Applied Mathematics Letters 24 (2011) 1527–1532

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Application of Laguerre matrix polynomials to the numerical inversion of Laplace transforms of matrix functions☆

J. Sastre [a,*], E. Defez [b], L. Jódar [b]

[a] Instituto de Telecomunicaciones y Aplicaciones Multimedia, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain

[b] Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain

ARTICLE INFO

Article history:
Received 21 May 2010
Received in revised form 23 March 2011
Accepted 23 March 2011

Keywords:
Laguerre matrix polynomial
Series expansion
Inverse Laplace transform

ABSTRACT

This paper presents an application of Laguerre matrix polynomial series to the numerical inversion of Laplace transforms of matrix functions.

1. Introduction and notation

The use of the Laplace transform to calculate the solution of different types of problems is very well known. In these problems it is necessary to calculate the Laplace inverse transform of certain functions. This inverse can be difficult to compute using techniques of complex analysis, and there exist numerous numerical methods for its evaluation [1,2]. Laguerre scalar polynomials $L_n^{\alpha}(t)$ have been used for the Laplace inverse transform of scalar functions [1,2, p. 11–12]. In particular, Davies and Martin in their review and comparison of methods indicate that the method based on Laguerre polynomials offers exceptional precision for a wide range of functions [2, p. 30]. In the solution of matrix problems using the method of the Laplace transform it is equally necessary to calculate the inverse Laplace transform of certain matrix functions. The common method for matrix problems is to try to decouple the system to scalar equations. However, in many circumstances this is impossible or it increases the computational cost.

In this paper, the application of Laguerre matrix polynomial series to the numerical inversion of the Laplace transform of matrix functions is studied. Laguerre matrix polynomials were introduced in [3] and theorems for the expansion of matrix functions in series of Laguerre matrix polynomials have been given in [4,5] recently.

This paper is organized as follows. Section 2 deals with notation and some preliminary results on Laguerre matrix polynomials. Section 3 deals with the numerical inversion of Laplace transforms using Laguerre matrix polynomials. Section 4 provides an illustrative example from circuit theory. Conclusions are given in Section 5.

2. Preliminaries

Throughout this paper for a complex number z, $\Re(z)$ denotes its real part. For a matrix A in $\mathbb{C}^{r\times r}$ its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A and $\beta(A) = \min\{\Re(z);\ z \in \sigma(A)\}$. If $f(z)$ and $g(z)$ are holomorphic functions of

☆ This work has been supported by the Programa de Apoyo a la Investigación y el Desarrollo of the Universitat Politècnica de València grant PAID-05-09-4338 and the Spanish Ministerio de Educación grant MTM2009-08587.

* Corresponding author. Tel.: +34 963879719; fax: +34 963877309.
E-mail addresses: jorsasma@iteam.upv.es (J. Sastre), edefez@imm.upv.es (E. Defez), ljodar@mat.upv.es (L. Jódar).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.03.039

the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus [6, p. 558], it follows that $f(A)\,g(A)=g(A)\,f(A)$. The reciprocal gamma function denoted by $\Gamma^{-1}(z)=1/\Gamma(z)$ is an entire function of the complex variable z. Then, the image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Furthermore, if $A+nI$ is invertible for every integer $n\geq 0$, where I is the identity matrix in $\mathbb{C}^{r\times r}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$, and by [7, p. 253] it follows that $A\,(A+I)\,\cdots\,(A+(n-1)I)=\Gamma(A+nI)\,\Gamma^{-1}(A)$, $n\geq 1$.

We recall some properties of Laguerre matrix polynomials which have been stated in [3]. If λ is a complex number with $\Re(\lambda)>0$ and A is a matrix in $\mathbb{C}^{r\times r}$ with $A+nI$ invertible for every integer $n\geq 1$, the nth Laguerre matrix polynomial $L_n^{(A,\lambda)}(x)$ is defined by [3, p. 58]

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!\,(n-k)!}\,(A+I)_n\,[(A+I)_k]^{-1}\,(\lambda x)^k, \tag{2.1}$$

where $(A+I)_n=(A+I)\,(A+2I)\,\cdots\,(A+nI)$, $n\geq 1$, $(A+I)_0=I$. Note that the parameter λ is a scaling of variable x. Hence from (2.1), one gets

$$L_n^{(A,\lambda)}(x)=L_n^{(A,1)}\,(\lambda\,x)\,, \tag{2.2}$$

and from (4.17.2) [8, p. 77] it is easy to show that Laguerre scalar polynomials $L_n^{\alpha}(x)$ are a particular case of Laguerre matrix polynomials where

$$L_n^{\alpha}(x)=L_n^{(A,1)}(x),\quad A=\alpha. \tag{2.3}$$

The generating function of Laguerre matrix polynomials is given in [3, p. 57]

$$G(x,t,\lambda,A)=(1-t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1-t}\right)=\sum_{n\geq 0}L_n^{(A,\lambda)}(x)t^n,\quad t\in\mathbb{C},\ |t|<1,\ x\in\mathbb{C}, \tag{2.4}$$

and the next Rodrigues' formula is satisfied [3]:

$$L_n^{(A,\lambda)}(x)=\frac{x^{-A}\mathrm{e}^{\lambda x}}{n!}\frac{d^n}{dx^n}\left[\mathrm{e}^{-\lambda x}x^{A+nI}\right],\quad n\geq 0. \tag{2.5}$$

Note that from (2.1) one gets $L_0^{(A,\lambda)}(x)=I$ and in the following we take $L_{-1}^{(A,\lambda)}(x)=0$. Hence, Laguerre matrix polynomials satisfy the three-term recurrence relation (3.10) of [3, p. 59] for $n\geq 0$ (see Theorem 3.1 [3, p. 60]), resulting

$$(n+1)L_{n+1}^{(A,\lambda)}(x)+[\lambda xI-(A+(2n+1)I)]L_n^{(A,\lambda)}(x)+(A+nI)L_{n-1}^{(A,\lambda)}(x)=0,\quad n\geq 0, \tag{2.6}$$

which will be useful to calculate the nth polynomial in terms of the polynomials of order $n-1$ and $n-2$.

3. Application to the inverse Laplace transform of matrix functions

To develop the proposed method we start off with the Laplace transform $F(s)$ of an unknown $\mathbb{C}^{r\times m}$-valued matrix function $f(t)$, which we want to approximate. In the following, we will assume that such function $f(t)$ formally admits the following left-series expansion of Laguerre matrix polynomials [4]:

$$f(t)=t^A\mathrm{e}^{-ct}\sum_{k\geq 0}L_k^{(A,\lambda)}(t)k!\,\Gamma^{-1}(A+(k+1)I)\,b_k. \tag{3.1}$$

Here $c,\lambda>0$ and $A\in\mathbb{C}^{r\times r}$ are parameters, $b_k\in\mathbb{C}^{r\times m}$, and $\beta(A)>-1/2$ (see Theorem 5.1 [4, p. 1032]). In an analogous way, the right-series expansion of Laguerre matrix polynomials could have been chosen [4, p. 1032]. Taking into account (3.1) yields

$$t^{-A}\mathrm{e}^{ct}f(t)=\sum_{k\geq 0}L_k^{(A,\lambda)}(t)k!\,\Gamma^{-1}(A+(k+1)I)\,b_k, \tag{3.2}$$

and using Laguerre matrix polynomial series expansion theorem (Theorem 5.1 [4, p. 1032]) one can conclude that the coefficients b_k take the following form

$$b_k=\lambda^{A+I}\int_0^{\infty}\mathrm{e}^{-(\lambda-c)t}L_k^{(A,\lambda)}(t)\,f(t)\mathrm{d}t. \tag{3.3}$$

These coefficients are unknown because $f(t)$ is unknown. In order to determine them, we use $f(t)$ Laplace transform, i.e. $F(s)$, taking $s=\sigma+i\omega$. Then, using (3.1), one gets

$$F(s)=\int_0^{\infty}f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t=\int_0^{\infty}t^A\mathrm{e}^{-(s+c)t}\sum_{k\geq 0}L_k^{(A,\lambda)}(t)k!\,\Gamma^{-1}(A+(k+1)I)\,b_k\,\mathrm{d}t. \tag{3.4}$$

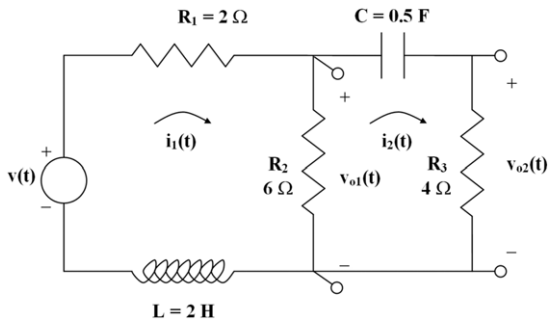

Fig. 1. Electrical circuit.

Taking into account (2.2) with the additional assumption that the series summation and integration can be interchanged, yields

$$F(s) = \sum_{k\geq 0}\left[\int_0^\infty t^A \mathrm{e}^{-(s+c)t} L_k^{(A,1)}(\lambda t)\,\mathrm{d}t\right] k!\,\Gamma^{-1}(A+(k+1)I)\,b_k.$$

Taking $u = \lambda t$ in the last expression, it follows that

$$F(s) = \sum_{k\geq 0}\lambda^{-A-I}\left[\int_0^\infty u^A \mathrm{e}^{-(s+c)u/\lambda} L_k^{(A,1)}(u)\,\mathrm{d}u\right] k!\,\Gamma^{-1}(A+(k+1)I)\,b_k. \tag{3.5}$$

Considering (2.5) and expression 7.414 (8) [9, p. 845], which states that

$$\int_0^\infty \mathrm{e}^{-pt}\,t^\alpha\,L_n^\alpha(t)\,\mathrm{d}t = \frac{\Gamma(\alpha+n+1)(p-1)^n}{n!\,p^{\alpha+n+1}}, \quad \Re(\alpha) > -1,\ \Re(p) > 0, \tag{3.6}$$

using (2.3) and taking into account that the right-hand side of (3.6) is a holomorphic function of α, by the matrix functional calculus [6] acting on α it follows that

$$\int_0^\infty \mathrm{e}^{-p\,t}\,t^A\,L_n^{(A,1)}(t)\,\mathrm{d}t = \frac{\Gamma(A+(n+1)I)(p-1)^n}{n!}\,p^{-A-(n+1)I}, \quad \beta(A) > -1,\ \Re(p) > 0. \tag{3.7}$$

Using (3.5) and (3.7), one obtains $F(s) = \sum_{k\geq 0}\left(\frac{s+c-\lambda}{s+c}\right)^k (s+c)^{-A-I}\,b_k$. Finally, after using the bilinear transformation $z = (s+c-\lambda)/(s+c)$ on the previous expression, the following approximation can be made

$$\Phi(z) = (s+c)^{A+I}F(s) = \sum_{k\geq 0} b_k z^k \approx \sum_{k=0}^{N-1} b_k z^k, \tag{3.8}$$

where the truncation error is supposed to be sufficiently small. This bilinear transformation projects the line $s = \sigma + iw$, $-\infty < \omega < \infty$ onto the unit circle and the right semi-plane delimited by the line to the interior of the unit circle. Thus, $\Phi(z)$ will be a periodic function on $z = \mathrm{e}^{i\theta}$ with period 2π, and clearly from the bilinear transformation one gets

$$s = \sigma + i\omega = \frac{\lambda}{1-\mathrm{e}^{i\theta}} - c = \frac{\lambda}{2} - c + \frac{i\lambda}{2}\cot(\theta/2). \tag{3.9}$$

Since $F(s)$ is known, the matrix coefficients b_k can be approximated from (3.8) employing trigonometric interpolation, in combination with the efficient Fast Fourier Transform (FFT) algorithm, see [2, p. 12, 29], [10, p. 246], and once these matrix coefficients are obtained, the unknown matrix function $f(t)$ can be approximated using series (3.1).

4. An example from circuit theory

Consider the electrical circuit of Fig. 1, formed with a source of $v(t)$ volts (V), three resistors of R_1, R_2 and R_3 Ohms (Ω), one inductance of L Henries (H) and a capacitor of C Farads (F). We want to approximate the voltage outputs $v_{o1}(t)$ and $v_{o2}(t)$. The circuit current intensities $i_1(t)$ and $i_2(t)$ in Amperes (A) satisfy [11]

$$\left.\begin{aligned} &D\,I'(t) + B\,I(t) = E\,v(t),\\ &D = \begin{pmatrix} L & 0\\ 0 & R_2+R_3\end{pmatrix}, \quad B = \begin{pmatrix} R_1+R_2 & -R_2\\ -\dfrac{(R_1+R_2)R_2}{L} & \dfrac{1}{C}-\dfrac{R_2^2}{L}\end{pmatrix}, \quad I(t) = \begin{pmatrix} i_1(t)\\ i_2(t)\end{pmatrix}, \quad E = \begin{pmatrix} 1\\ \dfrac{R_2}{L}\end{pmatrix}\end{aligned}\right\} \tag{4.1}$$

and then one can obtain the output voltages as $v_{o1}(t) = R_2\,(i_1(t) - i_2(t))$, $v_{o2}(t) = R_3\,i_2(t)$. Thus we need to obtain both currents $i_1(t)$ and $i_2(t)$. Denoting the Laplace transforms of $I(t)$ and $v(t)$ as $\mathcal{L}\{I(t)\}$ and $\mathcal{L}\{v(t)\}$, with $s = \sigma + j\omega$, taking Laplace transforms in (4.1) one obtains

$$\mathcal{L}\{I(t)\} = (sD + B)^{-1}[E\,\mathcal{L}\{v(t)\} + DI(0+)], \tag{4.2}$$

where $I(0+)$ denotes $\lim_{t\longrightarrow 0+} I(t)$. Let the circuit elements be $R_1 = 2\,\Omega$, $R_2 = 6\,\Omega$, $R_3 = 4\,\Omega$, $L = 2\,\mathrm{H}$, $C = 0.5\,\mathrm{F}$, and let the voltage circuit source be switched on at $t = 0$ generating a voltage of 12 V. Thus $v(t) = 0$ for $t < 0$, $i_1(0+) = 0$, $i_2(0+) = 0$, and

$$\mathcal{L}\{I(t)\} = (sD + B)^{-1}\frac{12}{s}\,E = 12\,F(s)\,E, \tag{4.3}$$

where $F(s) = \frac{(sD+B)^{-1}}{s}$ is the Laplace transform of certain matrix function $f(t)$ which we want to approximate. Using the minimal theorem [6, p. 571], the exact solution $I(t)$ is given by:

$$I(t) = \begin{pmatrix} i_1(t) \\ i_2(t) \end{pmatrix} = \begin{pmatrix} -3.375\mathrm{e}^{-2t} + 1.875\mathrm{e}^{-0.4t} & +1.5 \\ -2.250\mathrm{e}^{-2t} + 2.250\mathrm{e}^{-0.4t} & \end{pmatrix}. \tag{4.4}$$

Assuming that $f(t)$ admits a series expansion of type (3.1) it follows that

$$I(t) = 12 f(t)\,E = 12\left[t^A \mathrm{e}^{-ct}\sum_{k\geq 0} L_k^{(A,\lambda)}(t) k!\,\Gamma^{-1}(A + (k+1)I)\,b_k\right]E, \tag{4.5}$$

where $c, \lambda > 0$ and $A \in \mathbb{C}^{r\times r}$ are parameters, and $\beta(A) > -1/2$. Then, taking $z = \mathrm{e}^{i\theta}$ in (3.8) and considering (3.9) one gets

$$\Phi\left(\mathrm{e}^{i\theta}\right) = (s + c)^{A+I}F(s) \approx \sum_{k=0}^{N-1} b_k \mathrm{e}^{ik\theta}, \tag{4.6}$$

where s is given by (3.9) and $\Phi(\mathrm{e}^{i\theta})$ is a periodic function of period 2π. If this function has a smooth variation, b_k coefficients can be approximated using the Fast Fourier Transform algorithm by, see section 3.9.5 [10, p. 246]

$$b_k \approx \frac{1}{N}\sum_{n=0}^{N-1}(s_n + c)^{A+I}\,F(s_n)\,\mathrm{e}^{\frac{-i2\pi kn}{N}}, \qquad s_n = \frac{\lambda}{2} - c + \frac{i\lambda}{2}\cot(\theta_n/2), \quad \theta_n = \frac{2\pi n}{N},\ n = 0, 1, \ldots, N-1. \tag{4.7}$$

Note that θ_n values can be selected to avoid the evaluation of $(s + c)^{A+I}\,F(s)$ in $s = \lambda/2 - c \pm i\infty$. For example, we can take

$$s_n = \frac{\lambda}{2} - c + \frac{i\lambda}{2}\cot(\theta_n/2), \qquad \theta_n = \frac{(2n+1)\pi}{N}, \quad n = 0, 1, \ldots, N-1. \tag{4.8}$$

In this case, taking into account the Fourier transform properties, it follows that

$$\mathcal{F}\{x(\theta)\} = \mathrm{e}^{-i2\pi f\theta_0}\,\mathcal{F}\{x(\theta + \theta_0)\}, \tag{4.9}$$

where $\mathcal{F}\{x(\theta)\}$ denotes the Fourier transform of $x(\theta)$ and therefore

$$b_k \approx \mathrm{e}^{\frac{-i\pi k}{N}}\frac{1}{N}\sum_{n=0}^{N-1}(s_n + c)^{A+I}\,F(s_n)\,\mathrm{e}^{\frac{-i2\pi kn}{N}}, \tag{4.10}$$

which can be evaluated using the Fast Fourier Transform algorithm, and then, the desired intensities can be approximated using (4.5). Let us approximate intensities $i_1(t)$ and $i_2(t)$ using $N = 2^5 = 32$, with

$$A = \begin{pmatrix} 2/3 & -1/3 \\ -2/3 & 1/3 \end{pmatrix}, \qquad \lambda = 12, \qquad c = 1, \tag{4.11}$$

and the values of s_n given by (4.8). All calculations have been performed with MATLAB mathematic package. Fig. 2(a) presents the real and imaginary parts of the component $\Phi_{1,1}(\mathrm{e}^{i\theta})$ of $\Phi(\mathrm{e}^{i\theta}) = (s + c)^{A+I}\,F(s)$ for that parameters. The rest of the components vary in a similar smooth way.

Fig. 3(b) presents the variation in absolute value of component $i = j = 1$ of coefficients b_k. It decreases for $k = 1, 2, \ldots, 16$, and remains almost constant for $k > 16$, probably due to rounding errors. The rest of the components vary in a similar way. That suggests us to use only the first 16 coefficients to obtain a better precision. Fig. 3(b) presents the error for both intensities in $t = 0.5$ with the above-mentioned parameters A, c and λ, varying the number of coefficients used in the series evaluation, from the total 32 obtained. The error has a minimum around 16 coefficients as expected. In the scalar case, Piessens and Branders [12] allow to use fewer terms than the total N obtaining better results than using all of them.

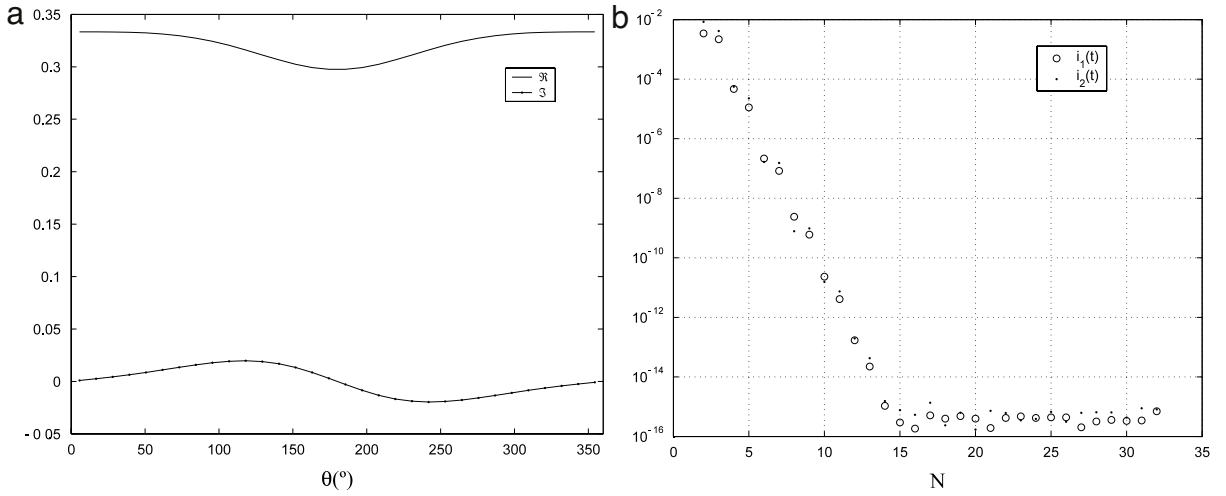

Fig. 2. (a) $\Phi_{11}(e^{i\theta})$. (b) Mean absolute error for intensities approximation in $t = 0, 0.1, \ldots, 0.5$, varying N.

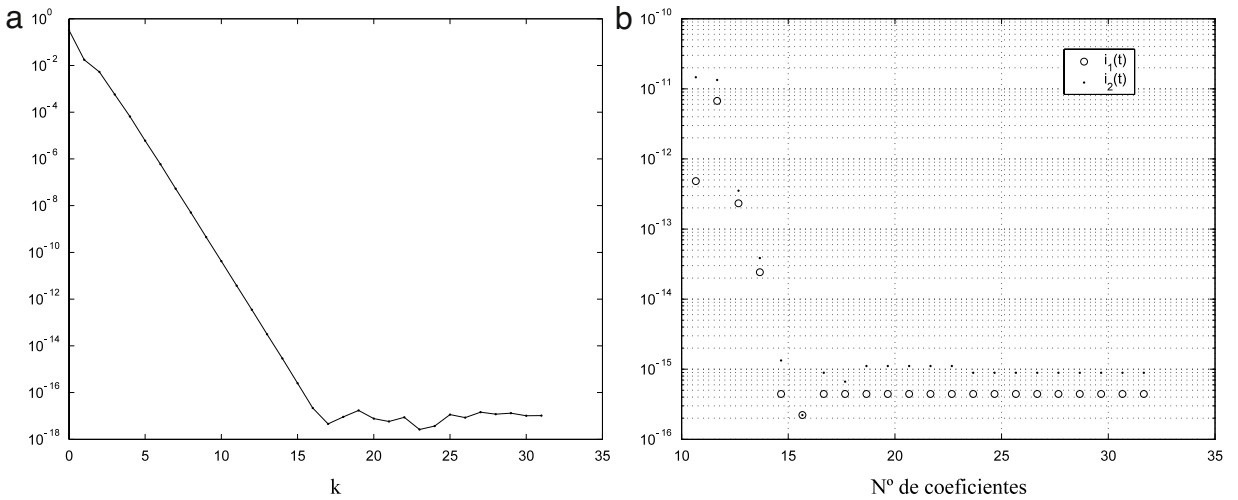

Fig. 3. (a) Absolute value of component $i = j = 1$ of b_k coefficients. (b) Mean absolute error for intensities approximation in $t = 0, 0.1, \ldots, 0.5$, varying the number of coefficients finally used in the series evaluation.

Table 1
Exact values of intensities $i_1(t)$ and $i_2(t)$ and their approximation error for $N = 6$ and $N = 14$.

			Error with Laguerre matrix polynomials			
t	$i_1(t)$	$i_2(t)$	$i_1(t)(N = 6)$	$i_2(t)(N = 6)$	$i_1(t)(N = 14)$	$i_2(t)(N = 14)$
0	0	0	0.1539e−6	0.3079e−6	0.8535e−15	0.1707e−14
0.1	0.5383	0.3196	0.4364e−7	0.1075e−6	0.1110e−14	0.1166e−14
0.2	0.9685	0.5688	0.2034e−6	0.1288e−6	0.6661e−15	0.5551e−15
0.3	1.3107	0.7607	0.4068e−6	0.1242e−6	0.1776e−14	0.1887e−14
0.4	1.5813	0.9063	0.3341e−6	0.1894e−6	0.2220e−15	0.8882e−15
0.5	1.7935	1.0144	0.1536e−6	0.1402e−6	0.1776e−14	0.3109e−14

Table 1 presents the exact values of intensities $i_1(t)$ and $i_2(t)$ for $t = 0, 0.1, \ldots, 0.5$ s, and the approximation error for Laguerre approximation with $N = 6$ and $N = 14$. In this case, the errors decrease from order 10^{-6} to order 10^{-14}. In all the computations we have eliminated the residual imaginary part of the approximated intensities, which is due to rounding errors in the evaluation of the Fast Fourier Transform. Note that once the coefficients b_k are obtained (4.5) provides a continuous approximated expression of the intensities depending on variable t. Fig. 2(b) presents the mean absolute error in the approximation of the intensities for the same values of variable t, varying N. It decreases quickly with N, and for $N > 16$ the error does not decrease any more probably due to the machine precision.

5. Conclusions

In this paper, an application of Laguerre matrix polynomial expansions to the inversion of Laplace transforms of matrix functions has been presented. The proposed method has the advantage that provides a continuous series which approximates the solution. Laguerre matrix polynomials involved in the series can be evaluated efficiently using the three-term recurrence relation (2.6) and, moreover, the evaluation of the coefficients b_k using the Fast Fourier Transform algorithm is efficient when using a number of points N of power 2.

References

[1] A.M. Cohen, Numerical Methods for Laplace Transform Inversion, Springer Verlag, 2007.
[2] B. Davies, B. Martin, Numerical inversion of Laplace transform: a survey and comparison of methods, J. Comput. Phys. 33 (1979) 1–32.
[3] L. Jódar, R. Company, E. Navarro, Laguerre matrix polynomials and systems of second order differential equations, Appl. Numer. Math. 15 (1994) 53–63.
[4] J. Sastre, E. Defez, L. Jódar, Laguerre matrix polynomials series expansion: theory and computer applications, Math. Comput. Model. 44 (2006) 1025–1043.
[5] J. Sastre, L. Jódar, On Laguerre matrix polynomials series, Util. Math. 71 (November 2006) 109–130.
[6] N. Dunford, J. Schwartz, Linear Operators, Part I, Interscience, New York, 1955.
[7] E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley, New York, 1969.
[8] N.N. Lebedev, Special Functions and their Applications, Dover, New York, 1972.
[9] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series and Products, Academic Press, Inc, 1980.
[10] P.J. Davis, Rabinowitz, Methods of Numerical Integration, Academic Press, 1984.
[11] A. Papoulis, M. Bertran, Digital and Analog Systems and Circuits, Marcombo, 1989.
[12] R. Piessens, M. Branders, Numerical inversion of the Laplace transform using generalized Laguerre polynomials, Proc. IEE 118 (1971) 1517–1522.