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Partition Theorems Related to the Rogers-Ramanujan Identities*

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ABSTRACT

In this paper a partition theorem is proved which contains the Rogers-Ramanujan identities and Euler's partition theorem as special cases. Other partition theorems of the Rogers-Ramanujan type are proved.

1. INTRODUCTION

In [3], Gordon proved the following striking generalization of the Rogers-Ramanujan identities.

THEOREM. *Let $0 < a \leq k$ be integers. Let $A_{k,a}(N)$ denote the number of partitions of N into parts not of the forms $(2k+1)m$, $(2k+1)m+a$, $(2k+1)m+(2k+1-a)$. Let $B_{k,a}(N)$ denote the number of partitions of N of the form $N = b_1 + \cdots + b_s$ ($s \geq 1$, otherwise arbitrary), $b_i \geq b_{i+1}$, $b_i - b_{i+k-1} \geq 2$, and 1 appearing as a summand at most $a-1$ times. Then,*

$$A_{k,a}(N) = B_{k,a}(N).$$

If $k = 2$, $a = 1, 2$, the above result reduces to the Rogers-Ramanujan identities [4, p. 291]. Gordon's method of proof was an extension of Schur's first proof of the Rogers-Ramanujan identities [6]. Gordon re-

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marks that his theorem parallels Glaisher's extension of Euler's theorem. Glaisher [2] proved:

THEOREM. *Let $r > 0$ be an integer. Let $A_r(N)$ denote the number of partitions of N into parts not of the form rm (i.e., parts not divisible by r). Let $B_r(N)$ denote the number of parttitions of N of the form $N = b_1 \cdots + b_s$ ($s \geq 1$, otherwise arbitrary), $b_i \geq b_{i+1}$, $b_i - b_{i+r-1} \geq 1$ (i.e., no part appears more than $r - 1$ times). Then*

$$A_r(N) = B_r(N).$$

If $r = 2$, the above theorem reduces to Euler's theorem [4, p. 277]. In [1], a proof of Gordon's theorem was given which was based on an extension of a technique of Rogers and Ramanujan [5] and A. Selberg [7]. In this paper we shall extend this method even further and shall obtain (among other results) a general partition theorem which contains not only the Rogers-Ramanujan-Gordon identities as a special case but also contains the Euler-Glaisher theorem.

2. Preliminary Lemmas

Our main results will be based on the three following lemmas.

LEMMA 1. *Let $0 = \alpha_1 < \cdots < \alpha_r < \lambda$, $0 < a \leq k$ all be integers. Let $c_{k,i}(M, N)$ be given for all integers M and N with $0 \leq i \leq k$, and*

$$c_{k,0}(M, N) = 0 \qquad \text{for all } k,\ M,\ N; \tag{2.1}$$

$$c_{k,i}(M, N) = \begin{cases} 1 & \text{if } M = N = 0 \quad \text{and} \quad 1 \leq i \leq k, \\ 0 & \text{if either } M \leq 0 \text{ or } N \leq 0 \text{ and not both are zero;} \end{cases} \tag{2.2}$$

$$c_{k,i}(M, N) - c_{k,i-1}(M, N) = \sum_{j=1}^{r} c_{k,k-i+1}(M-\lambda(i-1)-\alpha_j\,,\ N-M). \tag{2.3}$$

Then $c_{k,i}(M, N)$ is uniquely determined for all M and N ($0 \leq i \leq k$), and

$$\sum_{M=0}^{N} c_{k,a}(M, N)$$

is the coefficient of q^N in the power series expansion of

$$\prod_{\substack{n=1 \\ n\equiv 0 \pmod{\lambda} \\ n\equiv 0,\pm\lambda a \pmod{\lambda(2k+1)}}}^{\infty} (1 - q^n)^{-1} \prod_{m=1}^{\infty} (1 + q^{\alpha_2 m} + \cdots + q^{\alpha_r m}).$$

PROOF: We define

$$C_{k,i}(x;q) = 1 - x^i q^i + \sum_{\mu=1}^{\infty} (-1)^{\mu} x^{k\mu} q^{1/2(2k+1)\mu(\mu+1)-i\mu} \times (1 - x^i q^{(2\mu+1)i}) \frac{(1-xq)\cdots(1-xq^{\mu})}{(1-q)\cdots(1-q^{\mu})}.$$

Selberg [7, p. 4, Eq. 3] has proved

$$C_{k,i}(x;q) = C_{k,i-1}(x;q) + x^{i-1}q^{i-1}(1-xq)C_{k,k-i+1}(xq;q).$$

If we define

$$Q_{k,i}(x;q) = C_{k,i}(x^{\lambda};q^{\lambda}) \prod_{j=1}^{\infty} (1 - x^{\lambda}q^{j\lambda})^{-1} \prod_{n=1}^{\infty} (1 + x^{a_2}q^{a_2 n} + \cdots + x^{a_r}q^{a_r n}),$$

then

$$Q_{k,i}(x;q) = Q_{k,i-1}(x;q) + x^{\lambda(i-1)}q^{\lambda(i-1)}(1 + x^{a_2}q^{a_2} + \cdots + x^{a_r}q^{a_r}) \times Q_{k,k-i+1}(xq;q). \tag{$*$}$$

We may expand $Q_{k,i}(x;q)$ as follows

$$Q_{k,i}(x;q) = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} b_{k,i}(M,N) x^M q^N, \qquad |x| \leq 1,\ |q| < 1.$$

We then easily verify by means of the definition of $Q_{k,i}(x;q)$ and $(*)$ that the $b_{k,i}(M, N)$ satisfy (2.1), (2.2), and (2.3). It follows by mathematical induction that these three conditions define the $b_{k,i}(M, N)$ uniquely. Also $b_{k,i}(M, N) = 0$ if $M > N$ follows by mathematical induction. Therefore

$$c_{k,i}(M,N) = b_{k,i}(M,N),$$

and

$$\begin{aligned}\sum_{N=0}^{\infty} \left(\sum_{M=0}^{N} c_{k,i}(M,N) \right) q^N &= \sum_{N=0}^{\infty} \left(\sum_{M=0}^{N} b_{k,i}(M,N) \right) q^N \\ &= Q_{k,i}(1;q) \\ &= \prod_{\substack{n=1 \\ n\equiv 0 \pmod{\lambda} \\ n \equiv 0 \pm \lambda i \pmod{\lambda(2k+1)}}}^{\infty} (1-q^n)^{-1} \prod_{m=1}^{\infty} (1 + q^{a_2 m} + \cdots + q^{a_r m})\end{aligned}$$

(by Jacobi's identity [4, p. 282]). Thus the lemma follows.

LEMMA 2. *Let* $0 = \alpha_1 < \cdots < \alpha_r < \lambda$, $0 < a \leq k$ *all be integers. Let* $p_{k,a}(M, N)$ *denote the number of partitions of N into M parts of the form* $\sum_{i=1}^{\infty} f_i \cdot i$ *(where* $f_i \geq 0$ *denotes the number of times the summand i appears in the partition) with* (1) $f_1 \leq \lambda a - 1$; (2) *for all i,* $f_i \equiv \alpha_j \pmod{\lambda}$ *for some j*; (3) *if* $f_i \equiv \alpha_j \pmod{\lambda}$, *then* $f_i + f_{i+1} \leq \lambda k + \alpha_j - 1$. *We also define* $p_{k,a}(0, 0) = 1$, $p_{k,a}(M, N) = 0$ *if either M or N is non-positive and not both are zero. Finally we set* $p_{k,0}(M, N) = 0$ *for all k, M, N. Then the* $p_{k,i}(M, N)$ *satisfy* (2.1), (2.2), *and* (2.3) *of Lemma* 1.

PROOF: (2.1) and (2.2) are true by definition. We now prove (2.3).

$p_{k,i}(M, N) - p_{k,i-1}(M, N)$ counts the number of partitions of the type defined in the statement of the lemma with the added condition that 1 appears at least $\lambda(i-1)$ times and at most $\lambda i - 1$ times as a summand. Therefore there are exactly r types of partitions being enumerated; they are classified by

$$f_1 = \lambda(i-1) + \alpha_j \quad (1 \leq j \leq r),$$

Since $f_1 \equiv \alpha_j \pmod{\lambda}$, $f_1 + f_2 \leq \lambda k + \alpha_j - 1$ implies

$$f_2 \leq \lambda k + \alpha_j - 1 - \lambda(i-1) - \alpha_j = \lambda(k-i+1) - 1.$$

Now let us subtract 1 from every summand of the partition under consideration. Since 1 appeared exactly $\lambda(i-1) + \alpha_j$ times formerly, the number of summands has been reduced to $M - \lambda(i-1) - \alpha_j$. Since there were M summands originally, we are now partitioning $N - M$. Since 2 originally appeared at most $\lambda(k-i+1) - 1$ times, now 1 appears at most $\lambda(k-i+1) - 1$ times. Consequency the partition has been transformed into one enumerated by

$$p_{k,k-i+1}(M - \lambda(i-1) - \alpha_j, N - M).$$

The above process establishes a one-to-one correspondence between those partitions enumerated by

$$p_{k,i}(M, N) - p_{k,i-1}(M, N)$$

for which

$$f_1 = \lambda(i-1) + \alpha_j$$

and those partitions enumerated by

$$p_{k,k-i-1}(M - \lambda(i-1) - \alpha_j, N - M).$$

Thus

$$p_{k,i}(M, N) - p_{k,i-1}(M, N) = \sum_{j=1}^{r} p_{k,k-i+1}(M - \lambda(i-1) - \alpha_j, N-M).$$

Thus the lemma is established.

LEMMA 3. *Let* $0 = \alpha_1 < \cdots < \alpha_r < \lambda$, $0 < a \leq k$ *all be integers. Let* $B_{k,a}(N)$ *denote the number of partitions of N of the form* $\sum_{i=1}^{\infty} f_i \cdot i$, *with* (1) $f_1 \leq \lambda a - 1$; (2) *for all i*, $f_i \equiv \alpha_j \pmod{\lambda}$ *for some j*; (3) *if* $f_i \equiv \alpha_j$ $(\text{mod } \lambda)$, $f_i + f_{i+1} \leqq \lambda k + \alpha_j - 1$; $B_{k,a}(0) = 1$. *Then* $B_{k,a}(N)$ *is the coefficient of* q^N *in the power series expansion of*

$$\prod_{\substack{n=1 \\ n \equiv 0 \ (\text{mod } \lambda) \\ n \equiv 0, \pm\lambda a \ (\text{mod } \lambda(2k+1))}}^{\infty} (1 - q^n)^{-1} \prod_{m=1}^{\infty} (1 + q^{\alpha_2 m} + \cdots + q^{\alpha_r m}).$$

PROOF: We note that

$$B_{k,a}(N) = \sum_{M=0}^{N} p_{k,a}(M, N)$$

where $p_{k,a}(M, N)$ is defined in Lemma 2. By Lemma 2 applied to Lemma 1, we see that $\sum_{M=0}^{N} p_{k,a}(M,N)$ is the desired coefficient. Hence the result follows.

3. PARTITION THEOREMS

We may now prove a great number of partition theorems of the Rogers-Ramanujan type by noting that the infinite product in Lemma 3 is the generating function for partition functions related to partitions in which the summands are restricted to certain arithmetic progressions. We give two of many possible examples.

THEOREM 1. *Let* $\lambda > 0$, $0 < a \leq k$ *be integers. Let* $A_{\lambda,k,a}(N)$ *denote the number of partitions of N into parts not of the forms* $\lambda(2k+1)m$, $\lambda(2k+1)m + \lambda a$, $\lambda(2k+1)m + \lambda(2k+1-a)$. Let $B_{\lambda,k,a}(N)$ *denote the number of partitions of N of the form* $\sum_{i=1}^{\infty} f_i \cdot i$, where (1) $f_1 \leq \lambda a - 1$; (2) *if* $f_i \equiv \alpha (\text{mod } \lambda)$ $(0 \leq \alpha < \lambda)$, *then* $f_i + f_{i+1} \leq \lambda k + \alpha - 1$. *Then*

$$A_{\lambda,k,a}(N) = B_{\lambda,k,a}(N).$$

PROOF: In Lemma 3, take $r = \lambda$, $\alpha_j = j - 1$ $(1 \leq j \leq r)$. Then $B_{\lambda,k,a}(N)$ is the coefficient of q^N in the power series expansion of

$$\prod_{\substack{n=1 \\ n\equiv 0 \pmod{\lambda} \\ n\equiv 0, \pm\lambda a \pmod{\lambda(2k+1)}}}^{\infty} (1 - q^n)^{-1} \prod_{m=1}^{\infty} (1 + q^m + \cdots + q^{(\lambda-1)m})$$

$$= \prod_{n=0}^{\infty} \frac{(1 - q^{\lambda(2k+1)(n+1)})(1 - q^{\lambda(2k+1)n+\lambda a})(1 - q^{\lambda(2k+1)n+\lambda(2k+1-a)})}{(1 - q^{n+1})}$$

$$= \sum_{N=0}^{\infty} A_{\lambda,k,a}(N) q^N.$$

Therefore

$$A_{\lambda,k,a}(N) = B_{\lambda,k,a}(N).$$

COROLLARY 1.1. *The Rogers-Ramanujan-Gordon identities (given in Section* 1).

PROOF: In Theorem 1, take $\lambda = 1$.

COROLLARY 1.2. *The Euler-Glaisher theorem (given in Section* 1).

PROOF: Take $\lambda = r$, $k = a = 1$ in Theorem 1. $A_{r,1,1}(N)$ denotes the number of partitions of N into parts $\not\equiv 0$, $\pm r (\mathrm{mod}\ 3r)$, i.e., parts $\not\equiv 0 \pmod{r}$. Let us now consider a general partition $\sum_{i=1}^{\infty} f_i \cdot i$ enumerated by $B_{r,1,1}(N)$. We first note that if $f_i \geq 2r$, then $f_i + f_{i+1} \geq 2r$, which contradicts the restriction that $f_i + f_{i+1} \leq r + r - 1 = 2r - 1$. Suppose $f_i = r + \beta$ where $0 \leq \beta \leq r - 1$, then $f_i + f_{i+1} \geq r + \beta$, which contradicts the restriction that, since $f_i \equiv \beta \pmod{r}$, $f_i + f_{i+1} \leq r + \beta - 1$. Consequently $f_i < r$ for all i.

Now suppose $f_i < r$ for all i; I claim that such a partition is one of those enumerated by $B_{r,1,1}(N)$. Clearly the condition that $f_1 \leq r - 1$ is fulfilled. If $f_i \equiv \beta \pmod{r}$ $(0 \leq \beta < r)$, then since $f_i < r$, $f_i = \beta$; thus, $f_i + f_{i+1} \leq \beta + r - 1$. Thus the second condition is fulfilled. Therefore $B_{r,1,1}(N)$ enumerates the number of partitions of N in which each part appears at most $r - 1$ times. Thus the result follows.

We now give a corollary of a rather different nature.

COROLLARY 1.3. $B_{1,2rk+r+k,(2r+1)a}(N) = B_{2r+1,k,a}(N)$.

PROOF: This follows directly from Theorem 1 since both

$$A_{1,2rk+r+k,(2r+1)a}(N) \qquad \text{and} \qquad A_{2r+1,k,a}(N)$$

enumerate the number of partitions of N into parts

$$\not\equiv 0, \pm(2r+1)a \ (\mathrm{mod}(2r+1)\,(2k+1)).$$

It would be of interest to prove this result by a more direct means.

THEOREM 2. *Let $0 < \mu < \lambda$, $2\mu|\lambda$, $0 < a \leq k$ all be integers. Let $A_{\mu,\lambda,k,a}(N)$ denote the number of partitions of N into parts which are either $\equiv \mu (\mathrm{mod}\ 2\mu)$ or else $\equiv 0 (\mathrm{mod}\ \lambda)$ and $\not\equiv 0, \pm \lambda a (\mathrm{mod}\ \lambda(2k+1))$. Let $B_{\mu,\lambda,k,a}(N)$ denote the number of partitions of N of the form $\sum_{i=1}^{\infty} f_i \cdot i$ with (1) $f_1 \leq \lambda a - 1$; (2) for all i, $f_i \equiv 0$, or $\mu (\mathrm{mod}\ \lambda)$; (3) if $f_i \equiv \alpha$ $(\mathrm{mod}\ \lambda)$ (where α is either 0 or μ), then $f_i + f_{i+1} \leq \lambda k + \alpha - 1$.*

Then

$$A_{\mu,\lambda,k,a}(N) = B_{\mu,\lambda,k,a}(N).$$

PROOF: By Lemma 3, $B_{\mu,\lambda,k,a}(N)$ is the coefficient of q^N in the power series expansion of

$$\prod_{\substack{n=1 \\ n\equiv 0 \ (\mathrm{mod}\ \lambda) \\ n\not\equiv 0,\pm\lambda a \ (\mathrm{mod}\ \lambda(2k+1))}}^{\infty} (1-q^n)^{-1} \prod_{m=1}^{\infty} (1+q^{\mu m})$$

$$= \prod_{\substack{n=1 \\ n\equiv 0 \ (\mathrm{mod}\ \lambda) \\ n\not\equiv 0,\pm\lambda a \ (\mathrm{mod}\ \lambda(2k+1,)}}^{\infty} (1-q^n)^{-1} \prod_{m=0}^{\infty} (1-q^{2m\mu+\mu})^{-1}$$

(by Euler's identity [4, p. 277])

$$= \sum_{N=0}^{\infty} A_{\mu,\lambda,k,a}(N) q^N.$$

Therefore

$$B_{\mu,\lambda,k,a}(N) = A_{\mu,\lambda,k,a}(N).$$

If $\mu = k = a = 1$, $\lambda = 2$, then Theorem 2 reduces to Euler's theorem; however, I know of no other special cases of the above theorem having been proved before. I thus give an example with $\mu = 1$, $\lambda = k = a = 2$ (which incidentally also follows from Theorem 1 with $\lambda = k = a = 2$).

COROLLARY 2.1. *$A_{1,2,2,2}(N) = B_{1,2,2,2}(N)$. $A_{1,2,2,2}(N)$ is the number of partitions of N into parts $\equiv 1, 2, 3, 5, 7, 8, 9 \ (\mathrm{mod}\ 10)$. $B_{1,2,2,2}(N)$ denotes the number of partitions of N into parts such that each part appears* *at most three times with the restrictions that if n appears as a summand two or three times then* $(n + 1)$ *may appear at most once, and if n appears once or not at all then* $(n + 1)$ *may appear at most three times.*

For example, if $N = 10$, the partitions enumerated by $A_{1,2,2,2}(10)$ are

$9 + 1$, $8 + 2$, $8 + 1 + 1$, $7 + 3$, $7 + 2 + 1$, $7 + 1 + 1 + 1$, $5 + 5$,
$5 + 3 + 2$, $5 + 3 + 1 + 1$, $5 + 2 + 2 + 1$, $5 + 2 + 1 + 1 + 1$,
$5 + 1 + 1 + 1 + 1 + 1$, $3 + 3 + 3 + 1$, $3 + 3 + 2 + 2$,
$3 + 3 + 2 + 1 + 1$, $3 + 3 + 1 + 1 + 1 + 1$, $3 + 2 + 2 + 2 + 1$,
$3 + 2 + 2 + 1 + 1 + 1$, $3 + 2 + 1 + 1 + 1 + 1 + 1$,
$3 + 1 + 1 + 1 + 1 + 1 + 1 + 1$, $2 + 2 + 2 + 2 + 2$,
$2 + 2 + 2 + 2 + 1 + 1$, $2 + 2 + 2 + 1 + 1 + 1 + 1$,
$2 + 2 + 1 + 1 + 1 + 1 + 1 + 1$, $2 + 1 + 1 + 1 + 1 + 1 + 1 + 1$
$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$.

Thus $A_{1,2,2,2}(10) = 26$.

The pautitions enumerated by $B_{1,2,2,2}$ (10) are

10, $9 + 1$, $8 + 2$, $8 + 1 + 1$, $7 + 3$, $7 + 2 + 1$, $7 + 1 + 1 + 1$,
$6 + 4$, $6 + 3 + 1$, $6 + 2 + 2$, $6 + 2 + 1 + 1$, $5 + 5$, $5 + 4 + 1$,
$5 + 3 + 2$, $5 + 3 + 1 + 1$, $5 + 2 + 2 + 1$, $5 + 2 + 1 + 1 + 1$,
$4 + 4 + 2$, $4 + 4 + 1 + 1$, $4 + 3 + 3$, $4 + 3 + 2 + 1$,
$4 + 3 + 1 + 1 + 1$, $4 + 2 + 2 + 2$, $3 + 3 + 3 + 1$,
$3 + 3 - 2 + 1 + 1$, $3 + 2 + 2 + 2 + 1$.

Thus $B_{1,2,2,2}(10) = 26$ also.

We conclude with the following rather curious result.

COROLLARY 2.2. *Let* $a(N)$ *be the number of partitions of N into an odd number of parts of the form* $N = b_1 + \cdots + b_{2s+1}$ $(s \geq 0$, *otherwise arbitrary*) *such that* $b_i \geq b_{i+1}$, $b_{2i+1} > b_{2i+2}$, $a(0) = 0$. *Then*

$$B_{1,4,2,1}(N) = a(N + 1).$$

PROOF: By Theorem 2,

$$\sum_{N=0}^{\infty} B_{1,4,2,1}(N)\, q^N = \prod_{\substack{n=1 \\ n\not\equiv 0 \pmod 4 \\ n\not\equiv 0,\pm 4 \pmod{20}}}^{\infty} (1-q^n)^{-1} \prod_{m=1}^{\infty} (1+q^m)$$

$$= \prod_{n=0}^{\infty} (1-q^{20n+8})^{-1}(1-q^{20n+12})^{-1} \prod_{m=1}^{\infty} (1+q^m)$$

$$= q^{-1} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(1-q)\cdots(1-q^{2n+1})}$$

[8 p. 162, eq. 96]

$$= q^{-1} \sum_{N=1}^{\infty} a(N) q^N$$

$$= \sum_{N=0}^{\infty} a(N+1) q^N.$$

The identification of the generating function for $a(N)$ follows by the standard graph-theoretic technique [4, p. 291] using the fact that

$$(n+1)^2 = 1 + 1 + 2 + 2 + \cdots + n + n + (n+1).$$

Similar results may be obtained from some of the other identities of Slater [8].

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