Cauchy problem for quasi-linear wave equations with viscous damping

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Abstract

The paper studies the global existence and asymptotic behavior of weak solutions to the Cauchy problem for quasi-linear wave equations with viscous damping. It proves that when $p \geq \max\{m, \alpha\}$, where $m + 1, \alpha + 1$ and $p + 1$ are, respectively, the growth orders of the nonlinear strain terms, the nonlinear damping term and the source term, the Cauchy problem admits a global weak solution, which decays to zero according to the rate of polynomial as $t \to \infty$, as long as the initial data are taken in a certain potential well and the initial energy satisfies a bounded condition. Especially in the case of space dimension $N = 1$, the solutions are regularized and so generalized and classical solution both prove to be unique. Comparison of the results with previous ones shows that there exist clear boundaries similar to thresholds among the growth orders of the nonlinear terms, the states of the initial energy and the existence, asymptotic behavior and nonexistence of global solutions of the Cauchy problem.

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1. Introduction

We are concerned with the global existence and asymptotic behavior of weak solutions to the Cauchy problem for quasi-linear wave equations with viscous damping

$$u_{tt} - \Delta u_t + \lambda u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) + f(u_t) = g(u) \quad \text{on } \mathbb{R}^N \times (0, \infty),$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^N,$$

where $\lambda > 0$ is a real number, $\sigma_i (i = 1, \ldots, N)$, $f$ and $g$ are given nonlinear functions satisfying certain conditions to be specified later.

In the case of space dimension $N = 1$, equations of type (1.1) describe the spread of strain waves in a viscoelastic bar made up of the material of the rate type, see, e.g., [1,2,8,11,12]. They can also be seen as field equations governing the longitudinal motion of a viscoelastic bar obeying the nonlinear Voigt model, see [3]. For the initial boundary value problem (IBVP) of the equations of type (1.1), there have been many impressive works on the existence and nonexistence of global solutions and other properties, see [1–6, 8,9,11,12,15,17–19]. Recently, in [20–22], under certain polynomial growth assumptions on the nonlinearities $\sigma_i (i = 1, \ldots, N)$, $f$ and $g$, more specifically, $m+1$, $\alpha+1$ and $p+1$ are, respectively, the growth orders of the nonlinear strain terms, the nonlinear damping term and the source term, where in the sequel $m$, $\alpha$ and $p$ are nonnegative real numbers, the author has investigated the existence, asymptotic behavior and nonexistence of global weak solutions to the IBVP of Eq. (1.1). But for Cauchy problem (1.1), (1.2), do the similar results hold? In [23], the author has further considered the question and shown that Cauchy problem (1.1), (1.2), with $\lambda = 0$, the initial data $u_0 \in W^{1,m+2}(\mathbb{R}^N) \cap H^{1}(\mathbb{R}^N) \cap L_{p+2}(\mathbb{R}^N)$, $u_1 \in L_2(\mathbb{R}^N)$, admits a global weak solution, provided that $\alpha \geq \max\{m, p\}$, and if $m+2 < N$, also $\alpha + 2 < \frac{N(m+2)}{N - m - 2}$. The result shows that if the nonlinear damping term is assumed to be eventually dominating the strain terms and the source term, as expressed in the condition $\alpha \geq \max\{m, p\}$, then the global existence of a weak solution of Cauchy problem (1.1), (1.2), with $\lambda = 0$, can be obtained. If in contrast $p \geq \max\{m, \alpha\}$, however, it has been so far remained an open question if a global solution exists.

It is the purpose of the present paper to deal with the latter case. By a Galerkin approximation scheme combined with the potential well method, and via a limiting process of the solutions of a series of periodic boundary value problems (PBVP) we prove that also for $p \geq \max\{m, \alpha\}$ a global weak solution exists, provided that the initial data $u_0$ is taken in a certain potential well $\tilde{W}$, $u_1 \in L_2(\mathbb{R}^N)$ and the initial energy $\tilde{A}(0)$ satisfies a bounded condition to be given later. Moreover, it is shown that this solution features the asymptotic behavior

$$\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla u(t)\|_{L_{m+2}(\mathbb{R}^N)}^{m+2} + \|u(t)\|_{L_{p+2}(\mathbb{R}^N)}^{p+2} \leq O(t^{-\frac{2}{r-1}}) \quad \text{as } t \to \infty, \quad (1.3)$$

where $r \geq \frac{\alpha+1}{\alpha+2}$ is a constant. In particular, for the one-dimensional case, Eq. (1.1) becomes

$$u_{tt} - u_{xx} + \lambda u_t - \sigma(u_x)x + f(u_t) = g(u) \quad \text{on } \mathbb{R} \times (0, \infty),$$

(1.4)
and the weak solutions can be regularized and so generalized and classical solution both prove to be unique. And an example is shown. Comparison of the results with previous ones in [23] shows that there exist clear boundaries similar to thresholds among the growth orders of the nonlinear terms, the states of the initial energy and the existence, asymptotic behavior and nonexistence of global solutions of the Cauchy problem.

Since the linear part of Eq. (1.1) exists some time decay strong enough, it is possible to discuss the global existence and decay property of the solution to Cauchy problem (1.1), (1.2) according to the standard method, provided that the initial data is sufficiently small, see [14]. But where this kind of “smallness of initial data” is a theoretically qualitative concept rather than a quantitative one. When the initial data $u_0$ belongs to the potential well, as we do in this paper, the “smallness of initial data” is reduced to be that the initial energy satisfies a quantitative bounded condition.

The plan of the paper is as follows. The main results and some notations are stated in Section 2. The global existence and asymptotic behavior of weak solutions to the PBVP and Cauchy problem (1.1), (1.2) are studied in Sections 3 and 4, respectively. In Section 5, the weak solutions are regularized and so the generalized and classical solution both prove to be unique in the case of space dimension $N = 1$, and an example is shown.

2. Statement of main results

We first introduce the following abbreviations:

\[ \Omega = (-L, L)^N, \quad Q_T = \Omega \times (0, T), \quad L_p = L_p(\Omega), \]
\[ W^{m,p} = W^{m,p}(\Omega), \quad W_0^{m,p} = W_0^{m,p}(\Omega), \]
\[ C_0^\infty = C_0^\infty(\Omega), \quad H^m = W^{m,2}, \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \]
\[ \| \cdot \|_{k,p} = \| \cdot \|_{W^{k,p}}, \quad \| \cdot \| = \| \cdot \|_{L^2}. \]  \hspace{1cm} (2.0)

For every $p \in (1, \infty)$ we denote the dual of $W_0^{1,p}$ by $W^{-1,p'}$, with $p' = p/(p-1)$. The notation $(\cdot, \cdot)$ for the $L_2$-inner product will also be used for the notation of duality pairing between dual spaces. Let the functional space:

\[ U = L^\infty([0, T]; W^{1,m+2}(\mathbb{R}^N)) \cap H^1([0, T]; H^1(\mathbb{R}^N)) \cap L_\infty([0, T]; L_{p+2}(\mathbb{R}^N)). \]

**Definition.** The function $u \in U$, with $u_t \in L_{\alpha+2}(\tilde{Q}_T)$ and $u_{tt} \in L_1([0, T]; W^{-1,(m+2)'}(\mathbb{R}^N))$, where $\tilde{Q}_T = \mathbb{R}^N \times (0, T)$, is called a weak solution of problem (1.1), (1.2) on $[0, T]$, if for any $\chi \in W^{1,m+2}(\mathbb{R}^N)$, supp $\chi$ is a bounded set in $\mathbb{R}^N$, for a.e. $t \in [0, T]$,

\[ (u_{tt}, \chi) + (\nabla u_t, \nabla \chi) + \lambda (u_t, \chi) + \sum_{i=1}^N (\sigma_i(u_{x_i}), \chi_{x_i}) + (f(u_t), \chi) = (g(u), \chi), \]
\[ u(\cdot, 0) = u_0 \quad \text{in} \quad H^1(\mathbb{R}^N), \quad u_t(\cdot, 0) = u_1 \quad \text{in} \quad L_2(\mathbb{R}^N). \]

Define the potential well
\[
W = \{ u \in W^{1,m+2}_{0} \mid I(u) = C_1 \| \nabla u \|^{m+2}_{m+2} - (C_5 + 1) \| u \|^{p+2}_{p+2} > 0 \} \cup \{0\},
\]
\[
\tilde{W} = \{ u \in W^{1,m+2}(\mathbb{R}^N) \mid I(u) = C_1 \| \nabla u \|^{m+2}_{m+2}(\mathbb{R}^N) - (C_5 + 1) \| u \|^{p+2}_{p+2}(\mathbb{R}^N) > 0 \} \cup \{0\},
\]
\[
W^* = \{ u \in H^2(\mathbb{R}) \mid I(u) = C_1 \| u_x \|^{m+2}_{m+2}(\mathbb{R}) - (C_5 + 1) \| u \|^{p+2}_{p+2}(\mathbb{R}) > 0 \} \cup \{0\},
\]
where \( C_1 \) and \( C_5 \) are positive constants given in Theorem 2.1. For later purposes we introduce the functional \( J \) defined by
\[
J(u) := \frac{C_1}{m+2} \| \nabla u \|^{m+2}_{m+2} - \frac{C_5}{p+2} \| u \|^{p+2}_{p+2}
\] (2.2)

for suitable \( u \). Obviously,
\[
J(u) = \frac{1}{m+2} I(u) + b_1 \| u \|^{p+2}_{p+2} = \frac{C_5}{(p+2)(C_5+1)} I(u) + b_2 \| \nabla u \|^{m+2}_{m+2} 
\] (2.3)

for all such \( u \). Where and in the sequel we have
\[
b_1 = \frac{C_5 + 1}{m+2} - \frac{C_5}{p+2} (> 0), \quad b_2 = C_1 \left[ \frac{1}{m+2} - \frac{C_5}{(p+2)(C_5+1)} \right] (> 0).
\]

We first consider the PBVP of Eq. (1.1)
\[
\begin{align*}
&u(x, t) = u(x + 2Le_i, t), \quad x \in \mathbb{R}^N, \ t > 0, \\
&u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}^N,
\end{align*}
\] (2.4)

where \( x + 2Le_i = (x_1, \ldots, x_{i-1}, x_i + 2L, x_{i+1}, \ldots, x_N) \), \( L > 0 \) is a real number, \( \varphi|_{\partial \Omega} = \psi|_{\partial \Omega} = 0 \), \( \varphi(x) = \varphi(x + 2Le_i) \), \( \psi(x) = \psi(x + 2Le_i) \), \( x \in \mathbb{R}^N, \ i = 1, \ldots, N \).

**Definition.** The function \( u \in L_{\infty}([0, T]; W^{1,m+2}_{0} \cap H^1([0, T]; H^1_0)) \), with \( u_t \in L_{\alpha+2}(Q_T) \) and \( u_{tt} \in L_1([0, T]; W^{1,m+2}_{-1} \cap H^1(\mathbb{R}^N)) \), is called a weak solution of problem (1.1), (2.4), if for any \( \chi \in W^{1,m+2}_{0} \), for a.e. \( t \in [0, T] \),
\[
(u_{tt}, \chi) + (\nabla u_t, \nabla \chi) + \lambda(u_t, \chi) + \sum \sigma_i(u(x_i), x_{x_i}) + (f(u_t), \chi) = (g(u), \chi),
\]
\[
u (\cdot, 0) = \varphi \quad \text{in} \quad H^1, \quad u_t(\cdot, 0) = \psi \quad \text{in} \quad L_2.
\]

**Lemma 1.** [13,16] Let \( h(t) \ (t \geq 0) \) is a nonnegative and nonincreasing function. If there are two constants \( \nu \geq 0 \) and \( r > 0 \) such that
\[
\int_0^\infty h^{\nu+1}(s) \, ds \leq rh(t), \quad t \geq 0,
\]
then
\[
h(t) \leq \begin{cases} \left( \frac{\nu+1}{\nu} \right)^{\frac{1}{\nu}} r^{\frac{1}{\nu}} t^{-\frac{1}{\nu}}, & \nu > 0, \\ h(0) \exp(1 - t/r), & \nu = 0. \end{cases}
\]
Now we state the main results of the paper. (To simplify notation we shall not introduce
the range of summation if it is extending from 1, . . . , N.)

**Theorem 2.1.** Assume that

(i) \( \sigma_i \in C(\mathbb{R}) \), \((\sigma_i(s_1) - \sigma_i(s_2))(s_1 - s_2) \geq 0\), \( s_1, s_2 \in \mathbb{R} \), and
\[
C_1 |s|^{m+1} \leq |\sigma_i(s)| \leq C_2 |s|^{m+1}, \quad s \in \mathbb{R}, \quad i = 1, \ldots, N, \tag{2.6}
\]
where and in the sequel \( C_i > 0 \) (i = 1, . . . , 6) are constants.

(ii) \( f \in C(\mathbb{R}) \), \( f(s)s \geq 0 \) and \( |f(s)| \leq C_4 |s|^\alpha + 1 \), \( s \in \mathbb{R} \), and if \( \alpha > \frac{4}{N} \), also \( C_3 |s|^\alpha + 1 \leq |f(s)|, s \in \mathbb{R} \).

(iii) \( g \in C(\mathbb{R}) \), \( g(s)s \geq 0 \) and \( |g(s)| \leq C_5 |s|^\alpha + 1 \), \( s \in \mathbb{R} \), where and in the sequel \( \frac{p}{p+2} \leq \alpha \leq p, m + \frac{(m+2)\alpha}{N} \leq p \), and if \( m + 2 < N \), also \( p + 2 < \frac{N(m+2)}{N-m-2} \).

(iv) \( \varphi \in W, \psi \in L_2 \) such that
\[
\tilde{E}(0) \leq \frac{C_1}{2C_*(C_5 + 1)}, \tag{2.7}
\]
where and in the sequel \( C_* \) is an embedding constant from \( W_0^{1,m+2} \) to \( L_{p+2} \).

\[
\tilde{E}(0) = 2(C_{10} + 1/b_3)E(0) + 2(m + 2)\|\varphi\|_{L_2}^2 \left( \frac{p-m}{p+2} \right),
\]
\[
E(0) = \frac{1}{2} \|\psi\|^2 + \sum_i \int_0^\varphi \int_\Omega \sigma_i(s)ds dx - \int_\Omega \psi g(s) ds dx > 0, \tag{2.8}
\]
\[
b_3 = \frac{b_1 + b_2}{2}, \quad b_1, b_2 \text{ as shown in (2.3)},
\]
\[
C_{10} = \frac{m+2}{2} \left[ 4 \left( \frac{\alpha + 1}{2} \left( \frac{2}{p+2} \right) \right)^{\frac{1}{p+1}} \max\{C_{44}^{\frac{1}{p+1}}, C_{44}^{\frac{1}{p+1}} \} + 4 \right] + 1/b_3 \text{.}
\]

Then for any \( T > 0 \), problem (1.1), (2.4) admits a weak solution \( u \) on \([0, T]\). And the solution \( u \) features the asymptotic behavior
\[
\|u_t(t)\|^2 + \|\nabla u(t)\|_{m+2}^{m+2} + \|u(t)\|_{p+2}^{p+2} \leq O(t^{-\frac{2}{r-1}}) \quad \text{as } t \to +\infty, \tag{2.9}
\]
where \( r \geq \frac{\alpha+3}{\alpha+1} \) is a constant.

**Remark 1.** By assumption (iii) of Theorem 2.1 and the Gagliardo–Nirenberg inequality,
\[
W_0^{1,m+2} \hookrightarrow L_{p+2}, \quad W_0^{1,m+2} \hookrightarrow L_{\alpha+2}
\]
with the embedding constant \( C_* \) independent of \( \Omega \).

**Theorem 2.2.** Under conditions (i)–(iii) of Theorem 2.1, if \( u_0 \in \tilde{W} \cap H^1(\mathbb{R}^N) \), \( u_1 \in L_2(\mathbb{R}^N) \), such that
\[
\tilde{A}(0) \leq \frac{C_1}{2C_*(C_5 + 1)}, \tag{2.10}
\]
where
\[
\tilde{A}(0) = \left[2(C_{10} + 1/b_3)A(0) + 2(m + 2)\|u_0\|^2_{H^1(\mathbb{R}^N)}\right]^{\frac{\nu-m}{2-m}},
\]
\[
A(0) = \frac{1}{2}\|u_1\|^2_{L_2(\mathbb{R}^N)} + \sum_{i} \int_{\mathbb{R}^N} \int_{0}^{u_{0,i}} \sigma_i(s) \, ds \, dx - \int_{\mathbb{R}^N} \int_{0}^{u_0} g(s) \, ds \, dx > 0, \tag{2.11}
\]
then for any \( T > 0 \), Cauchy problem (1.1), (1.2) admits a weak solution \( u \) on \([0, T] \). And the solution \( u \) features asymptotic behavior (1.3).

In the case of space dimension \( N = 1 \), the weak solutions can be regularized.

**Theorem 2.3.** Assume that

(i) \( \sigma \in C^1(\mathbb{R}) \), \( \sigma'(s) \geq 0 \), and
\[
C_1|s|^{m+1} \leq |\sigma(s)| \leq C_2|s|^{m+1}, \quad s \in \mathbb{R}.
\]

(ii) One of the following conditions holds:

(ii_1) Condition (ii) of Theorem 2.1, where \( 0 < \alpha \leq 2 \), and \( f(s) \) is locally Lipschitz continuous.

(ii_2) \( f \in C^1(\mathbb{R}) \), \( f'(s) \geq 0 \), and
\[
|f(s)| \leq C_4|s|^\alpha+1, \quad |f'(s)| \leq C_6(|s|^\alpha + |s|^\alpha_1), \quad s \in \mathbb{R},
\]
where \( 0 < \alpha \leq 2 + \frac{2}{m+2}, \quad \alpha_1 = \frac{2(\alpha+1)}{\alpha+2} \).

(iii) \( g(s) \geq 0 \), \( |g(s)| \leq C_5|s|^{p+1} \), \( s \in \mathbb{R} \), where \( \frac{p}{p+2} \leq \alpha \leq p, \ m + (m + 2)^2 \leq p \), and \( g(s) \) is locally Lipschitz continuous.

(1) If \( u_0 \in W^* \), \( u_1 \in H^2(\mathbb{R}) \) such that (2.10) (with \( N = 1 \)) holds, then for any \( T > 0 \), Cauchy problem (1.4), (1.2) (with \( N = 1 \)) admits a unique generalized solution \( u \) on \([0, T] \), with \( u \in W^{1,\infty}([0, T]; H^2(\mathbb{R})) \cap W^{2,\infty}([0, T]; L_2(\mathbb{R})) \cap H^2([0, T]; H^1(\mathbb{R}))).

(2) If \( \sigma \in C^3(\mathbb{R}) \), \( f, g \in C^2(\mathbb{R}) \), \( u_0 \in W^* \cap H^4(\mathbb{R}) \), \( u_1 \in H^4(\mathbb{R}) \) such that (2.10) (with \( N = 1 \)) holds, then for any \( T > 0 \), Cauchy problem (1.4), (1.2) (with \( N = 1 \)) admits a unique classical solution \( u \) on \([0, T] \), with \( u \in H^3([0, T]; H^1(\mathbb{R})) \cap H^2([0, T]; H^3(\mathbb{R}))).

3. Global weak solutions to the PBVP

**Proof of Theorem 2.1.** For brevity, we take \( \lambda = 1 \) in (1.1). We look for approximate solutions \( u^n \) of problem (1.1), (2.4) of the form
\[
u^n(t) := \sum_{j=1}^{n} T_{jn}(t) w_j, \quad t \geq 0, \tag{3.0}
u^n(t) := \sum_{j=1}^{n} T_{jn}(t) w_j, \quad t \geq 0, \tag{3.0}
\]
where \( \{ w_j \}_{j=1}^\infty \) is a Schauder basis in \( W^{1,m+2}_0 \) and at the same time an orthonormal basis in \( L_2 \) (in fact, the existence of Schauder basis \( \{ w_j \}_{j=1}^\infty \) was proved in [7, 10]. Since \( \text{span}\{ w_j \} \) is dense in \( W^{1,m+2}_0 \) and \( W^{1,m+2}_0 \) is dense in \( L_2 \), \( \text{span}\{ w_j \} \) is dense in \( L_2 \), and \( \{ w_j \} \) is a basis in \( L_2 \). By standard orthogonalizing process, one can easily make out a sequence from \( \{ w_j \} \), still denoted by \( \{ w_j \} \), such that it composes an orthonormal basis in \( L_2 \), see [8], with \( w_j (x) = w_j (x + 2Le_j) \) \( (x \in \mathbb{R}^N, i = 1, \ldots, N) \), and the coefficients \( \{ T_{jn} \} \) satisfy

\[
\begin{align*}
(u^n_i, w_j) + (\nabla u^n_i, \nabla w_j) + (u^n_i, w_j) + \sum_i (\sigma_i (u^n_{xi}), w_{jxi}) + (f(u^n_i), w_j) - (g(u^n), w_j) &= 0, \quad (3.1) \\
u^n(0) &= \varphi^n, \quad u^n_i(0) = \psi^n \quad (3.2)
\end{align*}
\]

for \( t > 0, j = 1, \ldots, n \), and the suitably chosen appropriate initial data \( \varphi^n \) and \( \psi^n \). Since \( \mathcal{C} \) is dense in \( W^{1,m+2}_0 \cap H^1 \cap L^p \), we choose \( \varphi^n \) and \( \psi^n \) in \( \mathcal{C} \) in such a way that

\[
\varphi^n \to \varphi \quad \text{in} \quad W^{1,m+2} \cap H^1 \cap L^p, \quad \psi^n \to \psi \quad \text{in} \quad L_2 \quad (3.3)
\]

as \( n \to \infty \).

Substituting \( w_j \) in (3.1) by \( u^n_i \), one gets

\[
\frac{d}{dt} E_n(t) + (f(u^n_i), u^n_i) + \| u^n_i(t) \|_{1,2}^2 = 0. \quad (3.4)
\]

where

\[
E_n(t) = \frac{1}{2} \| u^n_i(t) \|_2^2 + \sum_i \int_\Omega \int_0^{u^n_i} \sigma_i(s) \, ds \, dx - \int_\Omega \int_0^{u^n} g(s) \, ds \, dx
\]

\[
\geq \frac{1}{2} \| u^n_i(t) \|_2^2 + J(u^n(t)), \quad (3.5)
\]

and where \( J(u^n) \) is shown in (2.2). By integral mean value theorem, the Hölder inequality, Remark 1 and (3.3) we easily get

\[
E_n(0) \to E(0)(>0) \quad \text{as} \quad n \to \infty. \quad (3.6)
\]

Hence, without loss of generality we assume that \( E_n(0) < 2E(0) \) for all \( n \), where

\[
E_n(0) = \frac{1}{2} \| \psi^n \|_2^2 + \sum_i \int_\Omega \int_0^{\varphi^n_{xi}} \sigma_i(s) \, ds \, dx - \int_\Omega \int_0^{\varphi^n} g(s) \, ds \, dx.
\]

Since \( \varphi \in W \), by (3.3),

\[
I(\psi^n) = C_1 \| \nabla \psi^n \|_{m+2}^{m+2} - (C_5 + 1) \| \psi^n \|_{p+2}^{p+2} \to I(\varphi) > 0 \quad (3.7)
\]

as \( n \to \infty \). Hence, without loss of generality we assume that \( I(\psi^n) > 0 \), i.e. \( \varphi^n \in W \) for all \( n \). And thus we claim
\[ u^n(t) \in W, \quad t > 0. \quad (3.8) \]

In fact, since \( u^n(0) = \varphi^n \in W \), if there exists a constant \( T > 0 \) such that \( I(u^n(t)) > 0 \), \( t \in [0, T) \), while \( I(u^n(T)) = 0 \), then by (2.3),
\[ J(u^n(t)) \geq b_3 \left( \| \nabla u^n(t) \|_{m+2}^{m+2} + \| u^n(t) \|_{p+2}^{p+2} \right), \quad t \in [0, T]. \quad (3.9) \]

Integrating (3.4) over \((0, t)\) and using (3.5) and (3.9), one gets
\[
\frac{1}{2} \left\| u^n_t(t) \right\|^2 + b_3 \left( \| \nabla u^n(t) \|_{m+2}^{m+2} + \| u^n(t) \|_{p+2}^{p+2} \right) \\
+ \int_0^t \left( \left\| u^n_{\tau}(\tau) \right\|^2_{1,2} + (f(u^n_\tau), u^n_\tau) \right) d\tau \leq E_n(0), \quad t \in [0, T]. \quad (3.10) \]

Substituting \( w_j \) in (3.1) by \( u^n \), one gets
\[
\frac{d}{dt} \left[ (u^n, u^n_t) + \frac{1}{2} \left\| u^n(t) \right\|_{1,2}^2 \right] + I(u^n(t)) + \left\| u^n(t) \right\|_{p+2}^{p+2} \\
\leq \left| (f(u^n_t), u^n) \right| + \left\| u^n_t(t) \right\|^2, \quad (3.11) \]

where the estimation
\[
\sum_{i} \left( \sigma_i(u^n_{x_i}), u^n_{x_i} \right) - (g(u^n), u^n) \geq I(u^n(t)) + \left\| u^n(t) \right\|_{p+2}^{p+2} 
\]

has been used. Since \( 2/(\alpha + 1) \leq (p + 2)' \leq (\alpha + 2)' \), note that \( F(a) = y^a/a \) \((a > 0, y > 0)\) is a concave function, by the Young inequality, assumption (ii) and the Cauchy inequality one obtains
\[
\left| (f(u^n_t), u^n) \right| \leq \frac{1}{2} \left\| u^n(t) \right\|_{p+2}^{p+2} + C_7 \left\| f(u^n_t(t)) \right\|_{(p+2)'}^{(p+2)'} \\
\leq \frac{1}{2} \left\| u^n(t) \right\|_{p+2}^{p+2} + C_8 \left( \left\| f(u^n_t(t)) \right\|_{2/((\alpha + 1))}^{2/(\alpha + 1)} + \left\| f(u^n_t(t)) \right\|_{(\alpha + 2)'}^{(\alpha + 2)'} \right) \\
\leq \frac{1}{2} \left\| u^n(t) \right\|_{p+2}^{p+2} + C_9 \left[ \left\| u^n_t(t) \right\|^2 + (f(u^n_t), u^n_t) \right], \quad (3.12) \]

\[
\left| (u^n, u^n_t) \right| \leq \frac{1}{4} \left\| u^n(t) \right\|^2 + \left\| u^n_t(t) \right\|^2, \quad (3.13) \]

where
\[
C_7 = \frac{p + 1}{p + 2} \left( \frac{2}{p + 2} \right)^{\frac{1}{p + 1}}, \quad C_8 = \frac{\alpha + 1}{2} \left( \frac{2}{p + 2} \right)^{\frac{1}{p + 1}}, \\
C_9 = C_8 \max\{C_4^{1/(\alpha + 1)}, C_4^{2/(\alpha + 1)}\}. \]

Integrating (3.11) over \((0, t)\), substituting (3.12), (3.13) into the resulting expression and using (3.10), one gets
$$\frac{1}{4}\|u^n(t)\|_{1,2}^2 + \int_0^t \left( I(u^n(\tau)) + \frac{1}{2} \|u^n(\tau)\|_{p+2}^{p+2} \right) d\tau$$

$$\leq \|u^n(t)\|^2 + (C_9 + 1) \int_0^t \left( (f(u^n_\tau), u^n) + \|u^n(\tau)\|^2 \right) d\tau + \|\varphi^n, \psi^n\| + \frac{1}{2} \|\varphi^n\|_{1,2}^2$$

$$\leq (4 + C_9) E_n(0) + \|\varphi^n\|_{1,2}^2, \quad t \in [0, T].$$

(3.14)

Since $2 \leq m + 2 < p + 2$, by (3.10) and (3.14) we have

$$\|u^n(t)\|_{m+2}^{m+2} \leq \frac{m + 2}{2} \|u^n(t)\|^{p+2} + \|u^n(t)\|_{p+2}^{p+2}$$

$$\leq C_{10} E_n(0) + 2(m+2) \|\varphi^n\|_{1,2}^2, \quad t \in [0, T],$$

(3.15)

where $C_{10} = \frac{m+2}{2} [4(4 + C_9) + 1/b_3]$. By the Gagliardo–Nirenberg inequality, (3.10), (3.15) and (2.7),

$$\|u^n(t)\|_{p+2}^{p+2} \leq C_* \|u^n(t)\|^{(p+2)(1-\delta)} \|\nabla u^n(t)\|^{(p+2)(\delta - (m+2))} \|\nabla u^n(t)\|_{m+2}^{m+2}$$

$$\leq C_* \|u^n(t)\|_{1, m+2}^{p-m} \|\nabla u^n(t)\|_{m+2}^{m+2}$$

$$\leq C_\cdot \tilde{E}(0) \|\nabla u^n(t)\|_{m+2}^{m+2}$$

$$\leq \frac{C_1}{2(C_5 + 1)} \|\nabla u^n(t)\|_{m+2}^{m+2}, \quad t \in [0, T],$$

(3.16)

where $\delta = \frac{(p-m)N}{(m+2)(p+2)}$. From (3.16) we see that $I(u^n(T)) > 0$, which violates the assumption. Hence, claim (3.8) is valid.

It follows from (3.8) that estimations (3.10), (3.14)–(3.16) hold for $t > 0$.

Define the operator $D : W_0^{1,m+2} \to W^{-1,(m+2)'}$,

$$(Du, v) = \sum_i (\sigma_i(u_{x_i}), v_{x_i}) \quad \text{for any } u, v \in W_0^{1,m+2}.$$  

(3.17)

By assumptions (i)–(iii) of Theorem 2.1, (3.10) and (3.14),

$$\left| (Du^n, v) \right| \leq \sum_i \|\sigma_i(u^n_{x_i}(t))\|_{m+2} \|v_{x_i}\|_{m+2}$$

$$\leq C \|\nabla u^n(t)\|_{m+2}^{m+1} \|\nabla v\|_{m+2}$$

$$\leq C \|v\|_{1,m+2},$$

(3.18)

$$\|Du^n(t)\|_{-1,(m+2)'} \leq C, \quad t > 0,$$

(3.19)

$$\|f(u^n_\tau(t))\|_{L^{(p+2)'}(Q_T)} \leq C_{14}^{\frac{1}{p+2}} \left( \int_0^T (f(u^n_\tau(t), u^n_\tau(t)) \, dt \right)^\frac{p+1}{p+2} \leq C,$$

(3.20)
where and in the sequel we denote by $C$ various positive constants independent of $\Omega$, $n$ and $t$.

If $\alpha \leq \frac{4}{N}$, then by the Gagliardo–Nirenberg inequality and (3.10),
\[
\begin{align*}
\|u^n_t(t)\|_{\alpha+2}^{\alpha+2} &\leq C \|u^n(t)\|^{\alpha} \|u^n(t)\|^{2-\frac{N\alpha}{2}} \|\nabla u^n(t)\|^{\frac{N\alpha}{2}} \leq C \|u^n(t)\|^{2}, \\
\int_0^T \|u^n_t(t)\|_{\alpha+2}^{\alpha+2} \, dt &\leq C \int_0^T \|u^n(t)\|_{1,2}^{2} \, dt \leq C.
\end{align*}
\] (3.21)

And if $\alpha > \frac{4}{N}$, then by assumption (ii) and (3.10),
\[
\int_0^T \|u^n_t(t)\|_{\alpha+2}^{\alpha+2} \, dt \leq \frac{1}{C_3} \int_0^T (f(u^n_t), u^n_t) \, dt \leq C.
\] (3.22)

Integrating (3.1) over $(0, t)$, one gets
\[
\begin{align*}
(u^n_t(t), w_j) + (\nabla u^n(t), \nabla w_j) + (u^n(t), w_j) + \sum_i \left( \int_0^t \sigma_i(u^n_{xi}(\tau)) \, d\tau, w_{jx_i} \right) \\
+ \left( \int_0^t [f(u^n_t(\tau)) - g(u^n(\tau))] \, d\tau, w_j \right) \\
= (\psi^n, w_j) + (\nabla \varphi^n, w_j) + (\varphi^n, w_j), \quad t > 0, \quad j = 1, \ldots, n,
\end{align*}
\] (3.23)

\[
u^n(0) = \varphi^n.
\] (3.24)

From (3.18)–(3.20) and Remark 1 we get that for any $T > 0$, the nonlinear terms in system of equations (3.23) are uniformly bounded on $[0, T]$, so the solution $u^n(t)$ of problem (3.23), (3.24) exists on $[0, T]$, and thus the solution $u^n(t)$ of problem (3.1), (3.2) exists on $[0, T]$ for each $n$.

By (3.10), (3.14)–(3.15), (3.18)–(3.22), we can extract a subsequence from $\{u^n\}$, still denoted by $\{u^n\}$, such that
\[
\begin{align*}
u^n &\to u \quad \text{in } L_\infty([0, T]; W^{1,m+2}_0) \cap L_\infty([0, T]; H^1) \cap L_\infty([0, T]; L^{p+2}) \quad \text{weak*}, \\
u^n_t &\to u_t \quad \text{in } L_\infty([0, T]; L^2) \cap L_\infty([0, T]; H^1) \cap L_{\alpha+2}(Q_T) \quad \text{weak*}, \\
D u^n &\to \xi \quad \text{in } L_\infty([0, T]; W^{-1,(m+2)'}) \quad \text{weak*}, \\
f(u^n_t) &\to \eta \quad \text{in } L_{(\alpha+2)'}(Q_T) \quad \text{weak*}, \\
g(u^n) &\to \gamma \quad \text{in } L_\infty([0, T]; L^{(p+2)'}) \quad \text{weak*}
\end{align*}
\] (3.25)–(3.29)
as $n \to \infty$. Repeating the similar arguments used in [20], we can easily prove that
\[
\begin{align*}
\eta &\equiv f(u_t), \\
\gamma &\equiv g(u), \\
\xi &\equiv D u,
\end{align*}
\] (3.30)
here we omit the process. Letting $n \to \infty$ in (3.23) one obtains, for any $\chi \in W^{1,m+2}_0$ and a.e. $t \in [0, T]$,

\[
(u_t(t), \chi) + (\nabla u(t), \nabla \chi) + (u(t), \chi) + \sum_i \left( \int_0^t \sigma_i(u_{x_i}(\tau)) d\tau, \chi_{x_i} \right) \\
+ \left( \int_0^t \left[ f(u_t(\tau)) - g(u(\tau)) \right] d\tau, \chi \right) \\
= (\psi, \chi) + (\nabla \varphi, \nabla \chi) + (\varphi, \chi).
\]

(3.31)

By (3.25)–(3.26),

\[
(u^n, w_j) \to (u, w_j), \quad (\nabla u^n, w_j) \to (\nabla u, w_j) \quad \text{in} \quad H^1[0, T] \quad \text{and in} \quad C[0, T] \quad \text{as} \quad n \to \infty,
\]

and hence by (3.3),

\[
u(0) = \varphi \quad \text{in} \quad H^1.
\]

(3.32)

Letting $t = 0$ in (3.31), and making use of (3.32) and the density of $W^{1,m+2}_0$ in $L^2$, one gets

\[
u_t(0) = \psi \quad \text{in} \quad L^2.
\]

(3.33)

Differentiating (3.31) with respect to $t$, we obtain that $u_{tt} \in L^1([0, T]; W^{-1,(m+2)'} \setminus H^1[0, T]$ and $u$ is a weak solution of problem (1.1), (2.4) on $[0, T]$.

Now, we discuss the asymptotic behavior of solutions of problem (1.1), (2.4).

It follows from (3.4) that

\[
\frac{d}{dt} E_n(t) = -(f(u^n_t), u^n_t) - \|u^n_t(t)\|_{1,2}^2 \leq 0, \\
\|u^n_t(t)\|_{1,2}^2 \leq -E_n'(t), \quad (f(u^n_t), u^n_t) \leq -E_n'(t), \quad t > 0.
\]

(3.34)

Substituting $w_j$ in (3.1) by $\frac{E_n^{r+1}}{r+1}(t)u^n(t)$ and integrating the resulting expression over $(S, T)$, one gets

\[
0 = \int_S^T \left[ E_n^{\frac{r+1}{2}}(\tau) \left( u^n_{tt}(\tau), u^n(\tau) \right) + (\nabla u^n, \nabla u^n) + (u^n_t, u^n) \\
+ \sum_i (\gamma_i(u^n_{x_i}), u^n_{x_i}) + (f(u^n_t), u^n) - (g(u^n), u^n) \right] d\tau \\
= \left[ E_n^{\frac{r+1}{2}}(\tau)(u^n_t, u^n) \right]_S^T - \frac{r+1}{2} \int_S^T E_n^{\frac{r-1}{2}}(\tau)E_n'(\tau)(u^n_t, u^n) d\tau \\
\quad - \int_S^T \left[ \frac{3}{2} \|u^n_t(\tau)\|^2 - E_n(\tau) \right] d\tau + \sum_i \int_S^T \left( \int_0^{u^n_{x_i}} \sigma_i(s) ds - \sigma_i(u^n_{x_i})u^n_{x_i} \right) dx
\]
\[
+ \int_\Omega \left( g(u^n)u^n - \int_0^u g(s) ds \right) dx - \frac{1}{2} \frac{d}{d\tau} \|u^n(\tau)\|_{1,2}^2 - (f(u^n_i), u^n) \right] d\tau,
\]
(3.35)

\[
\int_S^T E_n^{r+1}(\tau) d\tau = -\left[ E_n^{r+1}(\tau)(u^n_i, u^n) \right]_S^T + \frac{r + 1}{2} \int_S^T E_n^{r-1}(\tau) E_n'(\tau)(u^n_i, u^n) d\tau
\]

By (3.34),
\[
\int_S^T E_n^{r-1}(\tau)\|u^n_i(\tau)\|^2 d\tau \leq E_n^{r-1}(0) \int_S^T -E_n'(\tau) d\tau \leq CE_n^r(0)(E_n^1(0) + \|\varphi^n\|_{1,2})E_n(S).
\]
(3.39)

By assumption (i), (3.16) and (3.14),
\[
\int_S^T \left[ E_n^{r-1}(\tau) \sum_i \int_\Omega \left( \int_0^{u^n_i} \sigma_i(s) ds - \sigma_i(u^n_{x_i})u^n_{x_i} \right) dx \right] d\tau
\]
\[
\leq C \int_S^T E_n^{r-1}(\tau) \|\nabla u^n(\tau)\|_{m+2}^{m+2} d\tau
\]
\[ \leq C E_{n-\frac{3}{2}}(0) E_n(S) \int_S \mathcal{I}(u^n(\tau)) \, d\tau \]
\[ \leq C E_{n-\frac{3}{2}}(0) \left( E_n(0) + \left\| \phi_n^0 \right\|_{1,2}^2 \right) E_n(S). \] (3.40)

By assumption (iii) and (3.14),
\[ \left| \int_S E_{n-\frac{3}{2}}(\tau) \int_0^\tau \left( g(u^n) u^n - \int_0^\tau g(s) \, ds \right) \, dx \, d\tau \right| \]
\[ \leq C \int_S E_{n-\frac{3}{2}}(\tau) \left\| u^n(\tau) \right\|_{p+2} \, d\tau \]
\[ \leq C E_{n-\frac{3}{2}}(0) E_n(S) \int_S \left\| u^n(\tau) \right\|_{p+2} \, d\tau \]
\[ \leq C E_{n-\frac{3}{2}}(0) \left( E_n(0) + \left\| \phi_n^0 \right\|_{1,2}^2 \right) E_n(S), \] (3.41)

\[ \int_S E_{n-\frac{3}{2}}(\tau) \| u^n(\tau) \|_{1,2}^2 \]
\[ = \left[ E_{n-\frac{3}{2}}(\tau) \| u^n(\tau) \|_{1,2}^2 \right]_S - \frac{r-1}{2} \int_S \| u^n(\tau) \|_{1,2}^2 E_{n-\frac{3}{2}}(\tau) E'_n(\tau) \, d\tau \]
\[ \leq C E_{n-\frac{3}{2}}(0) \left( E_n(0) + \left\| \phi_n^0 \right\|_{1,2}^2 \right) E_n(S) \]
\[ + \frac{r-1}{2} E_{n-\frac{3}{2}}(0) \max_{\substack{S \leq t \leq T}} \| u^n(t) \|_{1,2}^2 \int_S E'_n(\tau) \, d\tau \]
\[ \leq C E_{n-\frac{3}{2}}(0) \left( E_n(0) + \left\| \phi_n^0 \right\|_{1,2}^2 \right) E_n(S). \] (3.42)

By assumption (ii), (3.34), the Young inequality, (3.10) and (3.14),
\[ \left| \int_S E_{n-\frac{3}{2}}(\tau) \left( f \left( u^n_t \right), u^n \right) \, d\tau \right| \]
\[ \leq \int_S E_{n-\frac{3}{2}}(\tau) \left\| u^n(\tau) \right\|_{\alpha+2} \left\| f \left( u^n_t(\tau) \right) \right\|_{(\alpha+2)} \, d\tau \]
\[ \leq C_4^{\frac{r+1}{2}} \int S^{r+1} \frac{r+1}{2} E_n^{\frac{r-1}{2}} (\tau) \| u^n(\tau) \|_{\alpha+2} ^{\frac{r+1}{\alpha+2}} d\tau \]

\[ \leq C_4^{\frac{r+1}{2}} \int S^{r+1} \frac{r+1}{2} E_n^{\frac{r-1}{2}} (\tau) E_n^{\frac{r-1}{2\alpha+2\gamma}} (\tau) \| u^n(\tau) \|_{\alpha+2} (-E_n'(\tau)) \frac{r+1}{\alpha+2} d\tau \]

\[ \leq \frac{1}{3} \int S^{r+1} \frac{r+1}{2} (\tau) d\tau + C \int S^{(r-1)(\alpha+2) - r-1} \frac{2(\alpha+1)}{2(\alpha+1)} (\tau) \| u^n(\tau) \|_{\alpha+2} (-E_n'(\tau)) d\tau \]

\[ \leq \frac{1}{3} \int S^{r+1} \frac{r+1}{2} (\tau) d\tau + CE_n^{(r-1)(\alpha+2) - r-1} \frac{2(\alpha+1)}{2(\alpha+1)} \alpha \| \varphi^n \|_{1,2} \frac{1}{\alpha+1} E(S), \quad (3.43) \]

where \( r \geq \frac{\alpha+3}{\alpha+1} \). Substituting (3.37)–(3.43) into (3.36), one obtains

\[ 2 \int S^{r+1} \frac{r+1}{2} (\tau) d\tau \leq C \left[ E_n^r (0) \left( E_n^1 (0) + \| \varphi^n \|_{1,2} \right) + E_n^{r-3} \left( 0 \right) \left( E_n (0) + \| \varphi^n \|_{1,2} \right) \right. \]

\[ + E_n \left( 2^{\alpha+1} \right) \left( 0 \right) \left( E_n (0) + \| \varphi^n \|_{1,2} \right) \frac{1}{\alpha+1} E(S). \quad (3.44) \]

Applying Lemma 1 to (3.44), one gets

\[ E_n(t) \leq C \left( \frac{r+1}{r-1} \right)^{\frac{2}{r+1}} \left[ E_n^r (0) \left( E_n^1 (0) + \| \varphi^n \|_{1,2} \right) + E_n^{r-3} (0) \left( E_n (0) + \| \varphi^n \|_{1,2} \right) \right. \]

\[ + E_n \left( 2^{\alpha+1} \right) (0) \left( E_n (0) + \| \varphi^n \|_{1,2} \right) \frac{1}{\alpha+1} E(S). \quad (3.45) \]

Since

\[ E_n(t) \geq \frac{1}{2} \left( \| u^n(t) \|_{m+2} + b_3 \left( \| \nabla u^n(t) \|_{m+2} + \| u^n(t) \|_{p+2} \right) \right), \quad (3.46) \]

letting \( n \to \infty \) in (3.45) and (3.46), by the sequential lower semi-continuity of the norm of the weak* limit, one gets

\[ \left\| u_t(t) \right\|_{1} + \left\| \nabla u(t) \right\|_{m+2} + \left\| u(t) \right\|_{p+2} \]

\[ \leq \liminf_{n \to \infty} \left[ \left\| u^n_t(t) \right\|_{1} + \left\| \nabla u^n(t) \right\|_{m+2} + \left\| u^n(t) \right\|_{p+2} \right] \]
4. Global weak solutions to the Cauchy problem

Proof of Theorem 2.2. We take a sequence \( \{L_s\} \), where \( L_s > 1 \) are real numbers, and \( L_s \to +\infty \) as \( s \to \infty \). Let \( \Omega_s = (-L_s, L_s)^N \). For each \( s \), we construct a potential well

\[
W_s = \left\{ u \in W^{1,m+2}_0(\Omega_s) \mid I_s(u) = C_1 \| \nabla u \|_{L^{m+2}(\Omega_s)}^{m+2} - (C_5 + 1) \| u \|_{L^{p+2}(\Omega_s)}^{p+2} > 0 \right\} \cup \{0\}
\]

and the periodic functions \( \varphi_s \) and \( \psi_s \), with \( \varphi_s|_{\partial \Omega_s} = \psi_s|_{\partial \Omega_s} = 0 \), and

(i) \( \varphi_s(x) = \varphi_s(x + 2L_s e_i), \psi_s(x) = \psi_s(x + 2L_s e_i), x \in \mathbb{R}^N, i = 1, \ldots, N, \)
(ii) \( \varphi_s(x) = u_0(x), \psi_s(x) = u_1(x) \) on \( \Omega_s^\pm = (-L_s + 1, L_s - 1)^N \) and

\[
\| \varphi_s \|_{W^{1,m+2}(\Omega_s)} \leqslant \| u_0 \|_{W^{1,m+2}(\mathbb{R}^N)}, \quad \| \varphi_s \|_{H^1(\Omega_s)} \leqslant \| u_0 \|_{H^1(\mathbb{R}^N)},
\]
\[
\| \varphi_s \|_{L^{p+2}(\Omega_s)} \leqslant \| u_0 \|_{L^{p+2}(\mathbb{R}^N)}, \quad \| \psi_s \|_{L^2(\Omega_s)} \leqslant \| u_1 \|_{L^2(\mathbb{R}^N)}.
\]

Let

\[
\tilde{\varphi}_s(x) = \begin{cases} \varphi_s(x), & x \in \Omega_s, \\ 0, & x \in \mathbb{R}^N - \Omega_s, \end{cases}
\]
\[
\tilde{\psi}_s(x) = \begin{cases} \psi_s(x), & x \in \Omega_s, \\ 0, & x \in \mathbb{R}^N - \Omega_s. \end{cases}
\]

By (4.2),

\[
\| \tilde{\varphi}_s - u_0 \|_{W^{1,m+2}(\mathbb{R}^N)} \leqslant 2 \| u_0 \|_{W^{1,m+2}(\mathbb{R}^N)}.
\]

Hence,

\[
\| \tilde{\varphi}_s - u_0 \|_{W^{1,m+2}(\mathbb{R}^N)} = \| \tilde{\varphi}_s - u_0 \|_{W^{1,m+2}(\mathbb{R}^N - \Omega_s^*)} \to 0
\]
as \( s \to \infty \), i.e., \( \tilde{\varphi}_s \to u_0 \) in \( W^{1,m+2}(\mathbb{R}^N) \). Similarly,

\[
\tilde{\varphi}_s \to u_0 \quad \text{in } H^1(\mathbb{R}^N) \cap L^{p+2}(\mathbb{R}^N), \quad \tilde{\psi}_s \to u_1 \quad \text{in } L^2(\mathbb{R}^N)
\]
as \( s \to \infty \). (4.6)

It follows from (4.5)–(4.6) that

\[
I_s(\varphi_s) \to I(u_0)(>0) \quad \text{as } s \to \infty.
\]

(4.7)

By integral mean value theorem, the Hölder inequality and (4.2),
\begin{align*}
\int_{\Omega_s} \int_{\Omega_s} \sigma_i(\tau) d\tau d\tau - \int_{\Omega_s} \int_{\Omega_s} \sigma_i(\tau) d\tau d\tau
&= \int_{\Omega_s} \int_{\Omega_s} \sigma_i(\xi_i^s) (\bar{\varphi}_{xxi} - u_{0xi}) d\tau d\tau
&\leq \|\sigma_i(\xi_i^s)\|_{L_{m+2}(\mathbb{R}^N)} \|\bar{\varphi}_{xxi} - u_{0xi}\|_{L_{m+2}(\mathbb{R}^N)}
&\leq C \|\xi_i^s\|_{L_{m+2}(\mathbb{R}^N)} \|\bar{\varphi}_{xxi} - u_{0xi}\|_{L_{m+2}(\mathbb{R}^N)} \rightarrow 0, \tag{4.8}
\end{align*}

where \(\xi_i^s = u_{0xi} + \theta_i^s \bar{\varphi}_{xxi}, \) \(0 < \theta_i^s < 1, i = 1, \ldots, N.\) Similarly,

\begin{align*}
\int_{\Omega_s} \int_{\Omega_s} g(\tau) d\tau d\tau - \int_{\Omega_s} \int_{\Omega_s} g(\tau) d\tau d\tau
&\rightarrow 0 \tag{4.9}
\end{align*}

as \(s \rightarrow \infty.\) It follows from (4.6), (4.8)–(4.9) that

\begin{align*}
E_s(0) \rightarrow A(0), \quad E_s^0(0) \rightarrow \tilde{A}(0) \tag{4.10}
\end{align*}

as \(s \rightarrow \infty,\) where

\begin{align*}
\tilde{E}_s(0) &= \left[2(C_{10} + 1/b_3) E_s(0) + 2(m + 2)\|\varphi_s\|_{H^1(\Omega_s)}^2 \right]^{\frac{p-m}{2}}
E_s(0) &= \frac{1}{2} \|\psi_s\|^2_{L^2(\Omega_s)} + \sum_i \int_{\Omega_s} \int_{\Omega_s} \sigma_i(\tau) d\tau d\tau - \int_{\Omega_s} \int_{\Omega_s} g(\tau) d\tau d\tau, \tag{4.11}
\end{align*}

\(A(0)\) and \(\tilde{A}(0)\) are shown in (2.11). By (4.7) and (4.10), without loss of generality we assume that \(I_{s}(\varphi_s) > 0\) for all \(s,\) and (2.10) holds for all \(E_s(0).\)

Repeating the arguments of Theorem 2.1, for each \(s,\) we get a weak solution \(u^s\) of corresponding problem (1.1), (2.4) (substituting \(L, \varphi \) and \(\psi \) there by \(L_s, \varphi_s \) and \(\psi_s,\) respectively), which is a weak* limit of a sequence \(\{u^{n_s}\}\), where \(u^{n_s}\) are the solutions of problem (3.1), (3.2) (substituting \(\Omega, u^n, \varphi^n, \psi^n, \varphi \) and \(\psi \) there by \(\Omega_s, u^{n_s}, \varphi^{n_s}, \psi^{n_s}, \varphi_s \) and \(\psi_s,\) respectively). And \(u^s\) has asymptotic behavior (3.47) (substituting \(\Omega, u^n, \varphi \) and \(E(0)\) there by \(\Omega_s, u^{n_s}, u^s, \varphi_s \) and \(E_s(0),\) respectively). By the sequential lower semi-continuity of the norm we conclude that inequalities (3.10), (3.14)–(3.15) still hold for \(u^{s},\) and so do inequalities (3.18)–(3.22) (substituting \(u^n\) and \(\Omega\) there by \(u^s\) and \(\Omega_s,\) respectively). Therefore, with the same method used in the proof of Theorem 2.1 we can extract a subsequence from \(\{u^s\},\) still denoted by \(\{u^s\},\) such that, for any \(\Omega_L \subset (-L, L)^N\) and \(Q_T = \Omega_L \times (0, T),\)

\begin{align*}
&u^s \rightarrow u \quad \text{in } L^\infty([0, T]; W^{1,m+2}(\Omega_L)) \cap L^\infty([0, T]; H^1(\Omega_L))
&\quad \cap L^\infty([0, T]; L^{p+2}(\Omega_L)) \quad \text{weak*};
&u^t \rightarrow u_t \quad \text{in } L^\infty([0, T]; L^2(\Omega_L)) \cap L^2([0, T]; H^1(\Omega_L)) \cap L^\infty(Q_T) \quad \text{weak*};
&Du^s \rightarrow Du \quad \text{in } L^\infty([0, T]; W^{-1,(m+2)'}(\Omega_L)) \quad \text{weak*};
\end{align*}
\[ f(u^t) \to f(u) \quad \text{in } L_{(\alpha+2)'(Q_T)} \quad \text{weak}^*; \]
\[ g(u^t) \to g(u) \quad \text{in } L_\infty([0, T]; L_{(p+2)'(\Omega_L)}) \quad \text{weak}^* \] (4.12)
as \( s \to \infty \), and inequalities (3.10), (3.14)–(3.15), (3.21)–(3.22) also hold for the limiting function \( u \). By the arbitrariness of \( L \) and the boundedness of the norm of \( u \) (see (3.10), (3.14)–(3.15) and (3.21)–(3.22)) we see that
\[ u \in U \quad \text{with } u_t \in L_{\alpha+2}(\tilde{Q}_T), \] (4.13)
and it follows from (3.47) that
\[
\|u_t(t)\|_{L_2(\mathbb{R}^N)}^2 + \|\nabla u(t)\|_{L_{m+2}(\mathbb{R}^N)}^{m+2} + \|u(t)\|_{L_{p+2}(\mathbb{R}^N)}^{p+2} \\
\leq \liminf_{s \to \infty} \left[ \|u^s(t)\|_{L_2(\Omega_s)}^2 + \|\nabla u^s(t)\|_{L_{m+2}(\Omega_s)}^{m+2} + \|u^s(t)\|_{L_{p+2}(\Omega_s)}^{p+2} \right] \\
\leq C \left[ \frac{r+1}{r-1} \right]^{\frac{2}{r-1}} \left[ A^\frac{2}{r^2}(0)(A^\frac{1}{2}(0) + \|u_0\|_{H^1(\mathbb{R}^N)})^2 + A^\frac{2}{r^2}(0)(A(0) + \|u_0\|_{H^1(\mathbb{R}^N)})^2 \\
+ A \left( \frac{(r-1)^2}{2(r+1)} \right) (A(0) + \|u_0\|_{H^1(\mathbb{R}^N)}) \right]^{\frac{2}{r^2}t^{1-r}} \quad t > 0, \quad (4.14)\]
i.e., (1.3) holds. For any \( \chi \in W^{1,m+2}(\mathbb{R}^N) \), supp \( \chi \) is a bounded set in \( \mathbb{R}^N \), there must be a \( L > 0 \) such that supp \( \chi \subset \Omega_L \) and \( \chi \in W^{1,m+2}(\Omega_L) \). Substituting \( u^n, w_j, \varphi^n \) and \( \psi^n \) in (3.23) by \( u^s, \chi, \varphi_s \) and \( \psi_s \), respectively, and letting \( s \to \infty \), one gets
\[
(u_t(t), \chi) + (\nabla u(t), \nabla \chi) + (u(t), \chi) + \sum_i \left( \int_0^t \sigma_i(u_{x_i}(\tau))d\tau, \chi_{x_i} \right) \\
+ \left( \int_0^t [f(u_t(\tau)) - g(u(\tau))]d\tau, \chi \right) \\
= (u_1, \chi) + (\nabla u_0, \nabla \chi) + (u_0, \chi). \quad (4.15)\]
By (4.12), for any \( \chi \in C_0^\infty(\mathbb{R}^N) \),
\[
(u^s, \chi) \to (u, \chi), \quad (\nabla u^s, \chi) \to (\nabla u, \chi) \quad \text{in } H^1[0, T] \text{ and in } C[0, T] \] (4.16)
as \( s \to \infty \). Therefore, by (4.6),
\[
(u(0), \chi) = (u_0, \chi), \quad (\nabla u(0), \chi) = (\nabla u_0, \chi). \quad (4.17)\]
Since \( C_0^\infty(\mathbb{R}^N) \) is dense in \( L_2(\mathbb{R}^N) \), by the continuity of inner-product in \( L_2 \) one gets that (4.17) holds for arbitrary \( \chi \in L_2(\mathbb{R}^N) \), and hence
\[
u(0) = u_0 \quad \text{in } H^1(\mathbb{R}^N). \quad (4.18)\]
For any \( \chi \in C_0^\infty(\mathbb{R}^N) \), letting \( t = 0 \) in (4.15) and making use of (4.18) one gets
\[
(u_t(0), \chi) = (u_1, \chi), \quad (4.19)\]
and by the same method used above one obtains
\[ u_t(0) = u_1 \text{ in } L_2(\mathbb{R}^N). \] (4.20)

Differentiating (4.15) with respect to \( t \) we see that \( u_{tt} \in L_1([0, T]; W^{-1,(m+2)}(\mathbb{R}^N)) \) and the limiting function \( u \) is a weak solution of problem (1.1), (1.2). And the solution \( u \) features asymptotic behavior (1.3). Theorem 2.2 is proved. \( \square \)

5. The case in one dimension

In the case of space dimension \( N = 1 \), the weak solutions of Cauchy problem (1.1), (1.2) can be regularized.

In this section, we still use the notations in (2.0), where \( \Omega = (-L, L) \). Moreover, \( \| \cdot \|_{\mathcal{C}^k} = \| \cdot \|_{\mathcal{C}^k(\Omega)} \).

Proof of Theorem 2.3. We still begin with the approximate solutions \( u^n \) of the form (3.0) of the PBVP of Eq. (1.4) (for brevity, we still take \( \lambda = 1 \))

\[ u(x, t) = u(x + 2L, t), \quad x \in \mathbb{R}, \quad t > 0, \] (5.1)

\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}, \] (5.2)

where \( \varphi(x) = \varphi(x + 2L), \quad \psi(x) = \psi(x + 2L), \quad x \in \mathbb{R}, \) \( w_j \) are eigenfunctions of the eigenvalue problem

\[ w_{xx} + \mu w = 0, \quad w(x) = w(x + 2L), \quad x \in \mathbb{R}, \] (5.3)

corresponding to eigenvalues \( \mu_j \) (\( j = 1, \ldots, \)), and \( \{w_j\}_{j=1}^\infty \) is an orthogonal basis in \( H^2 \), the coefficients \( \{T_{jn}\}_{j=1}^n \) satisfy \( T_{jn}(t) = (u^n(t), w_j) \) with

\[ (u^n_{tt} - u^n_{xxt} + u^n_t - \sigma(u^n)_x + f(u^n) - g(u^n), w_j) = 0, \] \( t > 0, \quad j = 1, \ldots, n, \) (5.4)

\[ u^n(0) = \varphi^n \to \varphi \quad \text{in } H^2, \quad u^n_t(0) = \psi^n \to \psi \quad \text{in } H^2. \] (5.5)

From the proof of Theorem 2.1 we deduce that estimations (3.10), (3.14)–(3.16) hold for \( t > 0, \) i.e.,

\[ \| u^n_t(t) \|_1^2 + \| u^n(t) \|_1^2 + \| u^n(t) \|_{1,m+2} + \| u^n(t) \|_{p+2}^2 \]
\[ + \int_0^t \left( \| u^n_t(\tau) \|_{1,2}^2 + (f(u^n_t), u^n_t) + \| \nabla u^n(\tau) \|_{m+2}^2 + \| u^n(\tau) \|_{p+2}^2 \right) d\tau \]
\[ \leq C, \quad t > 0. \] (5.6)

By the Gagliardo–Nirenberg inequality and (5.6),

\[ \| u^n(t) \|_C \leq C \| u^n(t) \|^{1/2} \| u^n(t) \|^{1/2} \leq C, \quad t > 0. \] (5.7)

Substituting \( w_j \) in (5.4) by \( -u^n_{xx} \) yields
\[
\begin{align*}
\frac{d}{dt} \left[ \frac{1}{2} \left( \| u^n_t(t) \|^2 + \| u^n_{xx}(t) \|^2 \right) - (u^n_t, u^n_{xx}) \right] + (\sigma'(u^n_x)u^n_{xx}, u^n_{xx}) \\
\leq \| u^n_{tt}(t) \|^2 + (f(u^n_t), u^n_{xx}) - (g(u^n), u^n_{xx}). & \quad (5.8)
\end{align*}
\]

(3.4) + \varepsilon \times (5.8) one gets

\[
\begin{align*}
\frac{d}{dt} \left[ \frac{1}{2} \| u^n_t(t) \|^2 + \int_\Omega \int_0^t \sigma(s) \, ds \, dx - \int_\Omega \int_0^t g(s) \, ds \, dx \right. \\
\left. + \frac{\varepsilon}{2} \left( \| u^n_{xx}(t) \|^2 + \| u^n_t(t) \|^2 \right) - \varepsilon (u^n_t, u^n_{xx}) \right] + \| u^n_t(t) \|^2_{1,2} + (f(u^n_t), u^n_t) \\
\leq \varepsilon \left[ \| u^n_{xx}(t) \|^2 + \| u^n_{xx}(t) \|^2 + |(f(u^n_t), u^n_{xx})| + \| g(u^n(t)) \|^2 \right]. & \quad (5.9)
\end{align*}
\]

1. If condition (ii_1) holds, then by the Hölder inequality, the Gagliardo–Nirenberg inequality and (5.6),

\[
| (f(u^n_t), u^n_{xx}) | \leq \| u^n_{xx}(t) \|^2 + C \| u^n_t(t) \|^{2\alpha_e+2} \\
\leq \| u^n_{xx}(t) \|^2 + C(\| u^n_t(t) \|^{\alpha+2} \| u^n_{xx}(t) \|^\alpha) \\
\leq \| u^n_{xx}(t) \|^2 + C(\| u^n_{xx}(t) \|^2 + 1), \quad t > 0. & \quad (5.10)
\]

2. If condition (ii_2) holds, then integrating by parts and using the same method one obtains

\[
| (f(u^n_t), u^n_{xx}) | \leq \left\| u^n_{xx}(t) \right\|_{m+2} \left\| f'(u^n_t) \right\|_{(m+2)'} \\
\leq C \left\| u^n_{xx}(t) \right\| \left( \left\| u^n_t(t) \right\|^{2\alpha\left(m+2\right)/m} + \left\| u^n_{xx}(t) \right\|^{\left(\alpha+2\right)\left(m+2\right)/m} \right) \\
\leq C \left\| u^n_{xx}(t) \right\| \left( \left\| u^n_t(t) \right\|^{\alpha\left(m+2\right)/m} + \left\| u^n_{xx}(t) \right\|^{\left(\alpha+2\right)\left(m+2\right)/m} \right) \\
\leq C \left\| u^n_{xx}(t) \right\| \left( \left\| u^n_{xx}(t) \right\|^{\alpha\left(m+2\right)/m} + \left\| u^n_{xx}(t) \right\|^{\left(\alpha+2\right)\left(m+2\right)/m} \right) \\
\leq C \left( \| u^n_{xx}(t) \|^2 + 1 \right), \quad t > 0. & \quad (5.11)
\]

By assumption (iii) of Theorem 2.3, (5.6) and (5.7),

\[
\left\| g(u^n(t)) \right\|^2 \leq C \| u^n(t) \|^p \left\| u^n(t) \right\|^{p+2}_{\infty} \leq C, \quad t > 0. & \quad (5.12)
\]

Noting that \( (u^n_t, u^n_{xx}) \leq \frac{1}{4} \| u^n_{xx}(t) \|^2 + \| u^n_t(t) \|^2 \), substituting (5.10) (if condition (ii_1) holds), (5.11) (if condition (ii_2) holds), and (5.12) into (5.9), integrating the resulting expression over \((0, t)\), exploiting (3.4), (3.9) (with \( N = 1 \)) and taking \( \varepsilon : 0 < \varepsilon \leq \frac{1}{2C^2+C} \), one gets

\[
\left( \frac{1}{2} - \varepsilon \right) \left\| u^n_t(t) \right\|^2 + \frac{\varepsilon}{4} \left( \left\| u^n_t(t) \right\|^2 + \left\| u^n_{xx}(t) \right\|^2 \right) + b_3 \left( \left\| u^n_{xx}(t) \right\|^{m+2}_{m+2} + \left\| u^n(t) \right\|^{p+2}_{p+2} \right)
\]
Applying the Gronwall inequality to (5.13) and exploiting (5.6) one obtains
\[
\|u^n(t)\|_{L^2} \leq C(T), \quad \|u^n(t)\|_{C^1} \leq C(T), \quad t \in [0, T],
\]  
(5.14)
where and in the sequel we denote by \(C(T)\) various positive constants depending only on \(T\). Substituting \(w_j\) in (5.4) by \(-u^n_{xxx}\), integrating by parts and using (5.14), one obtains:

1. If condition (ii1) holds, then by (5.10), (5.12) and (5.14),
\[
\frac{1}{2} \frac{d}{dt} \|u^n_{xt}(t)\|^2 + \|u^n_{xt}(t)\|^2 + \|u^n_{xxt}(t)\|^2 \leq \frac{1}{2} \|u^n_{xxt}(t)\|^2 + C \left( \|f(u^n(t))\|^2 + \|\sigma'(u^n_x(t))\|_\infty \right) \|u^n_{xx}(t)\|^2 \leq \frac{1}{2} \|u^n_{xxt}(t)\|^2 + C(T)(\|u^n_{xt}(t)\|^2 + 1), \quad t \in [0, T].
\]  
(5.15)

2. If condition (ii2) holds, then similarly,
\[
\frac{1}{2} \frac{d}{dt} \|u^n_{xt}(t)\|^2 + \|u^n_{xt}(t)\|^2 + \|u^n_{xxt}(t)\|^2 \leq \frac{1}{2} \|u^n_{xxt}(t)\|^2 + C \left( \|g(u^n(t))\|^2 + \|\sigma'(u^n_x(t))\|_\infty \right) \|u^n_{xx}(t)\|^2 \leq \frac{1}{2} \|u^n_{xxt}(t)\|^2 + C(T), \quad t \in [0, T].
\]  
(5.16)

Integrating (5.15) and (5.16) over \((0, t)\) and using (5.6), one gets
\[
\|u^n_{t}(t)\|_{L^2} + \int_0^t \|u^n_{\tau}(\tau)\|_{L^2}^2 d\tau \leq C(T),
\]
\[
\|u^n_{t}(t)\|_{C^1} \leq C(T), \quad t \in [0, T].
\]  
(5.17)
By (5.14) and (5.17), with the standard method we easily get
\[
\|u^n\|_{W^{1,\infty}(0,T;H^2)} + \|u^n\|_{W^{2,\infty}(0,T;L^1)} + \|u^n\|_{H^2(0,T);H^1} \leq C(T).
\]  
(5.18)
By (5.18), we can extract a subsequence from \(\{u^n\}\), still denoted by \(\{u^n\}\), such that
\[
\begin{align*}
\text{\(u^n_{tk}\)} & \to \text{\(u_{tk}\)} \quad \text{in \(L_\infty([0,T];H^2)\)} \quad \text{\(\text{weak}^*\);}
\text{\(u^n_{tt}\)} & \to \text{\(u_{tt}\)} \quad \text{in \(L_\infty([0,T];L^2) \cap L^2([0,T];H^1)\)} \quad \text{\(\text{weak}^*\);}
\text{\(u^n\)} & \to \text{\(u\)}, \quad \text{\(u^n_x\)} \to \text{\(u_x\)}, \quad \text{\(u^n_t\)} \to \text{\(u_t\)} \quad \text{in \(C(\bar{Q}_T)\)}
\end{align*}
\]  
(5.19)
as \(n \to \infty\), where \(u_{tk} = \frac{d^k u}{d t^k}, \ k = 0, 1\). From (5.19), the continuity of \(\sigma, f\) and \(g\) and the Lebesgue dominated convergence theorem we have, for any \(t \in [0, T]\),
\[
\begin{align*}
f(u^n_t) & \to f(u_t), \quad g(u^n) \to g(u), \quad \sigma(u^n_x) \to \sigma(u_x) \quad \text{in \(L^2\)}
\end{align*}
\]  
(5.20)
as $n \to \infty$. Letting $n \to \infty$ in (5.4), we deduce from the density of $\{w_j\}_{k=1}^{\infty}$ in $L_2$ that the limiting function

$$u \in W^{1,\infty}([0, T]; H^2) \cap W^{2,\infty}([0, T]; L_2) \cap H^2([0, T]; H^1)$$

and $u$ is a generalized solution of problem (1.4), (5.1)–(5.2).

Now, we restrict our attention to Cauchy problem (1.4), (1.2), with $N = 1$.

We still take a sequence $\{L_s\}$, and construct the periodic functions $\varphi_s, \psi_s$ satisfying conditions (i) and (ii) in the proof of Theorem 2.2, with $N = 1$. Moreover,

$$\|\varphi_s\|_{H^2(\Omega_s)} \leqslant \|u_0\|_{H^2(\mathbf{R})}, \quad \|\psi_s\|_{H^2(\Omega_s)} \leqslant \|u_1\|_{H^2(\mathbf{R})}. \quad (5.21)$$

Similarly, (4.5)–(4.10) hold, with $N = 1$, and

$$\varphi_s \to u_0 \text{ in } H^2(\mathbf{R}), \quad \psi_s \to u_1 \text{ in } H^2(\mathbf{R}) \text{ as } s \to \infty. \quad (5.22)$$

As shown above, for each $s$, we get a generalized solution $u^s$ of corresponding problem (1.4), (5.1)–(5.2) (substituting $L_s, \varphi_s$ and $\psi_s$, respectively), which is a weak* limit of a sequence $\{u^{n_s}\}$, where $u^{n_s}$ are the solutions of corresponding problem (5.4), (5.5) (substituting $u^n, \varphi^n, \psi^n, \varphi$ and $\psi$ there by $u^{n_s}, \varphi^{n_s}, \psi^{n_s}, \varphi_s$ and $\psi_s$, respectively). By the sequential lower semi-continuity of the norm we see that inequality (5.18) still holds for $u^s$. Therefore, we can select a subsequence from $\{u^s\}$, still denoted by $\{u^s\}$, such that for any $\Omega_L = (-L, L)$ and $Q_T = \Omega_L \times (0, T)$, (5.19)–(5.20) hold (substituting $u^n$ there by $u^s$). By the arbitrariness of $L$ and the boundedness of the norm of $u^s$ (see (5.18)), we see that the limiting function

$$u \in W^{1,\infty}([0, T]; H^2(\mathbf{R})) \cap W^{2,\infty}([0, T]; L_2(\mathbf{R})) \cap H^2([0, T]; H^1(\mathbf{R})). \quad (5.23)$$

And $u$ is a generalized solution of problem (1.4), (1.2), with $N = 1$.

By standard method we easily get the uniqueness of the generalized solution. Here we omit the process. The case (1) of Theorem 2.3 is proved.

Under the basis of estimation (5.18), by standard method we easily get conclusion (2) of Theorem 2.3. Here we omit the process. Theorem 2.3 is proved. □

**Example.** In Eq. (1.1), let

$$\sigma_i(s) = |s|^{m_i}s \quad (i = 1, \ldots, N), \quad f(s) = |s|^a s, \quad g(s) = |s|^p s,$$

where $m, a$ and $p$ are nonnegative real numbers. Obviously, $\sigma_i \ (i = 1, \ldots, N), f$ and $g$ belongs to $C(\mathbf{R})$, and Eq. (1.1) becomes

$$u_{tt} - \Delta u_t + \lambda u_t - \sum_i \frac{\partial}{\partial x_i}(|u_{x_i}|^{m_i}u_{x_i}) + |u_t|^a u_t = |u|^p u. \quad (5.24)$$

In particular, when the space dimension $N = 1$, Eq. (5.24) becomes

$$u_{tt} - u_{xxt} + \lambda u_t - \frac{\partial}{\partial x}(|u_x|^{m}u_x) + |u_t|^a u_t = |u|^p u. \quad (5.25)$$

1. If $\lambda \geqslant 0, \alpha \geqslant \max\{m, p\}$, and if $m + 2 < N$, also $\alpha + 2 < \frac{N(m+2)}{N-m-2}$, and the initial data $u_0 \in W^{1,m+2}(\mathbf{R}^N) \cap H^1(\mathbf{R}^N) \cap L_{p+2}(\mathbf{R}^N), u_1 \in L_2(\mathbf{R}^N)$, then for any $T > 0$, Cauchy problem (5.24), (1.2) admits a weak solution $u$ on $[0, T]$. See [23].
2. If $\lambda = 0$, $p = \frac{2\alpha}{\lambda_{\alpha}}$, $m + 1 = \alpha < \min\{1, m - 1\}$, and the initial data $u_0 \in W^{1, m+2}(R^N) \cap H^1(R^N) \cap L^{p+2}(R^N)$, $u_1 \in L^2(R^N)$ such that the initial energy $A(0) = \frac{1}{2} \|u_1\|^2_{L^2(R^N)} + \frac{1}{m+2} \|\nabla u_0\|^m_{L^{m+2}(R^N)} - \frac{1}{2} \|u_0\|_{L^{p+2}(R^N)} < 0$.

Then any weak solution $u$ of Cauchy problem (5.24), (1.2) blows up in finite time $\tilde{T}^-$, i.e.

$$\|u(t)\|^2_{L^2(R^N)} + \int_0^t \|\nabla u(s)\|^2_{L^2(R^N)} ds \to +\infty$$

as $t \to \tilde{T}^-$. See [23].

3. If $\lambda > 0$, $p > \alpha \geq \frac{p}{p+2}, p \geq m + \frac{(m+2)^2}{N}$, and if $m + 2 < N$, also $p + 2 < \frac{N(m+2)}{N-m-2}$, then a direct verification shows that conditions (i)–(iii) of Theorem 2.1 hold, where $C_1 = \cdots = C_5 = 1$. Hence, if the initial data $u_0 \in W \cap H^1(R^N), u_1 \in L^2(R^N)$ such that (2.10) holds, where

$$A(0) = \frac{1}{2} \|u_1\|^2_{L^2(R^N)} + \frac{1}{m+2} \|\nabla u_0\|^m_{L^{m+2}(R^N)} - \frac{1}{p+2} \|u_0\|^p_{L^{p+2}(R^N)}, \quad (5.26)$$

and

$$b_3 = \frac{3}{2} \left( \frac{1}{m+2} - \frac{1}{2(p+2)} \right),$$

$$C_{10} = \frac{m+2}{2} \left[ 4 \left( \frac{\alpha+1}{2} \left( \frac{2}{p+2} \right)^{\frac{1}{m+1}} + 4 \right) + 1/b_3 \right],$$

then by Theorem 2.2, for any $T > 0$, Cauchy problem (5.24), (1.2) admits a weak solution $u$ on $[0, T]$. And $u$ features asymptotic behavior (1.3).

4. In the case of space dimension $N = 1$, $\alpha > 0$. If $p \geq \alpha \geq \frac{p}{p+2}, p \geq m + (m+2)^2$, and $0 < \alpha \leq 2 + \frac{2}{m+2}$, then a simple verification shows that conditions (i), (ii2), (iii) of Theorem 2.3 hold, where $C_1 = \cdots = C_5 = 1, C_6 = \alpha + 1$.

(1) If the initial data $u_0 \in W^\ast, u_1 \in H^2(R)$ such that (2.10) holds, where $A(0)$ is shown in (5.26), with $N = 1$, then by Theorem 2.3, for any $T > 0$, Cauchy problem (5.25), (1.2), with $N = 1$, admits a unique generalized solution $u$ on $[0, T]$, with $u \in W^{1,\infty}([0, T]; H^2(R)) \cap W^{2,\infty}([0, T]; L^2(R)) \cap H^2([0, T]; H^1(R))$. And by Theorem 2.2, $u$ features asymptotic behavior (1.3).

(2) If $m \geq 2, \alpha, p > 1$, then $\sigma \in C^3(R), \ f, g \in C^2(R)$. And if the initial data $u_0 \in W^\ast \cap H^4(R), u_1 \in H^4(R)$, then by Theorem 2.3, for any $T > 0$, Cauchy problem (5.25), (1.2), with $N = 1$, admits a unique classical solution $u$ on $[0, T]$, with $u \in H^3([0, T]; H^1(R)) \cap H^2([0, T]; H^3(R))$. Moreover, $u$ features asymptotic behavior (1.3).

References


[18] Y. Yamada, Some remarks on the equation \( Y_{tt} - \sigma(Y_x)Y_{xx} - Y_{xxt} = f \), Osaka J. Math. 17 (1980) 303–323.


