Note

Planar Graphs with Square or Cube Root Are Four-Colorable

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A square (cube) coloring of a graph is an assignment of colors to its points so that no two points of distance less than or equal to two (three) get the same color. We prove that planar graphs with a square or cube root are four colorable using a square (cube) coloring of their square (cube) root.

1. Introduction

In general, we adhere to the terminology in Harary’s book [1] with the following additions. A square (cube) coloring of a graph $G$ is an assignment of colors to its points so that no two points of distance less than or equal to two (three) get the same color. Thus in a square coloring, for each point $v$, $v$ and its neighbors get distinct colors. In this paper we prove that planar graphs with a square or cube root are 4-colorable.

2. The Theorem

Theorem. If $G$ is a planar graph with a square root $H$, then $G$ is 4-colorable.

Proof. Coloring $G$ with four colors is same as square coloring $H$ with four colors. We will first show that each block $B$ of $H$ having at least three points together with the points adjacent to points of $B$ can be square colored with four colors so that the points in a constituent star get a prescribed coloring. Then it is shown that starting from any point $v$ of $H$, any square coloring of $v$ and its neighbors is extensible to the whole graph $H$.

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Since $H^2$ is planar, by a theorem in [2],

(a) $\Delta(H) \leq 3$.

(b) every block of $G$ with more than four points is a cycle of even length, and

(c) $H$ does not have three mutually adjacent cutpoints.

A block $B$ of $H$ with at least three points together with points adjacent to points of $B$ can be square colored with four colors as follows:

If $B$ is $C_3$ give colors 1, 2, and 3 to the three vertices. If $B$ is $K_4$ or $K_4 - x$ for some line $x$, give colors 1, 2, 3, and 4 to the four vertices. In all other cases $B = C_n$ for even $n$. If $n$ is a multiple of 4, color the vertices of $C_n$ cyclically with colors 1, 2, 3, and 4. If $n = 4r + 2$, then color the first $4r$ points cyclically with colors 1, 2, 3, and 4. Complete the cycle by coloring the remaining two vertices with colors 2 and 3, respectively. Color the point $u_i$ (if any) adjacent to $v_i$ of $B$ with the color other than those given to $v_i$ and points adjacent to $v_i$ in $B$. This is possible by (a). Since the four colors can be permuted among themselves, it is possible to complete the square coloring above even if the vertices of a constituent star are already square colored. (1)

Take any point $v$ of $H$. If $v$ lies on a block $B$ with at least three points, square color $B$ and the points adjacent to points of $B$ in toto with four colors as in the previous paragraph. If $v$ does not lie on such a block, then square color the star with center $v$ with four colors. (This is possible since $\Delta(H) \leq 3$.)

Next take a point $u$ such that $u$ is colored, but a neighbor of $u$ is not. If $u$ lies on a block $B_1$ with at least three points, then no other point of $B_1$ is already colored (if not $B_1$ contains $v$ and so all points of $B_1$ and their neighbors are already colored giving contradiction). Now square color $B_1$ and points adjacent to points of $B_1$ in toto with four colors as explained before without altering the colors of already colored points. (It is possible by (1).) If $u$ does not lie on such a block, square color the star with center $u$ with four colors without altering the colors of already-colored points. In each case, all the neighbors of $u$ get colored. Also in the resulting partially colored $H$, for each block with at least three points, either no point is colored or exactly one point is colored by (a) and (b). So repeating this process, the square coloring already done can be extended to the whole graph $H$. This completes the proof.

3. Conclusion

There are planar graphs which are not subgraphs of any planar graph with a square root. For $G$ given in Fig. 1, $G^2 + uv$ is such a graph. So the process of adding lines or points and lines to a planar graph and getting a planar
graph with a square root is not possible in general. Also if $G$ is connected, planar, has at least five points and has a cube root $H$, then $H$ is a path and hence $H$ can be cube colored with four colors. Thus every planar graph with a cube root is 4-colorable. It is of interest to note that connected planar graphs, with a cube root are maximal planar and they are uniquely 4-colorable when they have at least four points.

REFERENCES