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Exponential decay and resonances in a driven system

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ABSTRACT

We study the resonance phenomena for time periodic perturbations of a Hamiltonian H on the Hilbert space $L^2(\mathbb{R}^d)$. Here, resonances are characterized in terms of time behavior of the survival probability. Our approach uses the Floquet–Howland formalism combined with the results of L. Cattaneo, J.M. Graf and W. Hunziker on resonances for time independent perturbations.

1. Introduction

Let $\{H(t)\}$ be a periodic time dependent quantum Hamiltonian, that is, a family of self-adjoint operators acting on a complex Hilbert space $\mathcal{H}$. These operators describe a quantum driven system whose states are given by the solution of the Schrödinger equation:

$$\left(-\frac{\hbar}{i}\partial_t + H(t)\right)\psi(t) = 0, \quad \psi(t=0) = \psi_0. \tag{1.1}$$

The main question we want to address concerns the possibility of existence of metastable states (resonances) for such a system, in the case where $H(t)$ is a perturbation of a free time independent Hamiltonian H_0 having a bound state that disappears in the continuous spectrum when the perturbation is turned on.

Here, we characterize the presence of a resonance in a dynamical fashion, in terms of an approximate exponential behavior of the associated time evolution.

We assume that $\hbar = 1$ and consider the propagator $U(t, s)$; $t,\ s > 0$, associated to the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = H(t)\psi(t). \tag{1.2}$$

When $H_0\varphi = E_0\varphi$, we expect that φ becomes a resonant state for the Hamiltonian $H(t)$, in the sense that,

$$\langle\varphi, U(t, 0)\varphi\rangle \approx e^{-i\lambda|t|},$$

for some $\lambda \in \mathbb{C}$, $\Im\lambda < 0$ close to E_0.

The evolution for time-dependent Hamiltonians has been considered by many authors. For example, spectral and scattering theory for this problem has been treated by Howland in several articles (see e.g. [1,2]).

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More recently, in [3,4], the authors have considered a perturbation of the free Laplacian by a time-periodic potential and prove the absolute continuity of the Floquet spectrum.

There are also some results on the characterization of the resonance phenomenon for a time independent Hamiltonian H, in terms of local exponential decay in time of the evolution $\langle \varphi, e^{-iHt}\varphi\rangle$ for an adequate resonance state φ; see e.g. [5–11].

The relation between resonances and time decay of the evolution can be seen from the following formula, which expresses the evolution group as the Fourier transform of the derivative of the spectral measure,

$$\langle \varphi, e^{-iHt}\varphi\rangle = \frac{1}{2\pi i}\int_0^\infty e^{-itE}\langle \varphi, \mathrm{Im}(H-E-i0)^{-1}\varphi\rangle dE. \tag{1.3}$$

In several cases, the function $F(E) \equiv \langle \varphi,\ (H-E-i0)^{-1}\varphi\rangle$ can be continued in the lower half plane, through the cut due to the presence of the continuous spectrum. If this function has a pole at the complex number $E_0 - i\Gamma$, $\Gamma > 0$, then, by deforming the contour of integration and using residue calculus, it can be proven that

$$\langle \varphi, e^{-iHt}\varphi\rangle = e^{-itE_0 - t\Gamma}\|\varphi\|^2,$$

which is slowly (and exponentially) decaying, if Γ is small.

Mathematical justification of this result (the single-pole approximation) is quite difficult and requires strong conditions. We mention for instance [12,13,6,14,10]. For a concrete one dimensional model a different approach appears in [15].

A more recent result on the dynamical characterization of the resonance states was proposed in [16]. It is based on the positive commutator theory of Mourre [17] and in this paper we adopt this point of view. The correspondence between these resonance states and those defined from a meromorphic continuation of $F(z)$, $\Im z > 0$ was discussed in [12].

Dynamical resonance behavior of periodically perturbed Hamiltonians has already been obtained for example, in [18,19], in a formulation similar to ours.

Here, we first obtain the Fermi Golden Rule for a generic set of perturbations. Also, we prove directly the Mourre estimate for the corresponding Floquet Hamiltonian. For this reason, our results hold away from thresholds.

Previous works need a local decay, pointwise in time, which could hold at thresholds as well, (see e.g. Ref. [7] for discussion).

The article is organized as follows. First, we give a brief review of the results of [16] in Section 2 and of the Howland–Floquet formalism in Section 3. In Section 4, we describe the model studied in the paper. The resonance states for the associated Floquet operators are described in Section 5 and in Section 6 we show that the Fermi golden rule holds for a generic class of time dependent perturbations. Finally in Section 7, we derive a local decay in time on the propagator associated to the time dependent Schrödinger equation.

2. Mourre estimates and resonances

Let H be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$. For every Borel set Ω, denote by $E_\Omega(H)$ the spectral projector of the selfadjoint operator H associated with Ω. We will say that H satisfies a Mourre estimate [17] on an interval $I = (a, b) \subset \mathcal{R}$ if there exists a self adjoint operator A such that,

$$E_I(H)i[H, A]E_I(H) > cE_I(H) + K, \tag{2.4}$$

where $c > 0$ and K is a compact operator.

The commutator $i[H, A] = i(HA - AH)$ may be difficult to define, due to domain problems. Its definition requires the condition $e^{itA}\mathrm{Dom}(H) \subset D(H)$, for all $t \in \mathbb{R}$. Then the estimate,

$$|i\langle Hu, Av\rangle - i\langle Au, v\rangle| < C\|u\|\,\|(H+i)v\|$$

allows to define $i[H, A]$ in the quadratic form sense.

Next, we consider multiple order commutators,

$$ad_A^{(1)}(H) \equiv i[H, A]$$

and for $n \in \mathbb{N}$

$$ad_A^{(n+1)}(H) \equiv i[ad_A^{(n)}(H), A].$$

Suppose that $ad_A^{(j)}(H),\ j = 1\dots\nu$ are defined as H-bounded operators, and (2.4) holds with $K = 0$ (which implies that H has no eigenvalue in I). Then for some $s > 1/2$, the weighted resolvent

$$(A-i)^{-s}(H-z)^{-1}(A+i)^{-s}; \quad \Re z \in I,\ \Im z > 0$$

has a limit in the bounded operator sense on $\mathcal{H}$ as $\Im z$ approaches 0. Moreover, for all $\varphi \in \mathcal{H}$ the function,

$$\lambda \in I \to g_\varphi(\lambda) \equiv \langle (A-i)^{-s}(H-\lambda-i0)^{-1}(A+i)^{-s}\varphi, \varphi\rangle$$

admits derivatives up to order $n-1$ on I.

Further, suppose that H is a self-adjoint operator having a simple eigenvalue $E_0 \in I$, *embedded in the continuous spectrum*. Let φ_0 be the associated eigenvector, $H\varphi_0 = E_0\varphi_0$, $\|\varphi_0\| = 1$. Denote by P the corresponding eigenprojector and $Q = \mathbb{I} - P$. Consider the perturbed Hamiltonian,

$$H_\alpha = H + \alpha W.$$

Assume also that the operators $ad_A^{(j)}(W)$, $j = 1 \dots \nu$, are H-bounded operators. Then for α small enough, the function

$$F(z, \alpha) \equiv \langle QW\varphi_0, (QH_\alpha Q - z)^{-1} QW\varphi_0\rangle, \quad \Re z \in I$$

has a boundary value as $\Im z \to 0$. Moreover $E \in I \to F(E + i0, \alpha)$ admits derivatives up to order $n - 1$.

The main result in [16], states the following. Let $N \geq 1$ and $\nu > N + 5$ be some integers. Under above conditions, there exists a function $g \in C_0^\infty(\mathbb{R})$, such that $g(\lambda) = 1$ in a small interval around E_0 with $\sup|g| \leq 1$, and complex numbers E_α such that for α small enough

$$\langle \varphi_0, e^{-itH_\alpha} g(H_\alpha)\varphi_0\rangle = a(\alpha)e^{-iE_\alpha t} + b(\alpha, t), \tag{2.5}$$

where $a(\alpha) = 1 - O(\alpha^2)$ and $b(\alpha, t) = O\left(\alpha^2 |\log|\alpha|| (t+1)^{1-N}\right)$.

In the following, we write $\langle \varphi_0, e^{-iH_\alpha t} g(H_\alpha)\varphi_0\rangle \approx e^{-iE_\alpha t}$.

Moreover for $\alpha \in \mathbb{R}$, and small enough

$$E_\alpha = E_0 + \alpha\langle \varphi, W\varphi\rangle - \alpha^2 F(E_0 + i0, 0) + o(\alpha^2). \tag{2.6}$$

Note that for $\epsilon = \Im z > 0$ and $\alpha \neq 0$, we have that,

$$\Im F(z, \alpha) = \alpha^2 \epsilon \|Q(H_\alpha - z)^{-1} QW\varphi_0\|^2 \geq 0.$$

Hence, suppose that,

$$\frac{\Gamma}{2} := \Im F(E_0 + i0, 0) > 0.$$

This necessarily gives that E_0 must be embedded in the continuous spectrum of the operator H. Then $\langle \varphi_0, e^{-iH_\alpha} g(H_\alpha)\varphi_0\rangle$ exhibits a local exponential decay in time i.e. φ_0 is a metastable state associated to the Hamiltonian H_α.

Although the definition of resonances requires the strict positivity of Γ, we call the energy E_α in (2.6) a resonance for H_α

3. Howland formalism

In this section, we review basic facts of the time dependent theory initiated in [1]. Let $\{H(t), t \in \mathbb{R}\}$ be a family of selfadjoint operators in a Hilbert space $\mathcal{H}$. Suppose that for $t \in \mathbb{R}$, $H(t)$ has a constant domain $\mathcal{D}$. Furthermore, we assume that the family $\{H(t), t \in \mathbb{R}\}$ is T- periodic, $T > 0$, i.e. $H(t + T) = H(t)$. We notice that an important part of this theory also applies in the non periodic case.

Consider the abstract time-dependent Schrödinger equation,

$$i\frac{\partial \phi}{\partial t}(t) = H(t)\phi(t); \quad \phi(0) = \phi \in \mathcal{H}. \tag{3.7}$$

Then under adequate conditions (3.7) generates a unique propagator $\{U(t, s); (t, s) \in \mathbb{R}^2\}$.

Further, let $\mathcal{K} = L^2(\mathbb{T}; \mathcal{H})$, $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$ be the complex Hilbert space of weakly measurable, $\mathcal{H}$-valued functions with inner product,

$$\langle f, g\rangle = \int_0^T \langle f(t), g(t)\rangle_0 \, dt,$$

where $\langle ., .\rangle_0$ is the corresponding inner product in $\mathcal{H}$. Note that the enlarged space $\mathcal{K} = L^2(\mathbb{T}) \otimes \mathcal{H}$.

The propagator $U(\cdot, \cdot)$ induces a strongly continuous one parameter unitary group $\{W(\sigma); \sigma \in \mathbb{R}\}$ on the space $\mathcal{K}$, defined as

$$W(\sigma)\phi(t, \cdot) = U(t, t - \sigma)\phi(t - \sigma, \cdot); \quad \forall \phi \in \mathcal{H}. \tag{3.8}$$

Moreover, the Floquet Hamiltonian,

$$K = -i\frac{d}{dt} \otimes I_x + H(t)$$

with domain $\mathcal{D}(K) = \{\phi \in \mathcal{H};\ K\phi \in \mathcal{H}\}$ is precisely the infinitesimal generator of $W(\sigma)$, that is,

$$W(\sigma) = e^{-iK\sigma}, \quad \sigma \in \mathbb{R}.$$

The idea behind this construction is that the time-dependent evolution in $\mathcal{H}$ has been turned into a time-independent problem in the Floquet space $\mathcal{K}$.

4. Time dependent Hamiltonian

We now use the Floquet structure, combined with the results in [16], to study resonances for a time periodic family $H(t)$ of quantum Hamiltonians acting in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$.

Here, the resonant behavior will be characterized by the local decay in time of the survival probability,

$$P_s(t) \equiv |\langle \varphi, U(t,s)\varphi\rangle|^2,$$

for an adequate state $\varphi \in \mathcal{H}$ and where $U(t,s)$ is the corresponding propagator.

However, unless the Hamiltonian is time independent, in which case $U(t,s) = U(t-s, 0)$, the asymptotic behavior of the survival probability will depend on the initial time s. Actually, we shall obtain a result on the average value of this quantity on a time interval of length T.

From now on, $U_\alpha(t,s)$ denotes the propagator associated to a Hamiltonian $H_\alpha(t)$, $t \in \mathbb{T}$. We now define precisely this family of operators.

Fix an integer N and let $\nu := N + 6$. The Hamiltonian $H_\alpha(t)$ will be a time dependent perturbation of a free operator H acting in $\mathcal{H}$ and defined as

$$H = -\Delta + V, \tag{4.9}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth function satisfying the following assumptions. Let $\langle x\rangle := (1 + |x|^2)^{1/2}$.

hV1 : *$V \in C^\nu(\mathbb{R}^d)$ and it satisfies: there exists $p > 2$ such that for all α, $|\alpha| \le \nu$:*

$$\sup_{x\in\mathbb{R}^d} \langle x\rangle^{p+\alpha} |\partial^\alpha V(x)| < \infty. \tag{4.10}$$

Also from [20], the operator H with domain $D(H) = \mathcal{H}^2(\mathbb{R}^d)$ is a self-adjoint on $\mathcal{H}$; here $\mathcal{H}^2(\mathbb{R}^d)$ is the corresponding Sobolev spaces. Moreover, the spectrum $\sigma(H) = \sigma_d(H) \cup [0, +\infty)$.

The set $\sigma_d(H)$ consists of a discrete set of negative eigenvalues, they can accumulate at the threshold 0. On the other hand $\sigma_{ac}(H) = [0, +\infty)$, and because of our assumption there is neither singular continuous spectrum or positive eigenvalue embedded in $[0, +\infty)$.

In this work, we use standard notation to denote different types of spectrum of a selfadjoint operator (see e.g. [20]).

The operator H does not depend on t, but we can visualize it in the formalism described in the previous section. In this sense, the corresponding free Floquet Hamiltonian is

$$K = i\frac{\partial}{\partial t} \otimes I_x + I_t \otimes H,$$

acting on the extended Hilbert space

$$\mathcal{K} = L^2(\mathbb{T}; L^2(\mathbb{R}^d)) = L^2(\mathbb{T}) \otimes L^2(\mathbb{R}^d).$$

Here, $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$, I_t and I_x denote the identity operator on the spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{R}^d)$ respectively. It is easy to see that $D(K) = \mathcal{H}^1(\mathbb{T}) \otimes \mathcal{H}^2(\mathbb{R}^d)$.

Further, the operator $i\frac{\partial}{\partial t}$ in $L^2(\mathbb{T})$ has a discrete spectrum, with eigenvalues $n\omega \in \mathbb{Z}$, $\omega := 2\pi/T$ and eigenvectors $e_n(t) = \frac{1}{\sqrt{T}} e^{in\omega t}$. We denote the one dimensional projection $p_n = |e_n\rangle\langle e_n|$. Hence, the spectrum of K is

$$\sigma(K) = \sigma_{ac}(K) = \bigcup_{n\in\mathbb{Z}} [n\omega, \infty) = \mathbb{R}.$$

On the other hand, the pure point spectrum of K consists of the translation of the eigenvalues of H by any $n\omega$, $n \in \mathbb{N}$.

We also suppose that,

hV2: *the operator H has a simple eigenvalue E_0 with eigenvector φ_0 such that for all $n \in \mathbb{N}^*$, $\mu_n := E_0 + n\omega \notin \sigma_d(H) \cup \{0\}$.*

The assumption **hV2** means that first E_0 is also a simple eigenvalue of Floquet Hamiltonian K but it is embedded in its absolutely continuous spectrum. Actually, this is true for all eigenvalues of K. Moreover E_0 is not a spectral threshold of K. In fact without loss of generality we will suppose here:

hV′2: *the operator H has a simple eigenvalue E_0 with eigenvector φ_0 such that $|E_0| < 1$.*

Clearly this implies assumption **hV2**.

We now introduce the time dependent perturbation. Let

$$(t, x) \in \mathbb{T} \times \mathbb{R}^d \to W(t, x) \in \mathbb{R}$$

be a time periodic potential, $W(x, t + T) = W(x, t)$; $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ satisfying,

hW *$W \in C(\mathbb{T}; C^\nu(\mathbb{R}^d))$ and there exists $p > 2$ such that for all α, $|\alpha| \le 2$*

$$\sup_{t\in\mathbb{T}} \sup_{x\in\mathbb{R}^d} \{\langle x\rangle^{p+\alpha} |\partial_x^\alpha W(x,t)|\} < \infty. \tag{4.11}$$

Then the perturbed Hamiltonian,

$$H_\alpha(t) = H + \alpha W(x,t); \quad \alpha \in \mathbb{R},\ t \in \mathbb{T} \tag{4.12}$$

is a self adjoint operator with a time independent domain, $\mathcal{D}(H_\alpha(t)) = \mathcal{H}^2(\mathbb{R}^d)$.

The corresponding selfadjoint Floquet Hamiltonian is

$$K_\alpha = K + \alpha W(x,t), \tag{4.13}$$

acting on the enlarged space $\mathcal{K}$ with domain $\mathcal{D}(K_\alpha) = \mathcal{D}(K)$, for all $\alpha \in \mathbb{R}$.

4.1. Mourre estimate for the Floquet operator

Consider the following operator $D := -i\nabla(-\Delta + 1)^{-1}$, it is a bounded operator on $L^2(\mathbb{R}^d)$. Also, set

$$A = \frac{1}{2}(x \cdot D + D \cdot x).$$

Then A is an essentially selfadjoint operator on $L^2(\mathbb{R}^d)$ such that $e^{itA}\mathcal{H}^2(\mathbb{R}^d) \subset \mathcal{H}^2(\mathbb{R}^d)$ (see e.g. [21]). We denote by

$$B = I_t \otimes A,$$

the corresponding conjugate operator acting on the space $\mathcal{K}$. It is easy to see that in the form sense on $C_0^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d)$

$$i[-\Delta, A] = -2\Delta(-\Delta + 1)^{-1}, \tag{4.14}$$

and then it extends to a bounded selfadjoint operator in $L^2(\mathbb{R}^d)$.

We also have the following.

Lemma 4.1. *The commutator $i[H, A]$, defined in the form sense on $C_0^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d)$, extends to a bounded selfadjoint operator in $L^2(\mathbb{R}^d)$. Moreover, the multiple commutators $ad_A^j(H)$ are bounded, for $j = 1 \dots \nu$.*

Proof. Formally we have that,

$$i[V, A] = i[xV, D] + i[D, x]V.$$

Clearly, the commutator,

$$[xV, D] = i(xV)'(-\Delta + 1)^{-1} + i\nabla(-\Delta + 1)^{-1}(2\nabla \cdot (xV)' - (xV)'')(-\Delta + 1)^{-1},$$

extends to a compact operator in $L^2(\mathbb{R}^d)$.

Indeed, on the one hand our assumption hV1 implies that the operators $(xV)'(-\Delta+1)^{-1}$, $(xV)''(-\Delta+1)^{-1}$ are compact; hence, $[D, xV]$ as a sum of two compact operators, is also compact.

Now by using

$$[x, D] = i(-\Delta + 1)^{-1} - 2i\Delta(-\Delta + 1)^{-2}$$

and the assumption hV1, it is easy to see that $[x, D]V$ as well as $V[x, D]$ are compact operators.

Therefore, these arguments together with the identity (4.14) prove the first part of the lemma.

Now computing $ad_A^j(-\Delta)$; $j = 1 \dots \nu$. We get

$$ad_A^j(-\Delta) = (-\Delta + 1)^{-j} q\left(-2\Delta(-\Delta + 1)^{-1}\right)$$

where q is a polynomial of degree j, so it is a bounded operator on $\mathcal{H}$.

Finally, the multiple commutators $ad_A^j(V)$; $j = 1 \dots \nu$ involve some combination of higher order derivatives of V and bounded operators. Thus, these are also bounded up to the order ν, by our assumption hV1. □

Since we have

$$i[K, B] = I_t \otimes i[H, A],$$

then, the Lemma 4.1 implies that the commutator $i[K, B]$ is a bounded selfadjoint operator in $\mathcal{K}$. Moreover, the higher order commutators exist as bounded operators. Similarly, by using assumption **hW**, we can prove that $ad_B^j(W)$, $j = 1 \dots \nu$ also are bounded operators.

We now construct a Mourre estimate for the free Floquet Hamiltonian K. Let $J_{n_0} = (e_{n_0} + J_-, e_{n_0} + J_+)$, $J_- < 0$, $J_+ > 0$ be a small interval around the energy $e_{n_0} = E_0 + n_0\omega$ so that J_{n_0} contains no other eigenvalues of H.

Next let

$$E_{J_{n_0}}(K) = \oplus_{n\in\mathbb{Z}}\, p_n \otimes E_{J_{n_0}}(H + n\omega),$$

be the spectral projector of K associated with the interval J_{n_0}. Note also that $E_{J_{n_0}}(K) = E_{J_0}(K - n_0\omega)$.

Lemma 4.2. *Suppose* **hV1** *and* **hV′2**. *Let* $n_0 \in \mathbb{Z}$ *and* J_{n_0} *be the energy interval defined above. If* $|J_{n_0}|$ *is small enough, then the operator* K *satisfies a local Mourre estimate. Explicitly, there exists a constant* $c > 0$ *independent of* n_0 *and a compact operator* L *such that*

$$E_{J_{n_0}}(K)i[K, B]E_{J_{n_0}}(K) \geq cE_{J_{n_0}}(K) + L. \tag{4.15}$$

Proof. We can choose $n_0 = 0$, the lemma follows for any $n_0 \in \mathbb{Z}$ by the same arguments.

From (4.14), we need to bound from below modulo a compact operator, the following operator

$$\begin{aligned} E_{J_0}(K)I_t \otimes -\Delta(-\Delta+1)^{-1}E_{J_0}(K) &= E_{J_0}(K)I_t \otimes H(-\Delta+1)^{-1}E_{J_0}(K) - E_{J_0}(K)I_t \otimes V(-\Delta+1)^{-1}E_{J_0}(K) \\ &= E_{J_0}(K)I_t \otimes H(H+1)^{-1}E_{J_0}(K) + E_{J_0}(K)I_t \otimes H(H+1)^{-1}V(-\Delta+1)^{-1} \\ &\quad \times E_{J_0}(K) - E_{J_0}(K)I_t \otimes V(-\Delta+1)^{-1}E_{J_0}(K). \end{aligned}$$

We first consider the term

$$E_{J_0}(K)I_t \otimes H(H+1)^{-1}E_{J_0}(K) = \sum_{n\in\mathbb{Z}} p_n \otimes H(H+1)^{-1}E_{J_0}(H+n).$$

We note that if $|J_0|$ is small enough then $E_{J_0}(H+n) = 0$, for $n \geq 1$.

On the other hand, if $n < 0$ then,

$$H(H+1)^{-1}E_{J_0}(H+n) \geq (J_- - n)(1 + J_- - n)^{-1}E_{J_0}(H+n) \geq cE_{J_0}(H+n)$$

where $c = (1+J_-)(2+J_-)^{-1} > 0$.

Hence,

$$\begin{aligned} E_{J_0}(K)I_t \otimes H(H+1)^{-1}E_{J_0}(K) &\geq c\sum_{n<0} p_n \otimes E_{J_0}(H+n) + L_1 \\ &= c\sum_{n\in\mathbb{Z}} p_n \otimes E_{J_0}(H+n) + L_1 = cE_{J_0}(K) + L_1 \end{aligned}$$

where,

$$L_1 := (H(H+1)^{-1} - c)p_0 \otimes E_{J_0}(H) = (E_0(E_0+1)^{-1} - c)p_0 \otimes E_{J_0}(H).$$

For $|J_0|$ small enough. L_1 is a rank one operator as the product of two rank one operators, p_0 and $E_{J_0}(H)$.

Now, let

$$L_2 := E_{J_0}(K)I_t \otimes H(H+1)^{-1}V(-\Delta+1)^{-1}E_{J_0}(K).$$

Because of our assumptions, the operator $H(H+1)^{-1}V(-\Delta+1)^{-1}$ is compact.

On the other hand, for negative n,

$$\|H(H+1)^{-1}V(-\Delta+1)^{-1}E_{J_0}(H+n)\| = o(1/|n|).$$

Indeed, since $\|E_{J_0}(H+n)(H+1)^{-1}\| = o(1/|n|)$, by using the resolvent equation, $\|E_{J_0}(H+n)(-\Delta+1)^{-1}\|$ and then $\|H(H+1)^{-1}V(-\Delta+1)^{-1}E_{J_0}(H+n)\| = o(1/|n|)$. Finally

$$L_2 = \sum_{n<0} p_n \otimes E_{J_0}(H+n)H(H+1)^{-1}V(-\Delta+1)^{-1}E_{J_0}(H+n)$$

is compact, as a uniform norm limit of compact operators. Clearly, these arguments lead to the compacity of $L_3 = E_{J_0}(K)I_t \otimes V(-\Delta+1)^{-1}E_{J_0}(K)$. Hence, we conclude that in the quadratic form sense in $\mathcal{K}$, there exist a positive constant c and three compact operators, L_1, L_2, L_3 such that,

$$E_{J_0}(K)I_t \otimes -\Delta(-\Delta+1)^{-1}E_{J_0}(K) \geq cE_{J_0}(K) + L_1 + L_2 + L_3. \tag{4.16}$$

Now from Lemma 4.1 we can repeat the same lines of arguments as above to the operator $L_4 := E_{J_0}(K)I_t \otimes [V, A]E_{J_0}(K)$ and then L_4 is again compact. This finishes the proof of the lemma. □

5. Resonances ladder for the Floquet operator

In this section, we want to derive the localization of resonances for the Floquet Hamiltonian (4.13) associated to the eigenvalue E_0 of H.

To this end, we introduce further notations concerning the spectrum of the free Floquet operator. Let φ_0 be the eigenvector of H associated with $E_0, H\varphi_0 = E_0\varphi_0$. We denote the orthogonal eigenprojector $\pi_0 := |\varphi_0\rangle\langle\varphi_0|$ on $\mathcal{H}$, onto the one dimensional subspace generated by φ_0.

Then $\{E_0 + n\omega; n \in \mathbb{Z}\} \subset \sigma_{pp}(K)$ and from **hV′2**, if E is any other eigenvalue of H then $\{E + n\omega; n \in \mathbb{Z}\} \cap \{E_0 + n\omega; n \in \mathbb{Z}\} = \emptyset$.

Next let $n_0 \in \mathbb{Z}$ and J_{n_0} be the interval around $E_0 + n_0\omega$ defined in the previous section. Denote by $f_{n_0} = e_{n_0}(t) \otimes \varphi_0(x)$ the eigenvector of the operator K, associated to the eigenvalue $E_0 + n_0\omega$.

We also use $P_{n_0} = |f_{n_0}\rangle \langle f_{n_0}| = p_{n_0} \otimes \pi_0$ and $Q_{n_0} = I_{\mathcal{K}} - P_{n_0}$.

Following Section 2, we need to consider the function,

$$F(z, \alpha) = \langle f_{n_0}, WQ_{n_0}(K_\alpha - z)^{-1}Q_{n_0}Wf_{n_0}\rangle, \tag{5.17}$$

where $\Re z \in J_{n_0}$, $\Im z = \epsilon > 0$. Set

$$W_n(x) = \frac{1}{\sqrt{T}}\int_0^T e^{-\mathrm{int}}W(x, t)dt. \tag{5.18}$$

We have the following.

Lemma 5.3. (i) $F(E_0 + n_0\omega + i0, 0)$ *exists and it is independent of* n_0. *For any* $\epsilon > 0$, *this quantity is given by*

$$F(E_0 + i\epsilon, 0) = \sum_{n\neq 0}\langle W_n\varphi_0, (H + n\omega - E_0 - i\epsilon)^{-1}W_n\varphi_0\rangle + \langle W_0\varphi_0, (1 - \pi_0)(H - i\epsilon)^{-1}W_0\varphi_0\rangle. \tag{5.19}$$

(ii) *We also have*

$$\frac{\Gamma}{2} = \lim_{\epsilon\to 0}\Im F(E_0 + i\epsilon, 0) = \sum_{n>0}\Im\langle W_n\varphi_0, (H + n\omega - E_0 - i0)^{-1}W_n\varphi_0\rangle. \tag{5.20}$$

Proof. Let $z = E_0+n_0\omega+i\epsilon$; $\epsilon > 0$. The existence of the boundary value is a consequence of Section 2 and our assumptions. Clearly,

$$(K - z)^{-1} = \sum_{n\in\mathbb{Z}} p_n \otimes (H + n\omega - z)^{-1}. \tag{5.21}$$

Since $P_{n_0} = p_{n_0} \otimes \pi_0$ and $Q_{n_0} = I_t \otimes I_x - P_{n_0}$, we have that

$$Q_{n_0}(K - z)^{-1} = \sum_{n\in\mathbb{Z}}(p_n \otimes (H + n\omega - z)^{-1} - p_{n_0}p_n \otimes \pi_0(H + n\omega - z)^{-1}).$$

For $n = n_0$, the quantity in the sum is just, $p_{n_0} \otimes (1-\pi_0)(H+n_0\omega-z)^{-1}$, while for $n \neq n_0$, this term is, $p_n \otimes (H+n\omega-z)^{-1}$.

Then, we obtain,

$$F(z, 0) = \sum_{n\neq n_0}\langle e_{n_0} \otimes \varphi_0, Wp_n \otimes (H + (n - n_0)\omega - E_0 - i\epsilon)^{-1}We_{n_0} \otimes \varphi_0\rangle$$
$$+ \langle e_{n_0} \otimes \varphi_0, Wp_{n_0} \otimes (1 - \pi_0)(H - E_0 - i\epsilon)^{-1}We_{n_0} \otimes \varphi_0\rangle,$$

and then

$$F(z, 0) = \sum_{n\neq n_0}\langle W_{n-n_0}\varphi_0, (H + (n - n_0)\omega - E_0 - i\epsilon)^{-1}W_{n-n_0}\varphi_0\rangle + \langle W_0\varphi_0, (1 - \pi_0)(H - E_0 - i\epsilon)^{-1}W_0\varphi_0\rangle.$$

This proves (5.19).

By assumption **hV2**, for $n > 0$, $E_0 - n\omega \in \rho(H)$, where $\rho(H)$ denotes the resolvent set of H.

Therefore,

$$\lim_{\epsilon\to 0}\Im\langle W_n\varphi_0, (H + n\omega - E_0 - i\epsilon)^{-1}W_n\varphi_0\rangle = 0.$$

Also, because $E_0 \in \rho((1 - \pi_0)H)$ we have

$$\lim_{\epsilon\to 0}\langle W_0\varphi_0, (1 - \pi_0)(H - E_0 - i\epsilon)^{-1}W_0\varphi_0\rangle = 0.$$

On the other hand for $n < 0$ and for all positive numbers ϵ

$$\Im\langle W_n\varphi_0, (H + n\omega - E_0 - i\epsilon)^{-1}W_n\varphi_0\rangle \geq 0.$$

Hence these last three estimates prove (5.20). □

Remark 5.4. Since $E_0 - n\omega > 0 = \inf \sigma_c(H)$ if $n < 0$ we can express the strict positivity of Γ in terms of strict positivity of the derivative of the spectral measure $E_{(-\infty,\lambda]}(H) = E_{(E_0,\lambda]}(H)$. Indeed we know that in the distributional sense

$$\Im\langle W_n\varphi_0, (H + n\omega - E_0 - i0)^{-1} W_n\varphi_0\rangle = \left.\frac{d\langle W_n\varphi_0, E_{(-\infty,\lambda]}(H)W_n\varphi_0\rangle}{d\lambda}\right|_{\lambda=E_0-n\omega}.$$

This last quantity is strictly positive if $W_n\varphi_0$ has its spectral support around the energy $E_0 - n\omega$. We show below that the condition

$$E_{(E_0,\lambda]}(H)W_n\varphi \neq 0$$

is satisfied by a generic class of potentials W.

Then we conclude the following.

Theorem 5.5. *Under conditions* **hV1**, **hV′2**, (**hV2**) *for α small enough, the Floquet Hamiltonian admits resonances of the form*

$$E_{n,\alpha} = E_0 + n\omega + \alpha c_1 - \alpha^2 c_2 + o_n(\alpha^2); \quad n \in \mathbb{Z} \tag{5.22}$$

where $c_1 = \frac{1}{T}\int_{[0,T)\times\mathbb{R}^d} |\varphi_0(x)|^2 W(x,t)dtdx$ and $c_2 = F(E_0 + i0, 0)$. In particular the width of these resonances is

$$\Gamma = 2\alpha^2 \Im F(E_0 + i0, 0) \tag{5.23}$$

where $\Im F(E_0 + i0, 0)$ is given by (5.20).

Moreover, we have that for each $n \in \mathbb{Z}$,

$$\langle e_n \otimes \varphi_0, g(K_\alpha)e^{-isK_\alpha} e_n \otimes \varphi_0\rangle \approx e^{-iE_{n,\alpha}s}, \tag{5.24}$$

in the sense of (2.5).

6. The Fermi golden rule

In this section we want to show that the width Γ defined by (5.20) is strictly positive for a generic class of perturbations W. To this end we use an eigenfunction expansion for the operator H.

From [22], we know that the operator H has a complete set of real generalized eigenfunctions, $\{\varphi(k,.); k \in \mathbb{R}\}$ which are bounded and uniformly continuous.

By using standard arguments of the eigenfunction expansion theory (see e.g. [20]) we have the following lemma.

Lemma 6.6. *Suppose* **hV** *and* **hW**. *Then*

$$\frac{\Gamma}{2} = \sum_{n>0} \frac{1}{2\sqrt{e_n}}\left(\left|\int_{\mathbb{R}^d} dxW_n(x)\varphi_0(x)\varphi(\sqrt{e_n},x)\right|^2 + \left|\int_{\mathbb{R}^d} dxW_n(x)\varphi_0(x)\varphi(-\sqrt{e_n},x)\right|^2\right) \tag{6.25}$$

where W_n are defined in (5.18).

Now introduce the normed space $\mathbf{W}$ as the set of real perturbations $W \in C(\mathbb{T}; C^\nu(\mathbb{R}^d))$ satisfying,

$$\|W\|_{\mathbf{W}} := \sup_{x\in\mathbb{R}^d}\left(\frac{1}{T}\sum_{\alpha,|\alpha|<\nu}\langle x\rangle^{2+\alpha}\left(\int_0^T |\partial_x^\alpha W(x,t)|^2 dt\right)^{1/2}\right) < \infty. \tag{6.26}$$

For each $n > 0$, introduce the sets:

$$D_{\pm,n} := \left\{W \in \mathbf{W} \text{ s.t. } \int dxW_n(x)\varphi_0(x)\varphi(\pm\sqrt{e_n},x) \neq 0\right\}.$$

Lemma 6.7. *For each $n \in \mathbb{N}$, $D_{\pm,n}$ is a dense open subset of* **W**.

Proof. Consider the linear application $I : \mathbf{W} \to \mathbb{R}$ defined as,

$$I(W) := 1/T\int dxW_n(x)\varphi_0(x)\varphi(\sqrt{e_n},x); \quad W \in \mathbf{W}.$$

Then I is a continuous map, since we have,

$$|I(W)|^2 \le C\sup_{x\in\mathbb{R}^d}\left(\langle x\rangle^2\int_0^T |W(x,t)|^2 dt\right) \le C\|W\|^2_{\mathbf{W}}$$

where $C := \int dx\varphi_0(x)\cdot|\frac{\varphi(\sqrt{e_n},x)}{\langle x\rangle^2}|$. Note that C can be bounded independently of n. Then, $D_{+,n}$ is an open subset of **W**.

Moreover, suppose that $W \notin D_{+,n}$. We know that there exists some real point $x_0 \in \mathbb{R}^d$ such that $\varphi_0(x)\varphi(\sqrt{e_n}, x) \neq 0$ and by a simple continuity argument the same is true on some neighborhood $\nu(x_0)$ of x_0. Denote by χ a C^∞ positive function with support in $\nu(x_0)$; then for each $l \in \mathbb{N}$, $W^l := W + \frac{1}{l+1} \cdot e^{-\mathrm{int}}\chi \in D_{+,n}$ and $\|W - W^l\|_{\mathbf{W}} \to 0$ as $l \to \infty$. This shows that $D_{+,n}$ is dense in $\mathbf{W}$. Evidently, the same arguments hold for $D_{-,n}$. □

Then we obtain the following theorem.

Theorem 6.8. *There exists a dense open subset D of* $\mathbf{W}$*, such that for any* $W \in D$, $\frac{\Gamma}{2} > 0$.

Proof. Let $D := \left(\cup_{n\in\mathbb{N}} D_{+,n}\right) \cup \left(\cup_{n\in\mathbb{N}} D_{-,n}\right)$ and then apply the last lemma. □

7. Average decay of the propagator

In this section, we derive the result on exponential decay, but in terms of the original propagator.

Theorem 7.9. *Under conditions stated above, the propagator associated to the time dependent Hamiltonian* $H(t)$ *satisfies,*

$$\int_0^T \langle \varphi_0, U_\alpha(t+s,t)\varphi_0\rangle dt = \tilde{a}(\alpha)e^{-i\tilde{E}_\alpha s} + \tilde{b}(\alpha)$$

where $\tilde{E}_\alpha = E_0 + \alpha c_1 - \alpha^2 c_2 + o(\alpha^2)$, $\tilde{a}(\alpha) = 1 + O(\alpha^2)$ *and* $\tilde{b} = O(\alpha^2 |\log|\alpha||)$.

Proof. For any function $a \in L^2(\mathbb{T})$, we have that,

$$\langle a(t)\otimes\varphi_0, e^{-isK_\alpha}a(t)\otimes\varphi_0\rangle = \int_0^T \langle a(t)\varphi_0, U_\alpha(t,t-s)a(t-s)\varphi_0\rangle\, dt.$$

In particular, choose $a(t) := e_{n_0}(t)$. Then

$$\langle e_{n_0}\otimes\varphi_0, e^{-isK_\alpha}e_{n_0}\otimes\varphi_0\rangle = \frac{1}{T}e^{-in_0\omega s}\int_0^T \langle \varphi_0, U_\alpha(t,t-s)\varphi_0\rangle\, dt,$$

and since $t \in \mathbb{R} \to \langle\varphi, U_\alpha(t+s,t)\varphi\rangle$ is periodic with period T,

$$\langle e_{n_0}\otimes\varphi_0, e^{-isK_\alpha}e_{n_0}\otimes\varphi_0\rangle = \frac{1}{T}e^{-in_0\omega s}\int_0^T \langle \varphi_0, U_\alpha(t+s,t)\varphi_0\rangle\, dt. \tag{7.27}$$

Further by Theorem 5.5 and (5.24), we have that

$$\langle e_{n_0}\otimes\varphi_0, g(K_\alpha)e^{-isK_\alpha}e_{n_0}\otimes\varphi_0\rangle \approx e^{-isE_{n_0,\alpha}}.$$

Put $s = 0$ in this last relation; then $\langle e_n\otimes\varphi_0, g(K_\alpha)e_n\otimes\varphi_0\rangle = 1 - O(\alpha^2|\log|\alpha||)$ or equivalently $\langle e_{n_0}\otimes\varphi_0, (I_t - g(K_\alpha))e_n\otimes\varphi_0\rangle = O(\alpha^2|\log|\alpha||)$. Then

$$\langle e_{n_0}\otimes\varphi_0, e^{-isK_\alpha}e_{n_0}\otimes\varphi_0\rangle = (1+O(\alpha^2))e^{-isE_{n_0,\alpha}} + O(\alpha^2|\log|\alpha||).$$

So by (5.22) this proves the theorem. □

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References

[1] J. Howland, Stationary scattering theory for time-dependent Hamiltonians, Math. Ann. 207 (1974) 315–335.
[2] J. Howland, Scattering theory for Hamiltonian periodic in time, Indiana Univ. Math. J. 28 (3) (1979) 471–493.
[3] K. Yajima, Resonances for the AC = Stark effect, Comm. Math. Phys. 87 (1982) 331–352.
[4] S. Graffi, K. Yajima, Exterior complex scaling and the AC-Stark effect in a coulomb field, Comm. Math. Phys. 89 (1983) 277–301.
[5] O. Costin, A. Soffer, Resonance theory for Schrödinger operators, Comm. Math. Phys. 224 (2001) 133–152.
[6] W. Hunziker, Resonances, metastable states and exponential decay laws in perturbation theory, Comm. Math. Phys. 132 (1990) 177–188.
[7] A. Jensen, G. Nenciu, The Fermi golden rule and its form at thresholds in odd dimensions, Comm. Math. Phys. 261 (2006) 693–727.
[8] M. Merkli, I.M. Sigal, A time dependent theory of quantum resonances, Comm. Math. Phys. 201 (1999) 549–576.
[9] P. Pfeifer, J. Fröhlich, Generalized time-energy uncertainty relations and bounds on life-times of resonances, Rev. Modern Phys. 67 (1995) 759–779.
[10] E. Skibsted, On the evolution of resonances states, J. Math. Anal. Appl. 141 (1989) 27–48.
[11] A. Soffer, M.I. Weinstein, Time dependent resonance theory, GAFA, Geom. Funct. Anal. 8 (1998) 1086–1128.

[12] M.A. Astaburuaga, P. Briet, V. Bruneau, C. Fernández, G. Raikov, Dynamical resonances and SSF singularities for a magnetic Schrödinger operator, Serdica Math. J. 34 (1) (2008) 179–218.
[13] C. Gérard, I.M. Sigal, Space–time picture of semiclassical resonances, Comm. Math. Phys. 145 (1992) 281–328.
[14] E. Skibsted, Truncated Gamow functions, α-decay and the exponential law, Comm. Math. Phys. 104 (1986) 591–604.
[15] R. Lavine, Exponential decay, differential equations and mathematical physics, in: Proc. of the International Conference, U. of Alabama at Birmingham, 1994, pp. 132–141.
[16] L. Cattaneo, G.M. Graf, W. Hunziker, A general resonance theory based on Mourre's inequality, Ann. Henri Poincaré 7 (2006) 583–601.
[17] E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. 78 (1981) 391–408.
[18] O. Costin, R.D. Costin, J.L. Lebowitz, Time asymptotics of the Schrödinger wave function in time-periodic potentials, J. Stat. Phys. 116 (1–4) (2004) 283–310.
[19] A. Soffer, M.I. Weinstein, Nonautonomous Hamiltonians, J. Stat. Phys. 93 (1–2) (1998) 359–391.
[20] M. Reed, B. Simon, Methods of Modern Mathematical Physics I–IV, Academic Press, [Harcourt Brace Jovanovich, Publishers], New York, London, 1979.
[21] K. Yokoyama, Mourre theory for time-periodic systems, Nagoya Math. J. 149 (1998) 193–210.
[22] T. Ikebe, Eigenfunction expansions associated with the Schröedinger operators and their applications to scattering theory, Arch. Ration. Mech. Anal. 5 (1960) 134.