{1,\rho} =& [N_\rho^c A + \dot{N}_\rho^c] B_1,\\
&= N_\rho^c [A B_1 - \dot{B}_1] + [\dot{N}_\rho^c B_1 + N_\rho^c \dot{B}_1]\\
&= N_\rho^c W_3^c,
\end{aligned} \tag{14a}$$

$$\begin{aligned}
A_{1,\rho+1} =& [N_\rho^c A + \dot{N}_\rho^c] B_2,\\
&= N_\rho^c [A B_2 - \dot{B}_2] + [\dot{N}_\rho^c B_2 + N_\rho^c \dot{B}_2]\\
&= N_\rho^c W_4^c.
\end{aligned} \tag{14b}$$

Substituting (12) and (11c)–(11d) into (13c) and (14), after some manipulations we can obtain the following chain of equations

$$
\begin{array}{ll}
N_1^c B_1 = 0_{\beta,\gamma}; & N_2^c B_1 = 0_{\gamma,\gamma}, \\
N_1^c B_2 = 0_{\beta,\beta}; & N_2^c B_2 = 0_{\gamma,\beta}, \\
N_1^c W_3^c + \frac{d}{dt}[N_1^c B_1] = 0_{\beta,\gamma}; & N_2^c W_3^c + \frac{d}{dt}[N_2^c B_1] = 0_{\gamma,\gamma}, \\
N_1^c W_4^c + \frac{d}{dt}[N_1^c B_2] = 0_{\beta,\beta}; & N_2^c W_4^c + \frac{d}{dt}[N_2^c B_2] = 0_{\gamma,\beta}, \\
\vdots & \vdots \\
N_1^c W_{\rho-1}^c + \frac{d}{dt}[N_1^c W_{\rho-3}^c] = 0_{\beta,\gamma}; & N_2^c W_{\rho-1}^c + \frac{d}{dt}[N_2^c W_{\rho-3}^c] = I_\gamma, \\
N_1^c W_{\rho}^c + \frac{d}{dt}[N_1^c W_{\rho-2}^c] = 0_{\beta,\beta}; & N_2^c W_{\rho}^c + \frac{d}{dt}[N_2^c W_{\rho-2}^c] = 0_{\gamma,\beta}, \\
N_1^c W_{\rho+1}^c + \frac{d}{dt}[N_1^c W_{\rho-1}^c] = 0_{\beta,\gamma}; & N_2^c W_{\rho+1}^c + \frac{d}{dt}[N_2^c W_{\rho-1}^c] = A_{1,\rho}, \\
N_1^c W_{\rho+2}^c + \frac{d}{dt}[N_1^c W_{\rho}^c] = I_\beta; & N_2^c W_{\rho+2}^c + \frac{d}{dt}[N_2^c W_{\rho}^c] = A_{1,\rho+1}.
\end{array}
\tag{15}
$$

Considering the property of a block controllable system, it can be seen that $W_{\rho+1}^c$ is linearly dependent on $W_j^c$, for $j = 1, \ldots, \rho$. Therefore the equations with $W_{\rho+1}^c$ in (15) can be omitted. Solving (15) from top to bottom, recursively, it follows that

$$
\begin{array}{ll}
N_1^c W_1^c = 0_{\beta,\gamma}; & N_2^c W_1^c = 0_{\gamma,\gamma}, \\
N_1^c W_2^c = 0_{\beta,\beta}; & N_2^c W_2^c = 0_{\gamma,\beta}, \\
N_1^c W_3^c = 0_{\beta,\gamma}; & N_2^c W_3^c = 0_{\gamma,\gamma}, \\
N_1^c W_4^c = 0_{\beta,\beta}; & N_2^c W_4^c = 0_{\gamma,\beta}, \\
\vdots & \vdots \\
N_1^c W_{\rho-1}^c = 0_{\beta,\gamma}; & N_2^c W_{\rho-1}^c = I_\gamma, \\
N_1^c W_{\rho}^c = 0_{\beta,\beta}; & N_2^c W_{\rho}^c = 0_{\gamma,\beta}, \\
N_1^c W_{\rho+2}^c = I_\beta; & N_2^c W_{\rho+2}^c = A_{1,\rho+1}^c.
\end{array}
$$

Arrange above equations in vector form we obtain

$$
N_1^c W^c = (0_{\beta,\gamma}\ 0_{\beta,\beta}\ \ldots\ 0_{\beta,\beta}\ 0_{\beta,\gamma}\ I_\beta), \tag{16a}
$$

$$
N_2^c W^c = (0_{\gamma,\gamma}\ 0_{\gamma,\beta}\ \ldots\ I_\gamma\ 0_{\gamma,\beta}\ A_{1,\rho+1}^c). \tag{16b}
$$

Since (1) is block controllable, for an arbitrarily assigned submatrix $A_{1,\rho+1}^c$, $N_1^c$ and $N_2^c$ are uniquely determined by (11a) and (11b).

Second we need to show that $(N^c)^{-1}$ always exists. Similarly to the method of [3], let

$$
P = N^c W^c = \begin{pmatrix}
N_1^c W_1^c & N_1^c W_2^c & \ldots & N_1^c W_{\rho-1}^c & N_1^c W_\rho^c & N_1^c W_{\rho+2}^c \\
N_2^c W_1^c & N_2^c W_2^c & \ldots & N_2^c W_{\rho-1}^c & N_2^c W_\rho^c & N_2^c W_{\rho+2}^c \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
N_{\rho-1}^c W_1^c & N_{\rho-1}^c W_2^c & \ldots & N_{\rho-1}^c W_{\rho-1}^c & N_{\rho-1}^c W_\rho^c & N_{\rho-1}^c W_{\rho+2}^c \\
N_\rho^c W_1^c & N_\rho^c W_2^c & \ldots & N_\rho^c W_\rho^c & N_\rho^c W_\rho^c & N_\rho^c W_{\rho+2}^c \\
N_{\rho+1}^c W_1^c & N_{\rho+1}^c W_2^c & \ldots & N_{\rho+1}^c W_\rho^c & N_{\rho+1}^c W_\rho^c & N_{\rho+1}^c W_{\rho+2}^c
\end{pmatrix},
$$

From the property of block controllability and Equation (12) and (15)–(16), it can be easily shown that the $n \times n$ matrix $P$ has the following structure

$$
P = \begin{pmatrix}
0_{\beta,\gamma} & 0_\beta & 0_{\beta,\gamma} & 0_\beta & \ldots & 0_{\beta,\gamma} & 0_\beta & I_\beta \\
0_\gamma & 0_{\gamma,\beta} & 0_\gamma & 0_{\gamma,\beta} & \ldots & I_\gamma & 0_{\gamma,\beta} & \chi \\
0_{\beta,\gamma} & 0_\beta & 0_{\beta,\gamma} & 0_\beta & \ldots & 0_{\beta,\gamma} & I_\beta & \chi \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0_\gamma & 0_{\gamma,\beta} & I_\gamma & 0_{\gamma,\beta} & \ldots & \chi & \chi & \chi \\
0_{\beta,\gamma} & 0_\beta & 0_{\beta,\gamma} & I_\beta & \ldots & \chi & \chi & \chi \\
I_\gamma & 0_{\gamma,\beta} & \chi & \chi & \ldots & \chi & \chi & \chi \\
0_{\beta,\gamma} & I_\beta & \chi & \chi & \ldots & \chi & \chi & \chi
\end{pmatrix},
$$

where the $\chi$ entries in $P$ matrix are time-varying possibly nonzero block submatrices with compatible dimensions. From the structure of $P$, it is clear that the determinant value of $P$ is $\pm 1$, for all $t \in [0, \infty)$. Hence

$$(N^c)^{-1} = W^c P^{-1},$$

exists for all $t \in [0, \infty)$. ∎

Summarize the result of this section, the algorithms of controllable block companion canonical transformations for time-varying block controllable systems is given below.

ALGORITHM 1.

*Step 1:* Find the block controllable matrix $W^c(t, \alpha^c)$ as (5b).
*Step 2:* Find the transformation matrix by:
(i) Partition matrix $N^c(t)$ as (10);
(ii) Compute $N_i^c(t)$, for $i = 1, \ldots, \rho + 1$, by (11).
*Step 3:* Compute $(N^c(t))^{-1}$ and $\dot{N}^c(t)$.
*Step 4:* Transform original system (1) into controllable block companion canonical form (7) by (9).

## 4. OBSERVABLE BLOCK COMPANION CANONICAL FORMS

If system (1) is block observable, the argument of Section 3 can be easily extended to the transformation into observable block companion canonical form which can be described by

$$\dot{x}^o(t) = A^o(t)x^o(t) + B^o(t)u(t), \tag{17a}$$
$$y(t) = C^o(t)x^o(t), \tag{17b}$$

where

$$x^o(t) = (N^o(t))^{-1}x(t), \tag{17c}$$

$$A^o(t) = \begin{pmatrix} 0_{\beta,\beta} & 0_{\beta,\gamma} & 0_{\beta,\beta} & \ldots & 0_{\beta,\gamma} & 0_{\beta,\beta} & A^o_{1,1} & A^o_{1,2} \\ 0_{\gamma,\beta} & 0_{\gamma,\gamma} & 0_{\gamma,\beta} & \ldots & 0_{\gamma,\gamma} & 0_{\gamma,\beta} & A^o_{2,1} & A^o_{2,2} \\ I_\beta & 0_{\beta,\gamma} & 0_{\beta,\beta} & \ldots & 0_{\beta,\gamma} & 0_{\beta,\beta} & A^o_{3,1} & A^o_{3,2} \\ 0_{\gamma,\beta} & I_\gamma & 0_{\gamma,\beta} & \ldots & 0_{\gamma,\gamma} & 0_{\gamma,\beta} & A^o_{4,1} & A^o_{4,2} \\ 0_{\beta,\beta} & 0_{\beta,\gamma} & I_\beta & \ldots & 0_{\beta,\gamma} & 0_{\beta,\beta} & A^o_{5,1} & A^o_{5,2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0_{\gamma,\beta} & 0_{\gamma,\gamma} & 0_{\gamma,\beta} & \ldots & I_{\gamma,\gamma} & 0_{\gamma,\beta} & A^o_{\rho,1} & A^o_{\rho,2} \\ 0_{\beta,\beta} & 0_{\beta,\gamma} & 0_{\beta,\beta} & \ldots & 0_{\beta,\gamma} & I_\beta & A^o_{\rho+1,1} & A^o_{\rho+1,2} \end{pmatrix}, \tag{17d}$$

$$C^o = \begin{pmatrix} 0_{\gamma,\beta} & 0_{\gamma,\gamma} & 0_{\gamma,\beta} & 0_{\gamma,\gamma} & \ldots & 0_{\gamma,\beta} & I_\gamma & 0_{\gamma,\beta} \\ 0_{\beta,\beta} & 0_{\beta,\gamma} & 0_{\beta,\beta} & 0_{\beta,\gamma} & \ldots & 0_{\beta,\beta} & 0_{\beta,\gamma} & I_\beta \end{pmatrix}, \tag{17e}$$

$\rho = 2\alpha^o$, and

$$A^o_{i,1} \in \mathbf{R}^{\beta\times\gamma}, \quad A^o_{i,2} \in \mathbf{R}^{\beta\times\beta}, \quad \text{for } i \le \rho + 1 \text{ odd},$$

$$A^o_{i,1} \in \mathbf{R}^{\gamma\times\gamma}, \quad A^o_{i,2} \in \mathbf{R}^{\gamma\times\beta}, \quad \text{for } i \le \rho + 1 \text{ even},$$

are time-varying submatrices. The matrices of original system (1) and observable canonical system (16) have the following relations

$$A^o(t) = (N^o(t))^{-1}A(t)N^o(t) - (N^o(t))^{-1}\dot{N}^o(t), \tag{18a}$$
$$C^o(t) = C(t)N^o(t), \tag{18b}$$

and

$$B^o(t) = (N^o(t))^{-1}B(t). \tag{18c}$$

The determination of non-singular transformation $N^o(t)$ can be stated as the following theorem.

THEOREM 2. If system (1) is block observable, then there exists a non-singular transformation matrix $N^o(t)$

$$N^o = (N_1^o \; N_2^o \; \dots \; N_{\rho-1}^o \; N_\rho^o \; N_{\rho+1}^o) \tag{19}$$

where

$$N_i^o \in \mathbf{R}^{n\times\beta}, \qquad \text{for } i \le \rho+1 \text{ odd},$$
$$N_i^o \in \mathbf{R}^{n\times\gamma}, \qquad \text{for } i < \rho+1 \text{ even},$$

determined by

$$N_1^o = (W^o)^{-1} \begin{pmatrix} 0_{\gamma,\beta} \\ 0_{\beta,\beta} \\ \vdots \\ 0_{\beta,\beta} \\ 0_{\gamma,\beta} \\ I_\beta \end{pmatrix}, \tag{20a}$$

$$N_2^o = (W^o)^{-1} \begin{pmatrix} 0_{\gamma,\gamma} \\ 0_{\beta,\gamma} \\ \vdots \\ I_\gamma \\ 0_{\beta,\gamma} \\ A_{\rho+1,1}^o \end{pmatrix}, \tag{20b}$$

and

$$N_i^o = \mathbf{J}[N_{i-2}^c] \qquad \text{for } 3 \le i \le \rho+1, \tag{20c}$$

which transforms system (1) into equivalent observable block companion canonical structure (16).

PROOF. The proof is similar to that of Theorem 1. Premultiply $N^o$ on both sides of (17a) and then transpose both sides of (17a) and (17b) we can get

$$(A^o)^T(N^o)^T = (N^o)^T A^T - (\dot{N}^o)^T, \tag{21a}$$
$$(C^o)^T = (N^o)^T C^T, \tag{21b}$$

Then the argument in this theorem is transferred to the argument of Theorem 1. By the same procedure, the non-singular transformation matrix then is obtained as (20). ∎

The algorithm of observable block companion canonical transformations for time-varying block observable systems is similarly given below.

ALGORITHM 2.

*Step 1:* Find the block observable matrix $W^o(t, \alpha^o)$ as (6b).
*Step 2:* Find the transformation matrix by:
(i) Partition matrix $N^o(t)$ as (19);
(ii)Compute $N_i^o(t)$, for $i = 1, \dots, \rho+1$, by (20).
*Step 3:* Compute $(N^o(t))^{-1}$ and $\dot{N}^o(t)$.
*Step 4:* Transform original system (1) into observable block companion canonical form (16) by (17).

**Remark 1.** To use Algorithm 1 or Algorithm 2 only requires that the system (1) be block controllable (observable). However, for special case that if system (1) is block controllable (observable) and $q$ is an integer, $\beta = 0$ and $\gamma = r(m)$, the algorithm reduces to that of [3]. In this way, the canonical transformation technique of [3] is generalized.

**Remark 2.** The transformation procedure is characterized by an arbitrarily assigned submatrix $A_{1,\rho+1}^c$ $(A_{\rho+1,1}^o)$. This implies that the transformations are not unique. For a special case that $\bar{n}/r(m)$ is an integer, the free submatrix $A_{1,\rho+1}^c$ $(A_{\rho+1,1}^o)$ disappears and the transformation will be unique. More discussions on the non-uniqueness and the use of this property were given by Fahmy and O'Reilly [18].

## 5. EXAMPLE

Transform

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & e^{-t} & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u(t),$$

$$y(t) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} x(t),$$

into: i) Controllable block companion canonical form, ii) Observable block companion canonical form.

i) Controllable block companion canonical form

*Step 1.* The block controllable matrix is obtained as

$$W^c = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and

$$(W^c)^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

*Step 2.* By (11) and specify $A_{5,1}(k) = 1$, we get

$$N^c(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & e^{-t} & e^{-t} & 0 & 0 \end{pmatrix},$$

$$(N^c(t))^{-1} = \begin{pmatrix} 0 & -e^{-t} & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

*Step 3.* By (9), it is computed

$$A^c(t) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 \\ -1 & -e^{-t} & 0 & e^{-t} & 0 \end{pmatrix},$$

$$C^c(t) = \begin{pmatrix} -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

ii) Observable block companion canonical form

*Step 1.* The block observability matrix is obtained as

$$W^o(t) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1+e^{-t} & e^{-t} & 0 & -1 \end{pmatrix},$$

and

$$(W^o(t))^{-1} = \begin{pmatrix} 1 & -1-e^{-t} & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

*Step 2.* By specify $A_{5,1}(k) = 0$, we can get

$$N^o = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix},$$

$$(N^o)^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

*Step 3.* Then

$$A^o(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1+e^{-t} \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B^o(t) = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & -1 & 0 \end{pmatrix}^T. \quad \blacksquare$$

## 6. CONCLUSIONS

This paper presents a method for block companion canonical transformation for linear time-varying continuous systems. Without requiring $q$ to be an integer, the technique developed here can be applied to all block controllable (observable) linear time-varying systems, and thus generalized the results of [3]. For the case of non-integer $q$, the parameter in canonical forms are not unique and depend on the choice of $A^c_{1,\rho+1}$ $(A^o_{\rho+1,1})$. The canonical transformation developed in this paper can be applied to the state estimator and eigenvalue assignments for linear time-varying continuous system. For further research, it is also expected to develop the block companion canonical transformation techniques for the class of systems which are not block controllable (observable) or have time-varying controllability (observability) indices. These aspects are under investigation and will be reported in other papers.

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