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Cycle-magic graphs ☆

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Abstract

A simple graph $G=(V,E)$ admits a cycle-covering if every edge in E belongs at least to one subgraph of G isomorphic to a given cycle C. Then the graph G is C-magic if there exists a total labelling $f: V\cup E\to\{1,2,\ldots,|V|+|E|\}$ such that, for every subgraph $H'=(V',E')$ of G isomorphic to C, $\sum_{v\in V'} f(v)+\sum_{e\in E'} f(e)$ is constant. When $f(V)=\{1,\ldots,|V|\}$, then G is said to be C-supermagic.

We study the *cyclic*-magic and *cyclic*-supermagic behavior of several classes of connected graphs. We give several families of C_r-magic graphs for each $r\geqslant 3$. The results rely on a technique of partitioning sets of integers with special properties.

Keywords: Edge coverings; Magic labelings

1. Introduction

Let $G=(V,E)$ be a finite simple graph. An edge-covering of G is a family of subgraphs $H_1,\ldots,H_k$ such that each edge of E belongs to at least one of the subgraphs H_i, $1\leqslant i\leqslant k$. Then it is said that G admits an $(H_1,\ldots,H_k)$-(edge)covering. If every H_i is isomorphic to a given graph H, then G admits an H-covering.

Suppose that $G=(V,E)$ admits an H-covering. A bijective function

$$f: V\cup E\to\{1,2,\ldots,|V|+|E|\},$$

is an *H-magic labelling* of G whenever, for every subgraph $H'=(V',E')$ of G isomorphic to H,

$$f(H')=\sum_{v\in V'} f(v)+\sum_{e\in E'} f(e)$$

is constant. In this case we say that the graph G is H-magic. If $f(V)=\{1,\ldots,|V|\}$, G is said to be H-supermagic. The constant value that every copy of H takes under the labelling f is denoted by $m(f)$ in the magic case and by $s(f)$ in the supermagic case. Fig. 1 shows an example of a C_3-supermagic labelling.

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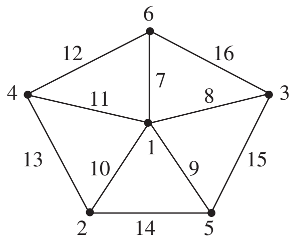

Fig. 1. C_3-supermagic covering.

The notion of H-magic graphs was introduced in [4] as an extension of the magic valuation given by Rosa [6] in 1967, see also [5], which corresponds to the case $H = K_2$. Supermagic labellings were treated in [2]. For these and other related labelling notions see the survey of Gallian [3].

When $H = K_2$ we say that a K_2-magic or supermagic graph is simply magic or supermagic. Many authors in this case use the terminology edge-magic or super edge-magic graph.

In this paper we study H-magic labellings when H is a cycle C_r. In this case we speak of *cycle-magic* labellings and *cycle-magic* graphs. A related notion of *face-magic* labellings of a planar graph G asks for a total labelling such that the sum over vertices and edges of each face of a planar embedding of G is constant; see, for instance, Baca [1]. When all faces have the same number r of edges, a C_r-magic labelling of G is also a face magic labelling of the graph.

The paper is organized as follows. In Section 3 we show that the wheel W_n with n odd is C_3-magic and that the Cartesian product of a C_4-free supermagic graph with K_2 is C_4-magic. In particular, the odd prisms and books are C_4-supermagic. In Section 4 we show that the windmill $W(r, k)$ is C_r-magic, thus providing a family of C_r-magic graphs for each $r \geqslant 3$. It is also shown that subdivided wheels and uniform Θ-graphs are cycle-magic. All these results rely on a technique of partitioning sets of integers with special properties introduced in [4]. This is explained in Section 2.

2. Notation and preliminary results

We will use the following notations.

For any two integers $n < m$ we denote by $[n, m]$ the set of all consecutive integers from n to m. For a set $I \subset \mathbb{N}$ we write, $\sum I = \sum_{x \in I} x$. Note that, for any $k \in \mathbb{N}$,

$$\sum (I + k) = \sum I + k|I|.$$

Finally, given a total labelling f of a graph $G = (V, E)$, we denote by

$$f(G) = \sum f(V) + \sum f(E).$$

However, we will use the same notation although G is not a graph but a set of vertices and edges.

Let $P = \{X_1, \ldots, X_k\}$ be a partition of a set X of integers. The set of subset sums of P is denoted by $\sum P = \{\sum X_1, \ldots, \sum X_k\}$. If all elements of P have the same cardinality, then P is said to be a *k-equipartition* of X.

We shall describe a partition $P = \{X_1, \ldots, X_k\}$ of a set $X = \{x_1, x_2, \ldots, x_n\}$ by giving a k-coloring on the elements of X in such a way that X_i contains all the elements with color i, $1 \leqslant i \leqslant k$. For example, the coloring $(1, 2, 1, 2, 2, 1)$ means that $X_1 = \{x_1, x_3, x_6\}$ and $X_2 = \{x_2, x_4, x_5\}$. When some pattern of colors $(c_1, c_2, \ldots, c_r)$ is repeated t times we write $(c_1, c_2, \ldots, c_r)^t$. For instance, the coloring $(1, 2, 1, 2, 2, 1)$ is denoted by $(1, 2)^2(2, 1)$.

We say that a k-equipartition $P = \{X_1, \ldots, X_k\}$ of a set of integers $X = \{x_1 < x_2 < \cdots < x_{hk}\}$ is *well-distributed* if for each $0 \leqslant j < h$, the elements $x_l \in X$, with $l \in [jk + 1, (j + 1)k]$, belong to distinct parts of P. For instance, $P_1 = \{\{1, 4, 5\}, \{2, 3, 6\}\}$ and $P_2 = \{\{1, 3, 5\}, \{2, 4, 6\}\}$, are well-distributed 2-equipartitions of $X = [1, 6]$ while $P_3 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$ is not.

We will use the next two lemmas proved in [4] for k-equipartitions. It is easily checked that the proofs given in [4] provide in fact well-distributed partitions.

Lemma 1 (*Gutiérrez and Lladó [4]*). *Let h and k be two positive integers. For each integer $0 \leqslant t \leqslant \lfloor h/2 \rfloor$, there exists a well-distributed k-equipartition P of $[1, hk]$ such that $\sum P$ is an arithmetic progression of difference $d = h - 2t$.*

Lemma 2 (*Gutiérrez and Lladó [4]*). *Let h and k be two positive integers. If h or k are not both even, there exists a well-distributed k-equipartition P of $[1, hk]$ such that $\sum P$ is a set of consecutive integers.*

Next lemma provide well-distributed equipartitions where all the parts have the same sum.

Lemma 3. *Let $h \geqslant 3$ be an odd integer. If either*

(1) *k is odd and $X = [1, hk]$, or*
(2) *k is even and $X = [1, hk+1] \backslash \{k/2+1\}$.*

there is a well-distributed k-equipartition P of X such that $|\sum P| = 1$.

Proof. (1) By Lemma 2 there is a well-distributed k-equipartition $P' = \{Y_1, \ldots, Y_k\}$ of the interval $Y = [1, (h-1)k]$ such that

$$\sum P' = \left\{ \sum Y_1 + (i-1) : 1 \leqslant i \leqslant k \right\}.$$

Consider the partition $P = \{X_1, \ldots, X_k\}$ of $[1, hk]$, where

$$X_i = Y_i \cup \{(1-i) + hk : \ 1 \leqslant i \leqslant k\}.$$

It is clear that P is a k-equipartition of $[1, hk]$.

As P' is a well-distributed k-equipartition of $[1, (h-1)k]$ and there is one element of each part in $[(h-1)k+1, hk]$, P is also well-distributed.

In addition, for any $1 \leqslant i \leqslant k$ we have,

$$\sum X_i = \sum Y_1 + (i-1) + (1-i) + hk = \sum Y_1 + hk,$$

which is independent of i and therefore $|\sum P| = 1$.

(2) Let k be an even number and $X = [1, hk+1] \backslash \{k/2+1\}$.

Set $A = [1, k+1] \backslash \{k/2+1\}$ and $B = [k+2, hk+1]$. Clearly, $|A| = k$, $|B| = (h-1)k$ and $X = A \cup B$.

Consider now the partition $P = \{X_1, \ldots, X_k\}$ given by the following k-coloring of $A \cup B$.

Color the k elements of A by

$$(k/2, k/2-1, \ldots, 1)(k, k-1, \ldots, k/2+1).$$

Now color the $(h-1)k$ elements of B by

$$(k/2+1, 1, k/2+2, 2, \ldots, k, k/2)(k, k-1, \ldots, 1)^{((h-3)/2)+1}(1, 2, \ldots, k)^{((h-3)/2)}.$$

It is clear by the coloring that P is well-distributed. Moreover, for $1 \leqslant i \leqslant k/2$, we have,

$$\sum X_i - \sum X_1 = (k/2 + 1 - i - k/2) + (k + 1 + 2i - k - 3) + \left(\frac{h-3}{2} + 1\right)(1-i) + \frac{h-3}{2}(i-1) = 0.$$

A similar computation shows that $\sum X_i - \sum X_1$ takes the same value when $k/2 < i \leqslant k$, so that $|\sum P| = 1$. $\square$

Remark 4. Note that the statements of the three above lemmas can be extended to any integer translation of the set X.

3. C_3 and C_4-magic graphs

Let $W_n = C_n + \{v\}$ denote the wheel with a rim of order n. Clearly W_n admits a covering by triangles. As an application of Lemma 2, we next show that any odd wheel is a C_3-supermagic graph.

Theorem 5. *The wheel W_n for $n \geqslant 5$ odd, is C_3-supermagic.*

Proof. Denote by $v_1, v_2, \ldots, v_n$ the vertices in the n-cycle of the wheel W_n and by v its central vertex. For $1 \leqslant i \leqslant n$ let $N_i = \{v_i, v_i v\}$.

Define a total labelling f of W_n on $[1, 3n+1]$ as follows. Set $f(v) = 1$, $f(v_n v_1) = 2n+2$ and for $1 \leqslant i < n$, $f(v_i v_{i+1}) = 3n + 2 - i$. Therefore, $f(E(C_n)) = [2n+2, 3n+1]$.

We have to define f on $N = \cup_{i=1}^n N_i$ in such a way that $f(N) = [2, 2n+1]$.

Since n is odd, by Lemma 2 there is a well-distributed n-equipartition $P = \{X_1, \ldots, X_n\}$ of the set $X = 1 + [1, 2n]$, such that $\sum X_i = \sum X_1 + (i-1)$. $X_i = \{x_{i,1} < x_{i,2}\}$. Since P is well-distributed, we have $1 < x_{i,1} \leqslant n+1$ and $n+1 < x_{i,2} \leqslant 1 + 2n$.

Let α be the permutation of $[1, n]$ given by

$$\alpha(i) = \begin{cases} i/2, & i \text{ even}, \\ (n+i)/2, & i \text{ odd}. \end{cases}$$

Since n is odd, α is a permutation of $[1, n]$. Moreover, $\alpha(i) + \alpha(i+1) = i + (n+1)/2$ for $1 \leqslant i \leqslant n-1$ and $\alpha(n) + \alpha(1) = (3n+1)/2$.

Define f on each N_i by the bijection from N_i to $X_{\alpha(i)}$ given by

$$f(v_i) = x_{\alpha(i),1} \quad \text{and} \quad f(vv_i) = x_{\alpha(i),2}.$$

Note that $1 < f(v_i) \leqslant n+1$ and $n+1 < f(vv_i) \leqslant 2n+1$, so that $f(V(N)) = [2, n+1]$ and $f(E(N)) = [n+2, 2n+1]$. Hence, if f is C_3-magic, then it is C_3-supermagic.

Let us show that $\sum f(H)$ is constant in every triangle H of W_n. Now we prove that f take the same sum in every subgraph H of W_n isomorphic to C_3. Since $n \geqslant 5$, each triangle H has vertex set either $\{v, v_i, v_{i+1}\}$ for some $1 \leqslant i < n$, or $\{v, v_n, v_1\}$. Therefore,

$$\begin{aligned} \sum f(H) &= \sum f(N_i) + \sum f(N_{i+1}) + f(v_i v_{i+1}) + f(v) \\ &= 2 \sum X_1 + \alpha(i) + \alpha(i+1) - 2 + (3n + 2 - i) + 1 \\ &= 2 \sum X_1 + i + (n+1)/2 + (3n+1) - i \\ &= 2 \sum X_1 + (7n+3)/2 \\ &= \sum f(N_n) + \sum f(N_1) + f(v_n v_1) + f(v). \end{aligned}$$

which is independent of i as claimed. This completes the proof. □

Fig. 2 shows an example of the C_3-supermagic labelling defined in the above proof.

Remark 6. In Fig. 3 two quite different C_3-magic labellings of the wheels W_4 and W_6 are displayed. We do not know if the wheel W_{2r} with $r > 3$ is a C_3-magic graph.

Another application of Lemma 2 provides a large family of C_4-supermagic graphs. Clearly, for any graph G, the Cartesian product $G \times K_2$ can be covered by 4-cycles.

Theorem 7. *Let G be a C_4-free supermagic graph of odd size. Then, the graph $G \times K_2$ is C_4-supermagic.*

Proof. Let n and m be, respectively, the order and size of $G = (V, E)$. We have to show a C_4-supermagic total labelling of $G \times K_2$ with the integers in $[1, 3n + 2m]$.

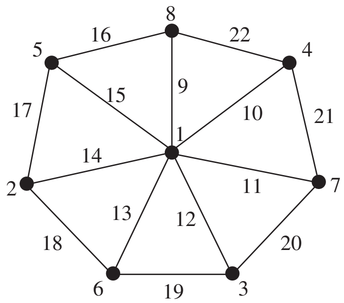

Fig. 2. C_3-supermagic labelling of W_7.

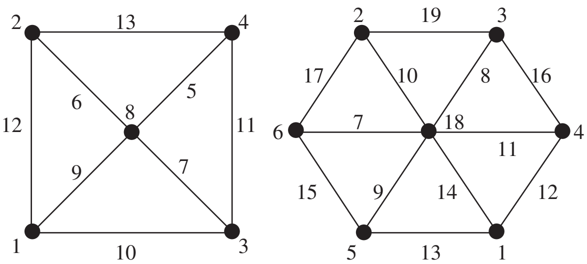

Fig. 3. C_3-magic labelings of W_4 and W_6.

For each vertex $x \in V(G)$ denote by $x_0, x_1 \in V(G \times K_2)$ the corresponding vertices in the two copies of G and $x_0x_1 \in E(G \times K_2)$ the edge joining them. Denote by $A_x = \{x_0, x_1, x_0x_1\}$ and by $A = \cup_{x \in V} A_x \subset V(G \times K_2) \cup E(G \times K_2)$. We have $|A| = 3n$. Now, for each edge $xy \in E(G)$, denote by $B_{xy} = \{x_0y_0, x_1y_1\}$ the corresponding edges in the two copies of G and $B = \cup_{xy \in E(G)} B_{xy}$. We have $|B| = 2m$. Clearly, $\{A, B\}$ is a partition of the set $V(G \times K_2) \cup E(G \times K_2)$.

By Lemma 2 there is a well-distributed n-equipartition $P_1 = \{X_1, \ldots, X_n\}$ of the set $[1, 3n]$, such that $\sum X_i = a + i$ for some integer a.

Since m is odd, Lemma 2 also ensures a well-distributed m-equipartition $P_2 = \{Y_1, \ldots, Y_m\}$ of $3n + [1, 2m]$ such that $\sum Y_i = b + i$ for some integer b.

Let f be a supermagic labelling of G with supermagic sum $s(f)$. Define a total labelling f' of $G \times K_2$ as follows. For $x \in V(G)$ define f' on A_x by any bijection from A_x to $X_{f(x)}$ (the bijection depends on f.) Similarly, for $xy \in E(G)$ define f' on B_{xy} by any bijection from B_{xy} to $Y_{f(xy)-n}$ (again the bijection depends on f.) Then, the map f' is a bijection from $V(G \times K_2) \cup E(G \times K_2)$ to $[1, 3n + 2m]$. In addition, as P_1 is well distributed in $[1, 3n]$, we can choose f' verifying $f'(V(G \times K_2)) = [1, 2n]$.

Now let H be a subgraph of $G \times K_2$ isomorphic to a 4-cycle. Since G is C_4-free, every 4-cycle H of $G \times K_2$ has the form,

$$V(H) \cup E(H) = A_x \cup A_y \cup B_{xy},$$

where $x, y \in V(G)$ and $xy \in E(G)$. Then, the sum of the elements in any 4-cycle H of $G \times K_2$ is

$$\begin{aligned} f'(H) &= f'(A_x) + f'(A_y) + f'(B_{xy}) \\ &= 2a + f(x) + f(y) + b + f(xy) - n \\ &= 2a + b + s(f) - n. \end{aligned}$$

independent of H. $\square$

As an application of Theorem 7 we have next Corollary.

Corollary 8. *The following two families of graphs are C_4-supermagic for n odd.*

(1) *The prims*, $C_n \times K_2$.
(2) *The books*, $K_{1,n} \times K_2$.

4. C_r-magic graphs

In this section we give a family of C_r-supermagic graphs for any integer $r \geqslant 3$. Let C_r be a cycle of length $r \geqslant 3$. Consider the graph $W(r, k)$ obtained by identifying one vertex in each of $k \geqslant 2$ disjoint copies of the cycle C_r. The resulting graphs are called *windmills*, and $W(3, k)$ is also known as the friendship graph. Note that windmills clearly admit a C_r-covering. We next show that they are C_r-supermagic graphs.

Theorem 9. *For any two integers $k \geqslant 2$ and $r \geqslant 3$, the windmill $W(r, k)$* is *C_r-supermagic.*

Proof. Let $G_1, \ldots, G_k$ be the r-cycles of $W(r, k)$ and let v their only common vertex. Denote by $G^* = W(r, k)\backslash\{v\}$ and its set of vertices and edges by V^* and E^*, respectively. Therefore we have, $|V^*| = (r-1)k$ and $|E^*| = rk$.

We want to define a C_r-supermagic total labelling f of $W(k, r)$ with the integers of $[1, (2r-1)k+1]$ such that $f(V(W(r, k))) = f(V^* \cup \{v\}) = [1, (r-1)k] + 1$.

Suppose first that k is odd. Let $f(v) = 1$.

By Lemma 3(i) there is a k-equipartition $P_1 = \{X_1, \ldots, X_k\}$ of the set $1 + [1, (2r-1)k]$ such that $|\sum P_1| = 1$. Furthermore, as it is well distributed, in each set X_i there are $r-1$ elements less or equal than $1 + (r-1)k$.

Define f on each $G_i^* = G_i\backslash\{v\}$, $1 \leqslant i \leqslant k$, by any bijection from G_i^* to X_i such that $f(V_i^*) \subset [1, (r-1)k] + 1$.

Suppose now that k is even. Now, let $f(v) = k/2 + 1$.

By Lemma 3(ii) there is a k-equipartition $P = \{X_1, \ldots, X_k\}$ of the set $[1, (2r-1)k+1]\backslash\{k/2+1\}$ such that $|\sum P| = 1$. Furthermore, there are $r-1$ elements less or equal than $1 + k(r-1)$ in each set X_i.

In this case, define also $f(G_i^*)$ by any bijection from V_i^* to X_i such that $f(V_i^*) \subset [1, (r-1)k+1]\backslash\{k/2+1\}$.

In both cases, for each $1 \leqslant i \leqslant k$,

$$f(G_i) = \sum X_i + f(v).$$

Hence f is a C_r-supermagic labelling of the windmill $W(k, r)$. □

Fig. 4 shows examples of cycle-supermagic labellings of windmills for different parities of the cycles.

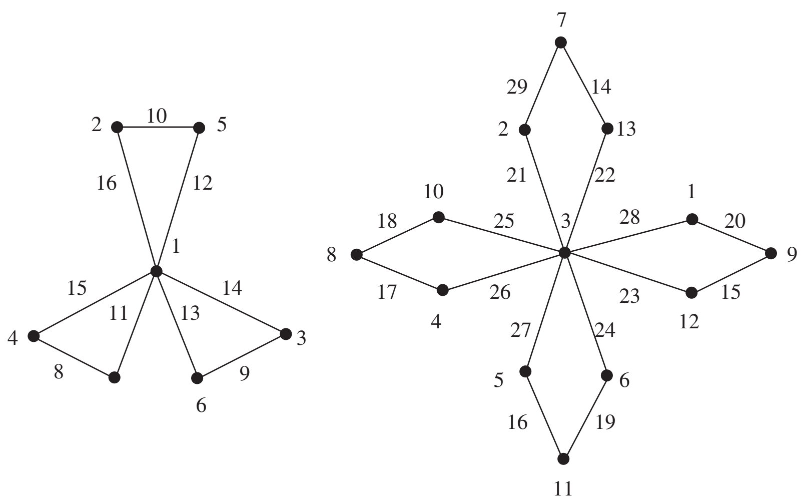

Fig. 4. C_k supermagic labelling of $W(k, k)$ for $k = 3, 4$.

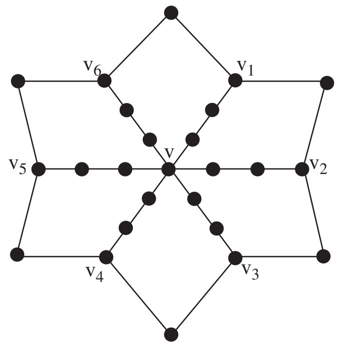

Fig. 5. The subdivided wheel $W_6(3, 2)$.

Next, we consider a family of graphs obtained by subdivisions of a wheel. The *subdivided wheel* $W_n(r, k)$ is the graph obtained from a wheel W_n by replacing each radial edge vv_i, $1 \leqslant i \leqslant n$ by a vv_i-path of size $r \geqslant 1$, and every external edge $v_i v_{i+1}$ by a $v_i v_{i+1}$-path of size $k \geqslant 1$. It is clear that, $|V(W_n(r, k))| = n(r + k) + 1$ and $|E(W_n(r, k))| = n(r + k)$. Fig. 5 shows a subdivided wheel of W_6.

Theorem 10. *Let r and k be two positive integers. The subdivided wheel $W_n(r, k)$ is C_{2r+k}-magic for any odd $n \neq 2r/k + 1$. Furthermore, $W_n(r, 1)$ is C_{2r+1}-supermagic.*

Proof. Let $n \geqslant 3$ be an odd integer.

Denote by v the central vertex of the subdivided wheel $W_n(r, k)$ and by $v_1, v_2, \ldots, v_n$ the remaining vertices of degree > 2. For $1 \leqslant i \leqslant n$ let P_i be the vv_i-path of length $r \geqslant 1$.

Let $P_i^* = P_i \setminus \{v\}$, $1 \leqslant i \leqslant n$ and $P^* = \cup_{i=1}^n P_i^*$.

Suppose first $k = 1$.

In this case, we want a C_{2r+1}-magic labelling f on $W_n(r, 1)$ with the integers in $[1, 1 + 2nr + n]$ such that $f(V) = [1, nr + 1]$. Let $f(v) = 1$ and $f(v_n v_1) = 2nr + 2$ and label the remaining edges of the external cycle of $W_n(r, 1)$ by $f(v_i v_{i+1}) = 2nr + 2 + n - i$, $1 \leqslant i < n$.

The only elements left to label are the ones in P^*, with $|P^*| = 2nr$. Since $n \geqslant 3$ is odd, Lemma 2 ensures the existence of a well-distributed n-equipartition $P = \{X_1, \ldots, X_n\}$ of the set $1 + [1, 2nr]$ such that $\sum X_i = a + i$, $1 \leqslant i \leqslant n$ for some constant a. Moreover, as P is well distributed, each X_i has r elements in $1 + [1, nr + 1]$. Now define f on each P_i^* by a bijection with $X_{\alpha(i)}$ which assigns the first r values of $[1, nr]$ to the vertices, where α is the following permutation of $[1, n]$.

$$\alpha(i) = \begin{cases} i/2, & i \text{ even}, \\ (n + i)/2, & i \text{ odd}. \end{cases}$$

Note that, as n is odd, $\alpha(n) + \alpha(1) = n + (n + 1)/2$ and, for $1 \leqslant i < n$, we have $\alpha(i) + \alpha(i + 1) = i + (n + 1)/2$.

Therefore, f is clearly a bijection from $W_n(r, 1)$ to $[1, n(2r + 1) + 1]$, and $f(V) = [1, nr + 1]$.

Now, since $n \neq 2r + 1$, for every subgraph H of $W_n(r, 1)$ isomorphic to C_{2r+1}, we have either $V(H) \cup E(H) = \{v\} \cup P_n^* \cup \{v_n v_1\} \cup P_1^*$ or

$$V(H) \cup E(H) = \{v\} \cup P_i^* \cup \{v_i v_{i+1}\} \cup P_{i+1}^*$$

for some $1 \leqslant i < n$.

Therefore, for each $1 \leqslant i < n$, we have

$$\begin{aligned} \sum f(H) &= \sum f(P_i^*) + \sum f(P_{i+1}^*) + f(v_i v_{i+1}) + f(v) \\ &= 2a + \alpha(i) + \alpha(i + 1) + (n(2r + 1) + 2 - i) + 1 \\ &= 2a + 2nr + \frac{3n + 7}{2}, \end{aligned}$$

which is independent of i. A similar computation shows that $\sum f(P_n^*) + \sum f(P_1^*) + f(v_n v_1) + f(v)$ has the same value. Hence f is a C_{2r+1}-supermagic labelling of $W_n(r, 1)$.

Suppose now $k > 1$.

In this case, for each $1 \leqslant i \leqslant n$, let Q_i be the $v_i v_{i+1}$-path of length $k \geqslant 1$. Denote by $Q_i^* = Q_i \setminus \{v_i, v_{i+1}\}$ and $Q^* = \cup_i Q_i^*$.

By Lemma 2 there is a well-distributed n-equipartition $P_2 = \{Y_1, \ldots, Y_n\}$, of the set $2nr + [1, n(2k-1)]$, such that for $1 \leqslant i \leqslant n$, $\sum Y_i = b + i$, for some constant b.

Define a total labelling f of $W_n(r, k)$ on $[1, n(2k-1)]$ as follows. Set $f(v) = 2n(r+k) - n + 1$. Define f on P_i^* by any bijection from P_i^* to $X_{\alpha(i)}$, where $P = \{X_1, \ldots, X_k\}$ and α are as in the above case. Define f on Q_i^* by any bijection to Y_{n+1-i}.

Since $n \neq 2r/k + 1$, every subgraph H of $W_n(r, k)$ isomorphic to C_{2r+k} verifies either $V(H) \cup E(H) = \{v\} \cup P_n^* \cup Q_n^* \cup P_1^*$ or

$$V(H) \cup E(H) = \{v\} \cup P_i^* \cup Q_i^* \cup P_{i+1}^*,$$

for some $1 \leqslant i < n$.

Then, for each $1 \leqslant i < n$ we have,

$$\begin{aligned}\sum f(H) &= \sum f(P_i^*) + \sum f(P_{i+1}^*) + f(Q_i^*) + f(v)\\ &= 2a + \alpha(i) + \alpha(i+1) + b + (n+1-i) + 2n(r+k) - n + 1\\ &= 2a + b + 2n(r+k) + \frac{n+5}{2}.\end{aligned}$$

It is also immediate to check that the labels of the remaining $(2r+k)$-cycle have also the same sum. □

Fig. 6 shows examples of cycle-supermagic labellings as defined in the above proof.

We finish by giving another family of cycle-supermagic graphs. Recall that, for a sequence $k_1, \ldots, k_n$ of positive integers, the graph $\Theta(k_1, \ldots, k_n)$ consists of n internally disjoint paths of orders $k_1 + 2, \ldots, k_n + 2$ joined by two end vertices u and v. When all the paths have the same size p, this graph, denoted by $\Theta_n(p)$, admits a C_{2p}-covering. We next show that such a graph is C_{2p}-supermagic.

Theorem 11. *The graph $\Theta_n(p)$ is C_{2p}-supermagic for $n, p \geqslant 2$.*

Proof. Let u and v be the common end vertices of the paths $P_1, \ldots, P_n$ in $\Theta_n(p) = (V, E)$. Denote by $\widetilde{P_i} = P_i \setminus \{u, v\}$, $1 \leqslant i \leqslant n$.

We want to define a total labelling f of $\Theta_n(p)$ with integers from the interval $[1, (2 + (p-1)n) + np]$ such that $f(V) = [1, 2 + (p-1)n]$.

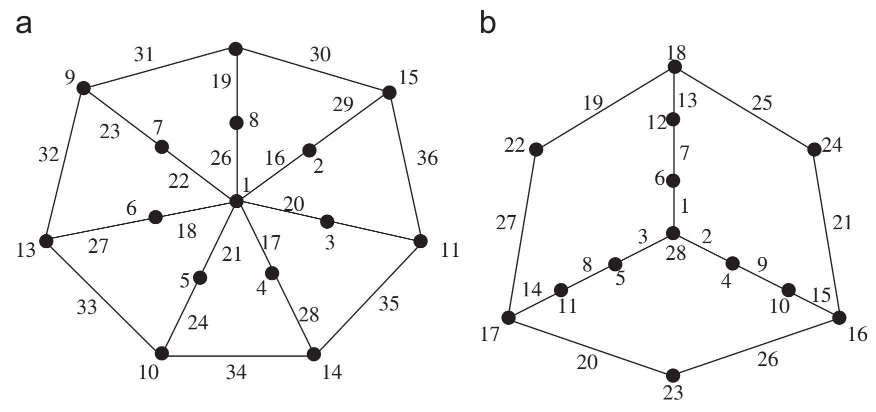

Fig. 6. (a) C_5-supermagic labelling of $W_7(2, 1)$. (b) C_8-magic labelling of $W_3(3, 2)$.

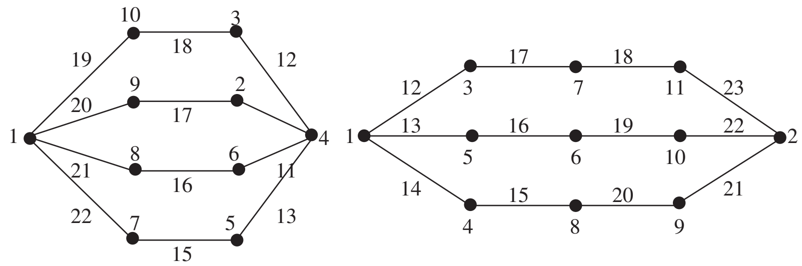

Fig. 7. C_6-supermagic labelling of $\Theta_4(3)$ and C_8-supermagic labelling of $\Theta_3(4)$.

If n is odd, Lemma 3(i) provides a n-equipartition $P = \{X_1, \ldots, X_n\}$ of the set $2 + [1, (2p-1)n]$ such that $\sum X_1 = \cdots = \sum X_n = a$, for some constant a, and, as it is well distributed, in each X_i there are $p-1$ integers less or equal than $2 + (p-1)n$.

Define a total labelling f on $\Theta_n(p)$ as follows. Set $f(u) = 1$, $f(v) = 2$ and f on $\widetilde{P_i}$ by a bijection from $\widetilde{P_i}$ to X_i such that the $p-1$ numbers in each X_i that are less or equal than $2 + (p-1)n$ are used for the $p-1$ vertices in each $\widetilde{P_i}$.

Every subgraph H of $\Theta_n(p)$ isomorphic to C_{2p} is of the form

$$V(H) \cup E(H) = \{u\} \cup \widetilde{P_i} \cup \{v\} \cup \widetilde{P_j}$$

for $1 \leqslant i < j \leqslant n$.

It is easy to check that $\sum f(H) = 2a + 3$.

Assume now that n is even. By Lemma 3(ii) there exists a n-equipartition $P = \{X_1, \ldots, X_n\}$ of the set $[1, (2p-1)n + 2] \backslash \{1, n/2 + 2\}$ such that $\sum X_1 = \cdots = \sum X_n = a$, for some constant a. Moreover, since P is well-distributed, in each X_i there are $p-1$ numbers less or equal than $2 + (p-1)n$.

Now, we proceed as before but setting $f(v) = n/2 + 2$. It is immediate to check that we indeed get a C_{2p}-supermagic labelling of $\Theta_n(p)$. $\square$

In Fig. 7 two supermagic labellings of $\Theta_n(p)$ for different parities of n and p are displayed.

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