Gluing of idempotents, radical embeddings and two classes of stable equivalences

Steffen Koenig a, Yuming Liu b,∗

a Mathematical Institute, University of Koeln, 50931 Koeln, Germany
b School of Mathematical Sciences, Beijing Normal University, 100875, Beijing, PR China

Received 10 August 2007
Available online 8 January 2008
Communicated by Kent R. Fuller

Abstract

Stable equivalences are studied between any finite dimensional algebra A with a simple projective module and a simple injective module and an algebra B obtained from A by ‘gluing’ the corresponding idempotents of A; this extends results by Martinez-Villa. Stable equivalences modulo projectives are compared to stable equivalences modulo semisimples, and in either situation a characterization is given for a radical embedding to induce such a stable equivalence.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Gluing of idempotents; Radical embedding; Stable equivalence modulo projectives; Stable equivalence modulo semisimples

1. Introduction

Given two Artin algebras A and B, classical results describe in detail when the algebras are Morita equivalent, that is, when their module categories are equivalent. For derived module categories, a Morita theory has been developed, too (see [13]). For stable equivalences, however, much less is known. Only one special class of stable equivalences, those of Morita type, has been described in more detail. These stable equivalences of Morita type are, by definition, given by a pair of bimodules that are projective on either side, and the equivalences are induced by a pair of adjoint functors between module categories (see [3,5]). Such stable equivalences
frequently occur in representation theory of finite groups (see [3,8]). All derived equivalences between symmetric algebras induce stable equivalences of Morita type. There are also many stable equivalences of Morita type between algebras of finite global dimension; for instance, given such equivalences between any algebras of finite type, the equivalences lift to stable equivalences of Morita type between the respective Auslander algebras (see [9]). Even for stable equivalences of Morita type, the fundamental conjecture of Auslander and Reiten is an open problem; it is not known if stable equivalences preserve the number of non-projective simple modules (see [2]).

Here, we will study a class of stable equivalences not of Morita type, but still close to Morita type, by using bimodules that are projective on one side, but not on the other. More precisely, we assume that $B$ is a subalgebra of $A$ having the same radical. We construct $B$ by a finite number of gluings of idempotents, that is by a pullback identifying two idempotents belonging to a simple projective module and to a simple injective module, respectively. We construct two bimodules inducing mutually inverse stable equivalences. In this way we recover, extend and reinterpret results by Martinez-Villa [10], who derived these stable equivalences in a different way, not using bimodules. One of our bimodules is $A$, thus the functor associated with it is induction. The other bimodule, however, is different from $A$ and $B$, and its functor is not restriction and thus not adjoint to the first functor. So the stable equivalences are not induced by a pair of adjoint functors, in contrast to the situation for stable equivalences of Morita type.

All stable equivalences mentioned so far refer to the stable category modulo projectives. We suggest also to study a second stable category, defined by quotienting out morphisms factoring through semisimple modules. For these stable categories modulo semisimples we easily prove the analogue of the conjecture of Auslander and Reiten; the number of non-simple indecomposable projectives up to isomorphism is left invariant. In our situation of radical embeddings and gluings of idempotents, it turns out that stable equivalences modulo semisimples exist precisely in the situation when we have constructed stable equivalences modulo projectives; but here the equivalences are induced from induction and restriction. Conversely, assuming the conjecture of Auslander and Reiten we can show that stable equivalences modulo projectives exist only when there are also stable equivalences modulo semisimples. So modulo the conjecture there is a close coincidence although the functors are quite different.

This article is organized as follows. Section 2 contains some background material. Section 3 is devoted to stable equivalences modulo semisimples, the main Theorem 3.8 being the characterization of when a radical embedding comes with a stable equivalence modulo semisimples. Section 4 then starts by constructing the bimodules used to give stable equivalences modulo projectives between $B$ and $A$ where $B$ is obtained from $A$ by gluing an injective vertex and a projective vertex (Theorem 4.10). At the end of the section this result gets combined with Theorem 3.8 to the main Theorem 4.12 of this article; modulo the Auslander–Reiten conjecture (for stable categories modulo projectives) a radical embedding $B \subset A$, that is gluing of idempotents, leads to equivalent stable categories modulo projectives if and only if it does so modulo semisimples if and only if $B$ is obtained from $A$ by a finite number of steps of gluing a simple injective vertex and a simple projective vertex.

2. Preliminaries

Throughout this paper we adopt the following convention. All the algebras considered are quiver algebras $A = kQ/I$, where $k$ is a field, $Q$ is a finite quiver and $I$ is an admissible ideal in $kQ$. Unless stated otherwise, by a module we shall mean a unitary finitely generated left module. The composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a given category will be denoted
by $fg$. Since the stable categories considered here are trivial for semisimple algebras, we also assume that the algebras considered have no semisimple summands.

We recall some definitions and notation from Auslander and Reiten [1]. Given an algebra $A$, we denote by $\text{mod} \ A$ the category of all finitely generated $A$-modules. Let $C$ be a full subcategory of $\text{mod} \ A$ which is closed under taking direct summands and direct sums. Related to $\text{mod} \ A$, we define two new categories. One is $\text{mod} \ C$, the full subcategory of $\text{mod} \ A$ consisting of modules without direct summands isomorphic to a module in $C$. Another one is the stable category $\text{mod} \ A/C$, the quotient category of $\text{mod} \ A$ by $C$. By definition, the objects of $\text{mod} \ A/C$ are the same as those of $\text{mod} \ A$, and the morphisms between two objects $X$ and $Y$ are given by the quotient space $\text{Hom}_A(X,Y)/\mathcal{C}(X,Y)$, where $\mathcal{C}(X,Y)$ is the subspace of $\text{Hom}_A(X,Y)$ consisting of those homomorphisms from $X$ to $Y$ that factor through an $A$-module in $C$. When $C$ is the category of projective modules (respectively, the category of injective modules), we get the usual category of projective modules (respectively, the categories of semisimple modules), we call $\text{mod} \ A/C$ and $\text{mod} \ C$ the category of projective modules (respectively, the category of semisimple modules). We now state the relationship between a radical embedding and gluing of idempotents. Let $A$ and $B$ be two algebras such that there is a radical embedding $f:B \rightarrow A$, that is, $f$ is a radical algebra monomorphism (that is, an injective algebra map) with $\text{rad} \ f(B) = \text{rad} \ A$. Without loss of generality we identify $B$ with its image in $A$.

**Lemma 2.1.**

(1) Let $f: \text{mod} \ A \rightarrow \text{mod} \ B$ be a functor such that $f(C_A) \subseteq C_B$. Then $f$ induces a functor $\overline{f}: \text{mod} \ A/C_A \rightarrow \text{mod} \ B/C_B$.

(2) Let $f: \text{mod} \ A/C_A \rightarrow \text{mod} \ B/C_B$ be an equivalence. Then $f$ induces a one-to-one correspondence between indecomposable modules in $\text{mod} \ C_A$ and in $\text{mod} \ C_B$.

(3) Let $f: \text{mod} \ A/C_A \rightarrow \text{mod} \ B/C_B$ be a morphism in $\text{mod} \ A$. If $f: X \rightarrow Y$ is a split epimorphism in $\text{mod} \ A/C$, then $f: X \rightarrow Y$ is a split epimorphism in $\text{mod} \ A$.

**Proof.** (1) and (3) are proved in [1]. For (2), compare with the proof of [2, Proposition 1.1, Chapter X]. □

We say that two algebras $A$ and $B$ are stably equivalent (with respect to the subcategories $C_A$ and $C_B$) if there is an equivalence $f: \text{mod} \ A/C_A \rightarrow \text{mod} \ B/C_B$. When $C_A$ and $C_B$ are the categories of projective modules (respectively, the categories of semisimple modules), we call $f$ a stable equivalence modulo projectives (respectively, modulo semisimple modules).

We now state the relationship between a radical embedding and gluing of idempotents. Let $A$ and $B$ be two algebras such that there is a radical embedding $f:B \rightarrow A$, that is, $f$ is an algebra monomorphism (that is, an injective algebra map) with $\text{rad} \ f(B) = \text{rad} \ A$. Without loss of generality we identify $B$ with its image in $A$. Thus we may and will view $B$ as a subalgebra of $A$. The identity of $A$ can be written as a sum of primitive orthogonal idempotents: $1 = e_1 + e_2 + \cdots + e_n$. Similarly, the identity of $B$ is a sum of primitive orthogonal idempotents: $1 = f_1 + f_2 + \cdots + f_m$. Since $1 = e_1 + e_2 + \cdots + e_n = f_1 + f_2 + \cdots + f_m$ in $A$, by [4, Theorem 3.4.1], we can assume without loss of generality that $f_1, f_2, \ldots, f_m$ is a partition of $e_1, e_2, \ldots, e_n$, that is, we can rearrange the order of $e_1, e_2, \ldots, e_n$ such that $f_1 = e_1 + \cdots + e_{i_1}$, $f_2 = e_{i_1+1} + \cdots + e_{i_2}$, and so on. It was pointed out by Xi in [14] that each radical embedding of $A$ is determined (up to isomorphism) by a partition of the complete set of primitive idempotents $e_1, e_2, \ldots, e_n$.
in A. It follows that each radical embedding $B$ of $A$ can be obtained by a finite sequence of subalgebras: $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_s = B$, where each $A_{i+1}$ is obtained from $A_i$ by gluing two primitive idempotents in $A_i$. More precisely, let $u_1, \ldots, u_r, v, w$ be a complete set of primitive idempotents in $A_i$. Then $A_{i+1}$ is the subalgebra of $A_i$ generated by $u_1, \ldots, u_r, v + w$ and all the arrows in $A_i$. Equivalently, $A_{i+1}$ is obtained from $A_i$ by identifying the vertices $v$ and $w$ in its quiver and putting the old relations plus all the newly formed paths through $v = w$.

We remark that the above construction of radical embedding is related to the notion of node in a special case. Recall from [10] that a simple non-projective non-injective module $S$ over an algebra $A$ is called a node if the middle term $E$ of the almost split sequence $0 \to S \to E \to TrD(S) \to 0$ is projective. By [10, Lemma 1], a simple non-projective non-injective module $S$ with projective cover $Q$ is a node if and only if the following condition holds: For all non-isomorphisms $f : P_i \to Q$, $g : Q \to P_j$ with $P_i, P_j$ indecomposable projective modules, we have $fg = 0$. It follows that in our situation, the above new vertex $v = w$ corresponds to a node in $A_{i+1}$ if $v$ is a sink and $w$ is a source in $A_i$.

3. Stable equivalences modulo semisimples

Let $A = kQ/I$ and $B = k\overline{Q}/\overline{I}$ be two finite dimensional algebras (without semisimple summands) such that there is a radical embedding $f : B \to A$. We identify $B$ as a subalgebra of $A$. So we have an induction functor $A_A \otimes_B - : \text{mod} B \to \text{mod} A$ and a restriction functor $B_A \otimes_A - \cong \text{Hom}_A(A_A B, -) : \text{mod} A \to \text{mod} B$. The induction functor $A_A \otimes_B -$ is left adjoint to the restriction functor $B_A \otimes_A -$. Clearly, $B_A \otimes_A -$ induces a functor: $\text{mod} B \to \text{mod} A$, which we also denote by $B_A \otimes_A -$. Note also that $B_A \otimes_A -$ is an exact faithful functor, but it does not induce a functor between $\text{mod} A$ and $\text{mod} B$ in general.

Since $A_A \otimes_B - : \text{mod} B \to \text{mod} A$ is left adjoint to $B_A \otimes_A - : \text{mod} A \to \text{mod} B$, for any $X \in \text{mod} A$, we have an exact sequence of $A$-modules:

$$0 \to \ker \delta_X \to A \otimes_B A \otimes_A X \xrightarrow{\delta_X} X \to 0,$$

where $\delta_X$ is the counit of this adjoint pair and $\delta_X$ is just given by the multiplication map. By [6, Lemma 5.1], (i) is a split exact sequence for any $A$-module $X$ and $\ker \delta_X$ is semisimple. Tensoring the exact sequence of $B$-bimodules: $0 \to B \to A \to A/B \to 0$ by any $Y \in \text{mod} B$, we get an exact sequence of $B$-modules:

$$Y \xrightarrow{\varepsilon_Y} A \otimes_B Y \to (A/B) \otimes_B Y \to 0,$$

where $\varepsilon_Y$ is the unit of this adjoint pair and it is given by $y \mapsto 1 \otimes y$. By a property of adjoint functors, $\varepsilon_Y$ is a split monomorphism for any $B$-module $Y$ that is restricted from an $A$-module $X$ (note that $\varepsilon_Y$ is not a monomorphism in general). Moreover, we have the following fact.

Lemma 3.1.

(1) The sequence (ii) is a split exact sequence if and only if the $B$-module $Y$ is in the subcategory $\mathcal{D} := \{ Y \in \text{mod} B \mid Y$ is a direct summand of $B(A \otimes_A X)$ for some $A X \}$.

(2) $\mathcal{D}$ contains all semisimple $B$-modules.

Proof. Claim (1) follows from the additivity of tensor product and the fact that $\varepsilon_Y$ is a split monomorphism for any $B$-module $Y$ that is restricted from an $A$-module $X$. 
(2) It suffices to prove that each simple $B$-module occurs as a direct summand of the restriction of a simple $A$-module, but this follows from the canonical radical embedding $B \hookrightarrow A$ inducing a (split) monomorphism $B/\text{rad } B \hookrightarrow A/\text{rad } A$ of $B$-modules. \hfill \Box

Similarly we can define a subcategory $\mathcal{D}^0$ for the right $B$-module category $B$-$\text{mod}$.

**Proposition 3.2.** Let $A$ and $B$ be two finite dimensional algebras such that there is a radical embedding $f : B \to A$. Then the restriction functor $\_A A \otimes_A - : \text{mod } A \to \text{mod } B$ and the induction functor $\_B A \otimes B - : \text{mod } B \to \text{mod } A$ induce mutually inverse equivalences between the stable category $\text{mod } A/\text{rad } A$ and the full subcategory $\mathcal{D}/\mathcal{S}_B$ of the stable category $\text{mod } B/\text{rad } B$.

**Proof.** First we show that $\_B A \otimes A -$ and $\_A A \otimes B -$ map semisimple modules to semisimple modules, therefore $\_B A \otimes A -$ and $\_A A \otimes B -$ induce functors between stable categories $\text{mod } A/\text{rad } A$ and $\text{mod } B/\text{rad } B$. Let $\mathcal{S}$ be a simple $A$-module. Then the restriction $\_B \mathcal{S}$ is a semisimple $B$-module since $(\text{rad } B)\mathcal{S} = (\text{rad } A)\mathcal{S} = 0$. Similarly, let $\_A \mathcal{S}$ be a simple $B$-module. To show that the induction $\_A A \otimes B \mathcal{S}$ is a semisimple $A$-module, it suffices to prove $(\text{rad } A) (\_A A \otimes B \mathcal{S}) = 0$, but this follows from the fact: $x(a \otimes s) = (xa) \otimes s = 1 \otimes (xas) = 0$ for any $x \in \text{rad } A$, $a \in A$, $s \in S$.

Since (i) is a split exact sequence for any $A$-module $X$ and $\ker \delta_X$ is semisimple, the counit $\delta_X$ induces a natural isomorphism $(\_A A \otimes B -) \circ (\_B A \otimes A -) \simeq (\_A A \otimes B \_A A -) \simeq \text{id}_{\text{mod } A/\text{rad } A} : \text{mod } A/\text{rad } A \to \text{mod } A/\text{rad } A$. On the other hand, since (ii) is a split exact sequence if and only if the $B$-module $Y \in \mathcal{D}$, and since the cokernel $(\_A A \otimes B Y)$ is semisimple for any $B$-module $Y$, the unit $\epsilon_Y$ induces a natural isomorphism $(\_B A \otimes A -) \circ (\_A A \otimes B -) \simeq (\_B A \otimes B -) \simeq \text{id}_{\mathcal{D}/\mathcal{S}_B} : \mathcal{D}/\mathcal{S}_B \to \mathcal{D}/\mathcal{S}_B$. \hfill \Box

By Proposition 3.2 and Lemma 2.1, there is a one-to-one correspondence between indecomposable modules in $\text{mod } S$ and in $\text{mod } \mathcal{D}$, which in general is a proper subcategory of $\text{mod } S$. Next we shall study the subcategory $\mathcal{D}$ of $\text{mod } B$ and give a criterion for $\mathcal{D} = \text{mod } B$.

We first consider the basic case of gluing two idempotents. More precisely, we fix the following notation: let $1 = e_1 + e_2 + \cdots + e_n$ be a decomposition of the unit into primitive idempotents in $A$, and let $B$ be the subalgebra obtained from $A$ by gluing $e_1$ and $e_n$. That is, $B$ is the unique subalgebra of $A$ which has primitive idempotents $f_1 = e_1 + e_n$, $f_i = e_i$ ($2 \leq i \leq n-1$) and the same radical as $A$.

**Lemma 3.3.** Let $A$ and $B$ be as above related by gluing of two idempotents. Then we have the following.

1. $A(A \otimes B f_1) \simeq Ae_1 \oplus Ae_n$ and $A(A \otimes B f_i) \simeq Ae_i$ for $2 \leq i \leq n-1$.
2. $A/B \simeq (Bf_1/\text{rad } Bf_1) \otimes_k (f_1 B/\text{rad } f_1 B)$ as $B$-$\text{bimodules}$.
3. $B(A \otimes A Ae_i) \simeq Bf_i$, $2 \leq i \leq n-1$. Moreover, let $\Lambda_1 := B(A \otimes A Ae_1)$ and let $\Lambda_2 := B(A \otimes A Ae_n)$. Then $\text{top}(\Lambda_1) \simeq \text{top}(\Lambda_2) \simeq Bf_1/\text{rad } Bf_1$, and we have an exact sequence of $B$-modules

$$0 \longrightarrow Bf_1 \longrightarrow A \otimes A \longrightarrow Bf_1/\text{rad } Bf_1 \longrightarrow 0.$$

**Proof.** 

1. $A \otimes B f_1 \simeq Af_1 = Ae_1 \oplus Ae_n$; $A \otimes B f_i \simeq Af_i = Ae_i$, $2 \leq i \leq n-1$.

2. We have the canonical exact sequence $0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$ of $B$-$\text{bimodules}$. It follows that $\text{dim}(A/B) = 1$ and $A/B$ is a simple $B$-$\text{bimodule}$. By the construction of $B$, we must have $A/B \simeq (Bf_1/\text{rad } Bf_1) \otimes_k (f_1 B/\text{rad } f_1 B)$ as $B$-$\text{bimodules}$.
(3) Tensoring the exact sequence in the proof (2) with \( Bf_i \in \text{mod } B \), we get an exact sequence of \( B \)-modules:

\[
0 \longrightarrow Bf_i \longrightarrow A \otimes_B Bf_i \longrightarrow (Bf_i / \text{rad } Bf_i) \otimes_k (f_1 B / \text{rad } f_1 B) \otimes_B Bf_i \longrightarrow 0.
\]

Since

\[
\dim_k ((f_1 B / \text{rad } f_1 B) \otimes_B Bf_i) = \begin{cases} 
1, & i = 1, \\
0, & 2 \leq i \leq n - 1,
\end{cases}
\]

we get that \( B(A \otimes_B A e_i) \simeq Bf_i \) for \( 2 \leq i \leq n - 1 \), and the conclusions for \( \Lambda_1 \) and \( \Lambda_2 \) hold true. \( \square \)

**Proposition 3.4.** Let \( A \) and \( B \) be as above related by gluing two idempotents. Then we have the following.

1. If the sequence (ii) of \( B \)-modules

\[
0 \longrightarrow Y \longrightarrow A \otimes_B Y \longrightarrow (A/B) \otimes_B Y \longrightarrow 0
\]

is exact for a \( B \)-module \( Y \), then the induced top-sequence of \( B \)-modules

\[
0 \longrightarrow \text{top}(Y) \longrightarrow \text{top}(A \otimes_B Y) \longrightarrow \text{top}((A/B) \otimes_B Y) \longrightarrow 0
\]

is exact.

2. If the exact sequence of \( B \)-bimodules

\[
0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0
\]

is split as a sequence of right \( B \)-modules, then the subcategory \( \mathcal{D} \) of \( \text{mod } B \) is closed under taking quotient modules.

3. \( \mathcal{D} = \text{mod } B \) if and only if \( Bf_1 \in \mathcal{D} \) and \( f_1 B \in \mathcal{D}^0 \).

**Proof.**

(1) It suffices to consider an indecomposable \( B \)-module \( Y \). By Lemma 3.1, the conclusion is true for a simple \( B \)-module. Now suppose \( Y \) is not simple. Since \( (A/B) \otimes_B Y \) is a semisimple \( B \)-module, we have the following exact commutative diagram

\[
\begin{array}{ccccccc}
0 & 0 & Z \\
& & & & & 0 \\
0 & \longrightarrow & \text{rad } Y & \longrightarrow & \text{rad}(A \otimes_B Y) & \longrightarrow & Z & \longrightarrow & 0 \\
& & & & & 0 \\
0 & \longrightarrow & Y & \longrightarrow & A \otimes_B Y & \longrightarrow & (A/B) \otimes_B Y & \longrightarrow & 0 \\
& & & & & 0 \\
(iii): & \text{top } Y & \longrightarrow & \text{top}(A \otimes_B Y) & \longrightarrow & \text{top}((A/B) \otimes_B Y) & \longrightarrow & 0, \\
& & & & & 0 \\
& & 0 & & & 0
\end{array}
\]
where $Z$ is the cokernel of the $B$-homomorphism $\text{rad} Y \rightarrow \text{rad}(A \otimes_B Y)$. Therefore the induced top-sequence (iii) is right exact. Now suppose $\text{top}(Y) \simeq (Bf_1/\text{rad}\, Bf_1)^{11} \oplus (Bf_2/\text{rad}\, Bf_2)^{12} \oplus \cdots \oplus (Bf_{n-1}/\text{rad}\, Bf_{n-1})^{1n-1}$. By Lemma 3.3 $\text{top}(A \otimes Y) \simeq (Bf_1/\text{rad}\, Bf_1)^{21} \oplus (Bf_2/\text{rad}\, Bf_2)^{22} \oplus \cdots \oplus (Bf_{n-1}/\text{rad}\, Bf_{n-1})^{2n-1}$ and $\text{top}(A/B \otimes Y) \simeq (Bf_1/\text{rad}\, Bf_1)^{11}$. Thus diagram chasing shows that (iii) is a (split) short exact sequence.

(2) Suppose that $Y \in \mathcal{D}$ and that $Y \rightarrow Y' \rightarrow 0$ is an epimorphism of $B$-modules. Then, under the assumption of (2), we have the following exact commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & Y & \rightarrow & A \otimes_B Y & \rightarrow & (A/B) \otimes_B Y & \rightarrow & 0 \\
0 & \rightarrow & Y' & \rightarrow & A \otimes_B Y' & \rightarrow & (A/B) \otimes_B Y' & \rightarrow & 0, \\
0 & & 0 & & 0 & & 0 & & 0,
\end{array}
\]

where the first row is a split exact sequence and $h$ is a split epimorphism. It follows that the second row is also split exact, therefore $Y' \in \mathcal{D}$.

(3) $f_1 B \in \mathcal{D}^0$ implies that the canonical sequence $0 \rightarrow f_1 B \rightarrow (e_1 A \oplus e_n A)_B \rightarrow f_1 B/\text{rad} f_1 B \rightarrow 0$ of right $B$-modules is split exact. This further implies that the exact sequence of $B$-bimodules $0 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 0$ is split as a sequence of right $B$-modules. Using (2) and the fact that each projective $B$-module is in $\mathcal{D}$ we get that $\mathcal{D} = \text{mod} B$.

Conversely, if $\mathcal{D} = \text{mod} B$ then clearly $Bf_1 \in \mathcal{D}$. To prove $f_1 B \in \mathcal{D}^0$, we need to prove that the induced sequence $0 \rightarrow f_1 B \rightarrow (e_1 A \oplus e_n A)_B \rightarrow f_1 B/\text{rad} f_1 B \rightarrow 0$ of right $B$-modules is split exact, or equivalently, to prove that the sequence $0 \rightarrow B \rightarrow B A_B \rightarrow A/B \rightarrow 0$ of $B$-bimodules is split as a sequence of right $B$-modules. Tensoring the exact sequence $0 \rightarrow B \rightarrow B A_B \rightarrow A/B \rightarrow 0$ by any $Y \in \text{mod} B$ from the right gives a (split) exact sequence $0 \rightarrow Y \rightarrow B(A \otimes_B Y) \rightarrow (A/B) \otimes_B Y \rightarrow 0$ of $B$-modules. Hence the sequence $0 \rightarrow B \rightarrow B A_B \rightarrow A/B \rightarrow 0$ is a pure exact sequence of right $B$-modules, and therefore it is split as a sequence of right $B$-modules. $\square$

The above proposition shows that if the radical embedding $B$ is obtained from $A$ by gluing two primitive idempotents, then $\mathcal{D} = \text{mod} B$ if and only if $Bf_1 \in \mathcal{D}$ and $f_1 B \in \mathcal{D}^0$, or equivalently, if and only if the associated two exact sequences

\[
0 \rightarrow Bf_1 \rightarrow B (Ae_1) \oplus_B (Ae_n) \rightarrow Bf_1/\text{rad} Bf_1 \rightarrow 0 \quad (iv)
\]

and

\[
0 \rightarrow f_1 B \rightarrow (e_1 A)_B \oplus (e_n A)_B \rightarrow f_1 B/\text{rad} f_1 B \rightarrow 0 \quad (v)
\]

are split. But sequence (iv) being split implies that $B(Ae_1)$ or $B(Ae_n)$ is simple. Without loss of generality we assume that $B(Ae_1)$ is simple. Therefore $A(Ae_1)$ is a simple projective $A$-module. Similarly, sequence (v) splits implies that $(e_1 A)_A$ or $(e_n A)_A$ is simple. Note that $(e_1 A)_A$ is not simple since otherwise the algebra $A$ would have a semisimple summand. It follows that $(e_n A)_A$ is a simple projective right $A$-module. Thus the radical embedding $B$ is obtained from $A$ by
gluing a simple projective vertex and a simple injective vertex. Using Proposition 3.2 we have the following.

**Proposition 3.5.** Suppose that $B \to A$ is a radical embedding such that $B$ is obtained from $A$ by gluing two vertices. If $B$ is obtained from $A$ by gluing a simple projective vertex and a simple injective vertex, then $A$ and $B$ are stably equivalent with respect to the subcategories of semisimple modules. Moreover, in this case the adjoint pair consisting of the induction functor and the restriction functor induce a pair of mutually inverse stable equivalences.

**Proof.** If $B$ is obtained from $A$ by gluing a simple projective vertex $e_1$ and a simple injective vertex $e_n$, then both of the above sequences (iv) and (v) are split and therefore $D = \text{mod } B$. It follows from Proposition 3.2 that the restriction functor $B A \otimes A -$ and the induction functor $A A \otimes_B -$ induce mutually inverse equivalences between the stable categories $\text{mod } A/S_A$ and $\text{mod } B/S_B$. □

Next we prove that the converse of the above proposition is also true. We need the following general fact on stable equivalence modulo semisimples.

**Lemma 3.6.** Let $A$ and $B$ be two finite dimensional algebras such that there is a stable equivalence modulo semisimples $F : \text{mod } A/S_A \to \text{mod } B/S_B$. Then $F$ gives a one-to-one correspondence between the non-isomorphic indecomposable non-simple projective modules in $\text{mod } A$ and in $\text{mod } B$.

**Proof.** Let $P$ be an indecomposable non-simple projective $A$-module. We want to show that $F(P)$ is an indecomposable non-simple projective $B$-module. By Lemma 2.1(2), it suffices to show that each epimorphism $f : Y \to F(P) \to 0$ in $\text{mod } B$ is split. We first show that $f : Y \to F(P)$ is an epimorphism in $\text{mod } B/S_B$. Otherwise, there exists a non-zero morphism $g : F(P) \to Z$ such that $fg = 0$ in $\text{mod } B/S_B$. This implies that the image of $fg$ is in $\text{soc}(Z)$. But $f$ is an epimorphism in $\text{mod } B$, and therefore the image of $g$ is also in $\text{soc}(Z)$. So $g = 0$ in $\text{mod } B/S_B$, and this is a contradiction! Denote by $G$ the inverse of the equivalence functor $F$ and by $h$ the image of $f$ under $G$. Then $h : G(Y) \to GF(P) \simeq P$ is an epimorphism in $\text{mod } A/S_A$. We claim that $h : G(Y) \to GF(P) \simeq P$ is an epimorphism in $\text{mod } A$. Otherwise, $h(G(Y)) \subseteq \text{rad } P$ and the composition $G(Y) \to GF(P) \simeq P \to P/\text{rad}^2 P$ factors through a semisimple module in $\text{mod } A$, this contradicts the fact that $h$ is an epimorphism in $\text{mod } A/S_A$! But $h : G(Y) \to GF(P) \simeq P$ is a split epimorphism in $\text{mod } A$ and therefore $h : G(Y) \to GF(P) \simeq P$ is a split epimorphism in $\text{mod } A/S_A$. It follows that $f : Y \to F(P)$ is a split epimorphism in $\text{mod } B/S_B$. By Lemma 2.1(3), $f : Y \to F(P)$ is a split epimorphism in $\text{mod } B$. This proves that $F(P)$ is an indecomposable non-simple projective $B$-module. Now the conclusion follows from the fact that $F$ is an equivalence. □

**Proposition 3.7.** Suppose that $B \to A$ is a radical embedding such that $B$ is obtained from $A$ by gluing two vertices. If $A$ and $B$ are stably equivalent with respect to the subcategories of semisimple modules, then $B$ is obtained from $A$ by gluing a simple projective vertex and a simple injective vertex.
Proof. Suppose $1 = e_1 + e_2 + \cdots + e_n$ decomposes the identity into primitive orthogonal idempotents in $A$. $B$ is a subalgebra obtained from $A$ by gluing $e_1$ and $e_n$, that is, $B$ has primitive idempotents $f_1 = e_1 + e_n$, $f_i = e_i$ $(2 \leq i \leq n - 1)$, and $B$ and $A$ have the same radical. Clearly $e_i$ is a sink (respectively, source) in $A$ if and only if $f_i$ is a sink (respectively, source) in $B$ for all $2 \leq i \leq n - 1$. Assume now that $A$ and $B$ are stably equivalent with respect to semisimple modules. By Lemma 3.6, at least one of $e_1$ and $e_n$ is a simple projective vertex. Without loss of generality, we assume that $e_1$ is a simple projective vertex. Since the opposite algebras $A^{\text{op}}$ and $B^{\text{op}}$ are also stably equivalent with respect to semisimple modules, the same reason shows that at least one of $e_1$ and $e_n$ is a simple injective vertex. $e_1$ cannot be a simple injective vertex since otherwise the algebra $A$ would contain a semisimple summand. It follows that $e_n$ is a simple injective vertex and the conclusion follows.

Now we state our main result in this section, which gives a characterization for a radical embedding to be a stable equivalence modulo semisimple modules.

Theorem 3.8. Let $A = kQ/I$ and $B = k\bar{Q}/\bar{I}$ be two finite dimensional algebras such that there is a radical embedding $f : B \to A$. Then $A$ and $B$ are stably equivalent with respect to semisimple modules if and only if $B$ is obtained from $A$ by a finite number of steps of gluing a simple projective vertex and a simple injective vertex. Moreover, in this case, the restriction functor and the induction functor induce inverse stable equivalences modulo semisimples between $A$ and $B$.

Proof. If $B$ is obtained from $A$ by a finite number of steps of gluing a simple projective vertex and a simple injective vertex, then by Proposition 3.5, each step is a stable equivalence modulo semisimples and therefore $A$ and $B$ are stably equivalent with respect to semisimple modules.

Conversely, suppose that $A$ and $B$ are stably equivalent with respect to semisimple modules. Since $f : B \to A$ is a radical embedding, by Section 2, we have a finite sequence: $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_s = B$, where each $A_{i+1}$ is obtained from $A_i$ by gluing two primitive idempotents $v$ and $w$ in $A_i$. By Lemma 3.6, $A$ and $B$ have the same number of non-isomorphic indecomposable non-simple projective modules. It follows that at least one of $v$ and $w$ is a simple projective vertex in $A_i$. Without loss of generality, we assume that $v$ is a simple projective vertex. Since the opposite algebras $A^{\text{op}}$ and $B^{\text{op}}$ are also stably equivalent with respect to semisimple modules, the same reason shows that at least one of $v$ and $w$ is a simple projective vertex in $A_i^{\text{op}}$, or equivalently, a simple injective vertex in $A_i$. The vertex $v$ cannot be simple injective since otherwise the algebra $A_i$ would contain a semisimple summand. Therefore $w$ is a simple injective vertex and $A_{i+1}$ is obtained from $A_i$ by gluing a simple projective vertex and a simple injective vertex.

As mentioned before, Martinez-Villa [10] has shown that if $B$ is an algebra obtained from $A$ by gluing a simple projective vertex and a simple injective vertex, then there is a stable equivalence modulo projectives between $A$ and $B$. Thus Proposition 3.5 indicates a potential intrinsic connection between the two types of stable equivalences. We will consider this problem in the next section.

4. Stable equivalences modulo projectives

As before, let $A = kQ/I$ and $B = k\bar{Q}/\bar{I}$ be two finite dimensional algebras (without semi-simple summands) such that there is a radical embedding $f : B \to A$. Suppose that $B$ is obtained from $A$ by gluing a simple projective vertex and a simple injective vertex. In this section, we will
show that there exist two bimodules such that they induce mutually inverse stable equivalences modulo projectives between $A$ and $B$. We often will write algebras as sets of matrices.

By assumption, $A$ has a simple projective module $S_0$ and a simple injective module $S_1$. Let $P_1$ be the projective cover of $S_1$. Then $A = S_0 \oplus Q \oplus P_1$, where the projective module $Q$ has no direct summand isomorphic to either $P_1$ or $S_0$. Therefore $A$ has the following matrix form:

$$A \simeq \text{End}_A(A) \simeq \begin{pmatrix} \text{End}_A(S_0) & \text{Hom}_A(S_0, Q) & \text{Hom}_A(S_0, P_1) \\ 0 & \text{End}_A(Q) & \text{Hom}_A(Q, P_1) \\ 0 & 0 & \text{End}_A(P_1) \end{pmatrix}$$

We will identify $A$ with the last matrix form. By construction, $B$ can be identified with the subalgebra

$$\left\{ \begin{pmatrix} x & y & z \\ 0 & u & v \\ 0 & 0 & x \end{pmatrix} \in A \ \bigg| \ x \in k, \ y \in \text{Hom}_A(S_0, Q), \ z \in \text{Hom}_A(S_0, P_1), \ u \in \text{End}_A(Q), \ v \in \text{Hom}_A(Q, P_1) \right\}.$$ 

Suppose that $AQ = Q_2 \oplus \cdots \oplus Q_{n-1}$, where $Q_i$ are indecomposable projective $A$-modules. Then the identity of $A$ can be written as a sum of primitive orthogonal idempotents:

$$1_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e_1 + e_2 + \cdots + e_{n-1} + e_n,$$

where $p_i$ denotes the canonical composition map $Q \rightarrow Q_i \hookrightarrow Q$ for $2 \leq i \leq n$. Write

$$e = e_2 + \cdots + e_{n-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{id}_Q & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

It follows that

$$A^*A = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_{n-1} \oplus Ae_n$$

$$= \begin{pmatrix} k & 0 & 0 \\ 0 & \text{Hom}_A(S_0, Q_2) & 0 \\ 0 & 0 & \text{Hom}_A(Q_2, Q_2) \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \text{Hom}_A(S_0, Q_{n-1}) & 0 \\ 0 & \text{Hom}_A(Q_{n-1}, Q_{n-1}) & 0 \\ 0 & 0 & \text{Hom}_A(Q_{n-1}, k) \end{pmatrix}.$$
Similarly,

\[ 1_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ p_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ p_{n-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = f_1 + f_2 + \cdots + f_{n-1} \]

is a sum of primitive orthogonal idempotents in \( B \). Write \( f_1 = e_1 + e_n, f_i = e_i \) for \( 2 \leq i \leq n - 1 \). Note also that, by our construction, the simple \( B \)-module \( Bf_1 / \text{rad} Bf_1 \) becomes a node (see the last paragraph of Section 2).

So

\[ bB = Bf_1 \oplus Bf_2 \oplus \cdots \oplus Bf_{n-1} \]

\[ = \begin{pmatrix} x & 0 & \text{Hom}_A(S_0, P_1) \\ 0 & 0 & \text{Hom}_A(Q, P_1) \\ 0 & 0 & x \end{pmatrix} \oplus \begin{pmatrix} 0 & \text{Hom}_A(S_0, Q_2) & 0 \\ 0 & \text{Hom}_A(Q, Q_2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \cdots \]

\[ \oplus \begin{pmatrix} 0 & \text{Hom}_A(S_0, Q_{n-1}) & 0 \\ 0 & \text{Hom}_A(Q, Q_{n-1}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Our aim is to construct a pair of functors between mod \( A \) and mod \( B \) inducing mutually inverse stable equivalences between mod \( A \) and mod \( B \). It may be tempting to choose the induction functor \( A_A \otimes B \) — and the restriction functor \( B_B \otimes A \) —. However, the restriction \( B_B \otimes A \) in general does not induce a functor between stable categories modulo projectives. So we have to find another bimodule to replace \( B_B \).

Set \( BT = Bf_1 \oplus Bf_2 \oplus \cdots \oplus Bf_{n-1} \oplus Bf_1 = Bf_1 \oplus Bf \oplus Bf_1 \). As a matrix algebra,

\[ \text{End}_B(T) = \begin{pmatrix} \text{Hom}_B(Bf_1, Bf_1) & \text{Hom}_B(Bf_1, Bf) & \text{Hom}_B(Bf_1, Bf_1) \\ \text{Hom}_B(Bf, Bf_1) & \text{Hom}_B(Bf, Bf) & \text{Hom}_B(Bf, Bf_1) \\ \text{Hom}_B(Bf_1, Bf_1) & \text{Hom}_B(Bf_1, Bf) & \text{Hom}_B(Bf_1, Bf_1) \end{pmatrix} \]

\[ \supseteq \begin{pmatrix} k \cdot \text{id}_{Bf_1} & \text{Hom}_B(Bf_1, Bf) & \text{rad(} \text{Hom}_B(Bf_1, Bf_1) \text{)} \\ 0 & \text{Hom}_B(Bf, Bf) & \text{Hom}_B(Bf, Bf_1) \\ 0 & 0 & k \cdot \text{id}_{Bf_1} \end{pmatrix} \]

\[ \simeq \begin{pmatrix} k f_1 & f_1 Bf & f_1 (\text{rad } B) f_1 \\ 0 & f Bf & f Bf_1 \\ 0 & 0 & k f_1 \end{pmatrix} := \tilde{A}. \]

**Lemma 4.1.** The two matrix algebras \( \tilde{A} \) and \( A \) are isomorphic.

**Proof.**

\[ f_1 Bf = \begin{pmatrix} 0 & \text{Hom}_A(S_0, Q) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_1 (\text{rad } B) f_1 = \begin{pmatrix} 0 & 0 & \text{Hom}_A(S_0, P_1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ f Bf = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \text{End}_A(Q) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f Bf_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \text{Hom}_A(Q, P_1) \\ 0 & 0 & 0 \end{pmatrix}. \]
So
\[
\bar{A} = \begin{pmatrix}
k f_1 & 0 & \text{Hom}_A(S_0, Q) & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \text{End}_A(Q) \\
0 & 0 & 0 & 0 & \text{Hom}_A(Q, P_1) \\
0 & 0 & k f_1
\end{pmatrix} \cong \begin{pmatrix}
\text{Hom}_A(S_0, Q) & 0 \\
0 & 0 \\
0 & 0 & \text{End}_A(Q) \\
0 & 0 & 0 & \text{Hom}_A(Q, P_1) \\
0 & 0 & 0 & k f_1
\end{pmatrix} \cong A.
\]

From now on we will identify $\bar{A}$ with $A$.

Since $T$ is a $B$-$\text{End}_B(T)$-bimodule and $\text{End}_B(T) \supseteq A$, $T$ becomes a $B$-$A$-bimodule. So we get a functor $B T \otimes_A - : \text{mod} A \rightarrow \text{mod} B$.

We have
\[
B T \otimes_A A e_i \cong T e_i = \begin{cases}
B f_1, & i = 1 \text{ or } n; \\
B f_i, & 2 \leq i \leq n - 1.
\end{cases}
\]

So $B T \otimes_A -$ induces a functor: $\text{mod} A \rightarrow \text{mod} B$, which we also denote by $B T \otimes_A -$.

**Lemma 4.2.** $(\text{rad } B f_1)(f_1 B f) = 0$, $(\text{rad } B f_1)(f_1 (\text{rad } B) f_1) = 0$ and $(f_1 B f)(f B f_1) \subseteq f_1 (\text{rad } B) f_1$.

**Proof.** This is a straightforward computation.

**Lemma 4.3.** $A_B \cong B \oplus (f_1 B / \text{rad } f_1 B)$ and $T_A \cong A \oplus (e_1 A / \text{rad } e_1 A)^{\dim_k \text{rad } B f_1}$. Therefore the functors $A_A \otimes_B - : \text{mod } B \rightarrow \text{mod } A$ and $B T \otimes_A - : \text{mod } A \rightarrow \text{mod } B$ are right exact, faithful and send projectives to projectives.

**Proof.** Using the matrix from, we can decompose
\[
A_B = B \oplus \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & k \end{pmatrix}
\]
as vector spaces. This is also a decomposition of right $B$-modules; indeed, the second term is a right $B$-submodule of $A$ and as such it is isomorphic to the simple module $(f_1 B / \text{rad } f_1 B)$. This proves the first isomorphism.

To prove the second isomorphism, we observe that, as a vector space, $T$ has the following decomposition:
\[
(\dagger) \quad T = \begin{pmatrix} k f_1 & 0 & f_1 B f & f_1 (\text{rad } B) f_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & k f_1 \\
\text{rad } B f_1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e_1 A \\
0 & 0 & 0 \\
0 & 0 & 0 & e_n A \\
\text{rad } B f_1 & 0 & 0 & 0 \end{pmatrix} = A \oplus (\text{rad } B f_1 \oplus 0 \oplus 0).\]
By Lemma 4.2, this is also a decomposition of right \( A \)-modules (with natural matrix action) and the last term is isomorphic to \( (e_1 A / \text{rad} \, e_1 A)^l \), where \( l = \dim_k(\text{rad} \, Bf_1) \).

**Lemma 4.4.**

\[
A \otimes_A (B_i / \text{rad} \, Bf_1) \simeq \begin{cases} 
Ae_1 \oplus (Ae_n / \text{rad} \, Ae_n), & i = 1; \\
Ae_i / \text{rad} \, Ae_i, & 2 \leq i \leq n - 1.
\end{cases}
\]

(1)

\[
T \otimes_A (Ae_i / \text{rad} \, Ae_i) \simeq \begin{cases} 
Bf_1, & i = 1; \\
Bf_i / \text{rad} \, Bf_1, & 2 \leq i \leq n - 1; \\
Bf_1 / \text{rad} \, Bf_1, & i = n.
\end{cases}
\]

(2)

**Proof.** (1) For any \( 2 \leq i \leq n - 1 \), we have an epimorphism \( Bf_i \to Bf_i / \text{rad} \, Bf_1 \). Tensoring with \( A A \otimes_B - \) we get \( Ae_i \simeq A Bf_i \to A B (Bf_i / \text{rad} \, Bf_1) \). Since \( A B \simeq B \oplus (f_1 B / \text{rad} \, f_1 B) \), and since \( (f_1 B / \text{rad} \, f_1 B) \oplus_B (Bf_i / \text{rad} \, Bf_1) = 0 (i \neq 1) \), we know that \( \dim_k(A \otimes_B (Bf_i / \text{rad} \, Bf_1)) = \dim_k(B \otimes_B (Bf_i / \text{rad} \, Bf_1)) = \dim_k(Bf_i / \text{rad} \, Bf_1) = 1 \). It follows that \( A \otimes_B (Bf_i / \text{rad} \, Bf_1) \simeq Ae_i / \text{rad} \, Ae_i \) \((2 \leq i \leq n - 1)\).

For \( i = 1 \), we have an exact sequence:

\[
0 \to \text{rad} \, Bf_1 \to Bf_1 \to Bf_1 / \text{rad} \, Bf_1 \to 0.
\]

Tensoring with \( A A \otimes_B - \) we get the following exact commutative diagram:

\[
\begin{array}{cccccc}
A \otimes_B \text{rad} \, Bf_1 & \xrightarrow{A \otimes_B v} & A \otimes_B Bf_1 & \xrightarrow{u} & A \otimes_B (Bf_i / \text{rad} \, Bf_1) & \to 0 \\
\| \downarrow \| \downarrow \| \downarrow \| \downarrow \downarrow \downarrow \downarrow \downarrow & & & & & \\
A \otimes_B \text{rad} \, Bf_1 & \xrightarrow{(A \otimes_B v)u} & Ae_1 \oplus Ae_n & \to & \text{coker}((A \otimes_B v)u) & \to 0,
\end{array}
\]

where \( u \) is induced from multiplication. Since \( \text{rad} \, B = \text{rad} \, A \), we have \( \text{im}((A \otimes_B v)u) = \text{rad} \, Ae_n \). Therefore \( \text{coker}((A \otimes_B v)u) \simeq Ae_1 \oplus (Ae_n / \text{rad} \, Ae_n) \).

(2) For \( i = 1 \), \( T \otimes_A Ae_1 \simeq Bf_1 \).

For \( 2 \leq i \leq n \), we have an epimorphism \( Ae_i \to Ae_i / \text{rad} \, Ae_i \). Tensoring with \( T \otimes_A - \) we get \( T \otimes_A Ae_i \to T \otimes_A Ae_i / \text{rad} \, Ae_i \). Since \( T_A \simeq A \oplus (e_1 A / \text{rad} \, e_1 A)^l \) (where \( l = \dim_k(\text{rad} \, Bf_1) \)), and since \( (e_1 A / \text{rad} \, e_1 A) \otimes_A (Ae_i / \text{rad} \, Ae_i) = 0 (i \neq 1) \), we know that \( \dim_k(T \otimes_A (Ae_i / \text{rad} \, Ae_i)) = \dim_k(A \otimes_A (Ae_i / \text{rad} \, Ae_i)) = \dim_k(Ae_i / \text{rad} \, Ae_i) = 1 \). It follows that

\[
T \otimes_A (Ae_i / \text{rad} \, Ae_i) \simeq \begin{cases} 
Bf_i / \text{rad} \, Bf_1, & 2 \leq i \leq n - 1; \\
Bf_1 / \text{rad} \, Bf_1, & i = n.
\end{cases}
\]

\[
\square
\]

Let \( C \) be the full subcategory of \( \text{mod} \, A \) consisting of \( A \)-modules which have no direct summand isomorphic to \( Ae_1 \). The next lemma shows that the restriction of the functor \( B T \otimes_A - \) to \( C \) is naturally isomorphic to \( B A \otimes_A - \).

**Lemma 4.5.** The functors \( B T \otimes_A - \) and \( B A \otimes_A - : C \to \text{mod} \, B \) are naturally isomorphic.
Proof. Recall the decomposition (†) of $T$ as a right $A$-module:

\[
T_A = \left( k f_1 \oplus f_1 B f \oplus f_1 (\text{rad}
B) f_1 \right) = e_1 A
\oplus \left( 0 \oplus f B f \oplus f B f_1 \right) = e A
\oplus \left( \text{rad}
B f_1 \oplus 0 \oplus 0 \right) = e_n A
\oplus \left( \text{rad}
B f_1 \oplus 0 \oplus 0 \right) \oplus (\text{rad}
B f_1 \oplus 0 \oplus 0)
\]

\[
= A \oplus (\text{rad}
B f_1 \oplus 0 \oplus 0).
\]

The submodule $K = (\text{rad}
B f_1 \oplus 0 \oplus 0)$ is in fact a $B$-$A$-bi-submodule of $T$ and therefore we get a quotient $B$-$A$-bimodule $T/K$. We have $T/K \simeq A_A$ as right $A$-modules, where the $A$-module structure on $(T/K)_A$ is the natural matrix action and $A_A$ is the regular right $A$-module. Observe that the left $B$-module structure on $B(T/K)$ is given by the natural matrix action (modulo the submodule $K$) of

\[
B = \begin{pmatrix} x f_1 & f_1 B f & f_1 (\text{rad}
B) f_1 \\ 0 & f B f & f B f_1 \\ 0 & 0 & x f_1 \end{pmatrix}.
\]

Therefore $T/K \simeq B A$ as left $B$-modules. Indeed, straightforward calculations show that the map

\[
\begin{pmatrix} a & b & c \\ 0 & u & v \\ 0 & 0 & w \end{pmatrix} \mapsto \begin{pmatrix} a & b & c \\ 0 & u & v \\ 0 & 0 & w \end{pmatrix}
\]

gives a $B$-$A$-bimodule isomorphism: $A \to T/K$, where $A$ has the natural $B$-$A$-bimodule structure. It follows that we have an exact sequence of $B$-$A$-bimodules:

\[
0 \to K \to T \to A \to 0,
\]

where $B K \simeq \text{rad}
B f_1$ and $K_A \simeq (e_1 A / \text{rad}
 e_1 A)^l$, $l = \dim_k(\text{rad}
B f_1)$. For any $X \in \mathcal{C}$, we have an exact sequence of $B$-modules:

\[
K \otimes_A X \to T \otimes_A X \to A \otimes_A X \to 0.
\]

Since $X$ has no direct summand isomorphic to $A e_1$, $(e_1 A / \text{rad}
 e_1 A) \otimes_A X)^{\dim_k \text{rad}
B f_1} = 0$ as a vector space. So $B T \otimes_A X \simeq B A \otimes_A X$. □

Recall from Section 3 that we have for any $X \in \text{mod}
 A$ an exact sequence of $A$-modules:

\[
0 \to \ker \delta_X \to A \otimes_B A \otimes_A X \xrightarrow{\delta_X} X \to 0, \quad (i)
\]

where $\delta_X$ is the counit of the adjoint pair. We have noted that (i) is a split exact sequence and $\ker \delta_X$ is semisimple. The following lemma shows that we can say more in our situation.

Lemma 4.6. The exact sequence (i) is split, and $\ker \delta_X$ is isomorphic to a direct sum of copies of $A e_1$ for any $X \in \mathcal{C}$. 

**Proof.** It is sufficient to prove the claim for $X$ indecomposable. There are two cases to be considered.

**Case 1.** $X \simeq Ae_1$. Then $A \otimes_B A \otimes_A X \simeq Ae_1 \oplus (Ae_n/\text{rad } Ae_n)$ and the sequence (i) splits.

**Case 2.** $X \not\simeq Ae_1$ and therefore $X \in C$. Then $\text{top } X \simeq (Ae_2/\text{rad } Ae_2)^{l_2} \oplus \cdots \oplus (Ae_n/\text{rad } Ae_n)^{l_n}$ for some $l_2, \ldots, l_n$. Therefore $\text{top } (BA \otimes_A X) \simeq (Bf_2/\text{rad } Bf_2)^{l_2} \oplus \cdots \oplus (Bf_{n-1}/\text{rad } Bf_{n-1})^{l_{n-1}} \oplus (Bf_1/\text{rad } Bf_1)^{l_1}$. Thus $\text{top } (A \otimes_B A \otimes_A X) \simeq (Ae_2/\text{rad } Ae_2)^{l_2} \oplus \cdots \oplus (Ae_n/\text{rad } Ae_n)^{l_n} \oplus (Ae_1)^{l_1}$. Write $A \otimes_B A \otimes_A X \simeq X_1 \oplus (Ae_1)^{l_1}$, where $X_1$ has no direct summand isomorphic to $Ae_1$. So the following exact sequence is isomorphic to (i):

$$0 \rightarrow \ker \delta_X \rightarrow X_1 \oplus (Ae_1)^{l_1} \xrightarrow{\gamma_1, \gamma_2} X \rightarrow 0.$$ 

It is now sufficient to show that $\gamma_1$ is an isomorphism. First, $\gamma_1$ is an epimorphism since $\gamma_2$ has image in $\text{rad } X$. Next, we compute the dimension of $A \otimes_B A \otimes_A X$. As vector spaces, $A \otimes_B A \otimes_A X \simeq (B \oplus (f_1B/\text{rad } f_1B)) \otimes_B A \otimes_A X \simeq (B \oplus (f_1B/\text{rad } f_1B)) \otimes_B X \simeq X \oplus ((f_1B/\text{rad } f_1B) \otimes_B X)$. Note that $(f_1B/\text{rad } f_1B) \otimes_B X \simeq (f_1B/\text{rad } f_1B) \otimes_B (Bf_1/\text{rad } Bf_1))^{l_1}$. So $\dim_k (A \otimes_B A \otimes_A X) = \dim_k X + l_1 = \dim_k X + l_n$. It follows that $\dim_k X_1 = \dim_k X$ and $\gamma_1$ is an isomorphism. □

**Lemma 4.7.** There are natural isomorphisms $(A A \otimes \_ \_ \_ B \_ \_ A B) \circ (B T \otimes \_ \_ A \_ \_ \_ \_ \_ \_ ) \simeq A A \otimes B T \otimes \_ \_ A \_ \_ B \_ \_ A - \simeq \text{id}_{\text{mod } A}: \text{mod } A \rightarrow \text{mod } A$.

**Remark.** On the level of module categories, the two tensor functors do not form an adjoint pair. The adjoint of induction does not pass to the stable category.

**Proof.** By 4.5 there is a natural isomorphism: $B T \otimes \_ \_ A \_ \_ B A \otimes \_ \_ \_ : C \rightarrow \text{mod } B$. Thus we have a natural isomorphism:

$$A A \otimes_B T \otimes \_ \_ A \_ \_ B A \otimes \_ \_ A \_ \_ B A \otimes \_ \_ A \_ \_ A - : C \rightarrow \text{mod } A.$$ 

This induces a natural isomorphism:

$$A A \otimes_B T \otimes \_ \_ A \_ \_ B A \otimes \_ \_ A \_ \_ B A \otimes \_ \_ A \_ \_ A - : C \rightarrow \text{mod } A,$$

where $C$ is the full subcategory of $\text{mod } A$, whose objects do not have direct summands isomorphic to $Ae_1$. The inclusion functor incl: $C \rightarrow \text{mod } A$ induces an equivalence: $C \rightarrow \underline{\text{mod } A}$, which we also denote by incl. There is also a natural transformation:

$$A \otimes_B A \otimes_A \_ \_ B A \otimes \_ \_ A \_ \_ B A \otimes \_ \_ A \_ \_ A - \rightarrow \text{id}_{\text{mod } A}: \text{mod } A \rightarrow \text{mod } A.$$ 

It follows that we have the following natural transformation:

$$A \otimes_B A \otimes_A \_ \_ B A \otimes \_ \_ A \_ \_ B A \otimes \_ \_ A \_ \_ A - \rightarrow \text{incl}: C \rightarrow \underline{\text{mod } A}.$$ 

By Lemma 4.6, $\delta$ induces a natural isomorphism:

$$A A \otimes_B A \otimes_A \_ \_ B A \otimes \_ \_ A \_ \_ B A \otimes \_ \_ A \_ \_ A - \simeq \text{incl}: C \rightarrow \underline{\text{mod } A}.$$
Therefore we have a natural isomorphism:

\[ A A \otimes_B T \otimes_A \simeq \text{incl} : \mathcal{C} \rightarrow \text{mod} A. \]

There is a commutative diagram

\[ \begin{array}{ccc}
\mathcal{C} & \overset{A A \otimes_B T \otimes_A}{\longrightarrow} & \text{mod} A \\
\text{incl} \downarrow \simeq \downarrow & & \downarrow \\
\text{mod} A & \rightarrow & \text{mod} A
\end{array} \]

This diagram and the natural isomorphism \( A A \otimes_B T \otimes_A \simeq \text{incl} : \mathcal{C} \rightarrow \text{mod} A. \) imply that

\[ A A \otimes_B T \otimes_A \simeq \text{id}_{\text{mod} A} : \text{mod} A \rightarrow \text{mod} A. \]

Recall from Section 3 that there is an exact sequence of \( B \)-bimodules:

\[ 0 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 0, \]

where \( A/B \simeq (Bf_1/\text{rad} Bf_1) \otimes_k (f_1 B/\text{rad} f_1 B). \) This sequence is split both as left and as right \( B \)-modules. Tensoring with any \( Y \in \text{mod} B \), we have an exact sequence of \( B \)-modules:

\[ 0 \rightarrow Y \overset{\varepsilon_Y}{\rightarrow} A \otimes_B Y \rightarrow (A/B) \otimes_B Y \rightarrow 0, \quad (ii) \]

where \( \varepsilon_Y \) is the unit of the adjoint pair. Note that \( (A/B) \otimes_B Y \simeq (Bf_1/\text{rad} Bf_1)^{m_Y} \), where \( m_Y = \dim_k((f_1 B/\text{rad} f_1 B) \otimes_B Y) = \) the multiplicity of the simple module \( Bf_1/\text{rad} Bf_1 \) in \( \text{top} Y \). By results in Section 3, the sequence (ii) is always split exact in our situation. The following lemma gives a more direct proof for this fact.

**Lemma 4.8.** For any \( Y \in \text{mod} B \), (ii) is a split exact sequence.

**Proof.** It is sufficient to prove it for \( Y \) indecomposable. The conclusion is true if \( Y \) is a simple module. Now suppose \( Y \) is not simple. We have \( B A \otimes_B Y \simeq B A \otimes_A (A \otimes_B Y) \). We write \( A A \otimes_B Y \simeq Y_1 \oplus (Ae_1)^{m_Y}, \) where \( Y_1 \) has no direct summand isomorphic to \( Ae_1. \) So \( B A \otimes_A (A \otimes_B Y) \simeq B A \otimes_A Y_1 \oplus (Bf_1/\text{rad} Bf_1)^{m_Y}. \) Therefore we get an exact sequence of \( B \)-modules which is isomorphic to (ii):

\[ (\ast) \quad 0 \rightarrow Y \overset{(\alpha, \beta)}{\rightarrow} A \otimes_A Y_1 \oplus (Bf_1/\text{rad} Bf_1)^{m_Y} \rightarrow (Bf_1/\text{rad} Bf_1)^{m_Y} \rightarrow 0. \]

We claim that \( \alpha : Y \rightarrow B A \otimes_A Y_1 \) is an injective map and therefore an isomorphism. It then follows that (\( \ast \)) is split.

In fact, \( \ker \alpha \) is a submodule of \( Y \). If \( \ker \alpha \neq 0 \), then there are two cases.

**Case 1.** \( \ker \alpha \cap \text{rad} Y \neq 0 \). This is impossible since \( \alpha|_{\text{rad} Y} \) is injective.

**Case 2.** \( \ker \alpha \cap \text{rad} Y = 0 \). Hence \( \ker \alpha \subseteq \text{top} Y \) and \( \ker \alpha \) is a semisimple summand of \( Y \). This is also impossible since \( Y \) is indecomposable non-simple.

So \( \ker \alpha = 0 \) and \( \alpha \) is an isomorphism. \( \square \)
Note that Lemma 4.8 also implies the following: for any \( Y \in \text{mod} \ B \), \( B T \otimes_A (A \otimes_B Y) \simeq B T \otimes_A (Y_1 \oplus (Ae_1)_m Y) \simeq (B A \otimes_A Y_1) \oplus (B f_1)_m Y \simeq B Y \oplus (B f_1)_m Y \).

**Lemma 4.9.** There is a natural equivalence \((B T \otimes_A -) \circ (A A \otimes B -) \simeq \text{id}_{\text{mod} \ B} : \text{mod} \ B \rightarrow \text{mod} \ B\).

**Proof.** By Lemma 4.7 the functor \( A A \otimes B - : \text{mod} \ B \rightarrow \text{mod} \ A \) is full and dense. It is sufficient to prove that \( A A \otimes B - : \text{mod} \ B \rightarrow \text{mod} \ A \) is a faithful functor.

Given a map \( f : X \rightarrow Y \) in \( \text{mod} \ B \) such that \( A \otimes_B f : A \otimes_B X \rightarrow A \otimes_B Y \) factors through some projective \( A \)-module \( P' \) in \( \text{mod} \ A \), we have to show \( f : X \rightarrow Y \) also factors through some projective \( B \)-module in \( \text{mod} \ B \). Without loss of generality, we can assume that both \( X \) and \( Y \) are indecomposable non-projective and \( P' \) is the projective cover of \( A \otimes_B Y \). There are two cases to be considered.

**Case 1.** \( Y \simeq (B f_1 / \text{rad} B f_1) \).

We show \( f \) must be zero in this case. Suppose that \( f \neq 0 \), then \( f \) is an epimorphism. Tensoring with \( A \otimes_B - \) we have the following commutative diagram:

\[
\begin{array}{ccc}
A \otimes_B X & \xrightarrow{A \otimes_B f} & A \otimes_B Y \\
\omega & & \simeq A e_1 \oplus (Ae_1 / \text{rad} A e_1) \\
\downarrow & & \\
P' \simeq (A e_1 \oplus A e_1)
\end{array}
\]

It is clear that \( \omega \) must be an epimorphism, too. But then \( \omega \) is a split epimorphism and \( A e_1 \) is a summand of \( A \otimes_B X \). However, by the proof of Lemma 4.8, \( A A \otimes B X \simeq X_1 \oplus (Ae_1)^m X \), where \( X_1 \) is an indecomposable non-projective \( A \)-module, a contradiction.

**Case 2.** \( Y \not\simeq (B f_1 / \text{rad} B f_1) \).

By the proof of Lemma 4.8, we have \( A A \otimes_B Y \simeq Y_1 \oplus (Ae_1)^m Y \), where \( Y_1 \) is a non-projective indecomposable \( A \)-module. Further we can assume that \( A \otimes_B f : A \otimes_B X \rightarrow A \otimes_B Y \) factors through \( P' = (P'_1 \oplus (Ae_1)^m Y) \rightarrow A \otimes_B Y \), where \( P'_1 \) has no summand isomorphic to \( A e_1 \) and is a projective cover of \( Y_1 \). It follows that \( B A \otimes_A A \otimes_B f : B A \otimes_A A \otimes_B X \rightarrow B A \otimes_A A \otimes_B Y \) factors through \( B A \otimes_A P' \simeq Q'_1 \oplus (B f_1 / \text{rad} B f_1)^m Y \), where \( Q'_1 \simeq B A \otimes_A P'_1 \) is a projective \( B \)-module.

We have a commutative diagram:

\[
\begin{array}{ccc}
A \otimes_A A \otimes_B X & \xrightarrow{A \otimes_A A \otimes_B f} & A \otimes_A A \otimes_B Y \\
\downarrow & & \\
Q'_1 \oplus (B f_1 / \text{rad} B f_1)^m Y
\end{array}
\]

On the other hand, we have a natural transformation \( \varepsilon : \)

\[
\begin{array}{c}
B B \otimes B \rightarrow B A \otimes B \simeq B A \otimes_A A \otimes_B - : \text{mod} \ B \rightarrow \text{mod} \ B,
\end{array}
\]
which is induced from the exact sequence (ii). We now consider the following commutative diagram of $B$-modules:

\[
\begin{array}{ccc}
B \otimes_B X & \xrightarrow{B \otimes_B f} & B \otimes_B Y \\
\downarrow{\varepsilon_X} & & \downarrow{\varepsilon_Y} \\
A \otimes_A A \otimes_B X & \xrightarrow{A \otimes_A A \otimes_B f} & A \otimes_A A \otimes_B Y.
\end{array}
\]

By Lemma 4.8, $\varepsilon_Y$ splits by a map $u : A \otimes_A A \otimes_B Y \to B \otimes_B Y$. Since the composition $t_2u : (Bf_1/\text{rad } Bf_1)^m_Y \to B \otimes_B Y$ is a radical morphism and the simple module $Bf_1/\text{rad } Bf_1$ is a node, it must factor through a projective $B$-module. It follows that $f \simeq B \otimes_B f$ also factors through a projective $B$-module. \qed

Combing Lemma 4.7 and Lemma 4.9 we get the following.

**Theorem 4.10.** Let $B$ be a radical embedding obtained from $A$ by gluing a simple projective vertex and a simple injective vertex, and let $T$ be the bimodule defined as above. Then the tensor functors $B_T \otimes_A -$ and $A \otimes_B -$ induce mutually inverse stable equivalences between $\text{mod } A$ and $\text{mod } B$.

**Remarks.** (1) We have $T \otimes_A (\text{add}(Ae_1 \oplus Ae_n)) \subseteq \text{add}(Bf_1)$ and $A \otimes_B (\text{add}(Bf_1)) \subseteq \text{add}(Ae_1 \oplus Ae_n)$. Thus, according to Lemmas 4.6 and 4.8, we can state the above theorem in a stronger form: Let $A$, $B$ and $T$ be as in Theorem 4.10. Then the tensor functors $B_T \otimes_A -$ and $A \otimes_B -$ induce mutually inverse equivalences between the stable categories $\text{mod } A/\text{add}(Ae_1 \oplus Ae_n)$ and $\text{mod } B/\text{add}(Bf_1)$.

(2) Let $A$ and $B$ be as in Theorem 4.10. Then $B_A \otimes_A (\text{add}(L_1 \oplus L_n)) \subseteq \text{add}(L)$ and $A_A \otimes_B (\text{add}(L)) \subseteq \text{add}(L_1 \oplus L_n)$, where $L_1, L_n, L$ are the simple modules corresponding to vertices $e_1, e_n, f_1$. Thus, by a similar reason as in (1), the restriction functor $B_A \otimes_A -$ and the induction functor $A_A \otimes_B -$ induce mutually inverse equivalences between the stable categories $\text{mod } A/\text{add}(L_1 \oplus L_n)$ and $\text{mod } B/\text{add}(L)$. Let $\mathcal{E}$ be the full subcategory of $\text{mod } A$ consisting of $A$-modules which have no direct summand isomorphic to $L_1$ or $L_n$. Since $\text{mod } A/\text{add}(L_1 \oplus L_n) \simeq \mathcal{E}$, we have an equivalence: $\mathcal{E} \to \text{mod } B/\text{add}(L)$.

(3) Let $A$ and $B$ be as in Theorem 4.10. Recall that in [10] (see also [12]), $A$ is viewed as a triangular matrix algebra

\[
A = \begin{pmatrix} B/b & a \\ 0 & B/a \end{pmatrix},
\]
where $a = \tau_L(B)$ is the trace of the simple module $L$ (corresponding to the vertex $f_1$) in $B$, and $b = \text{ann}(a)$ is the annihilator of $a$. Note that $a = f_1(\text{rad } B)$ is a $B/b$-$B/a$-bimodule and that $b = (\text{rad } B)f_1 \oplus Bf$. The map

$$x \mapsto \begin{pmatrix} \bar{x} & x_2 \\ 0 & \bar{x} \end{pmatrix}$$

gives a radical embedding from $B$ to $A$, where $x = (x_1, x_2, x_3) \in B = kf_1 \oplus f_1(\text{rad } B) \oplus f B$. Each $A$-module can be described as a triple $(X, Y, f)$, where $X$ is a $B/b$-module, $Y$ is a $B/a$-module, and $f$ is a $B/b$-homomorphism: $a \otimes_{B/a} Y \to X$. Each homomorphism from $(X, Y, f)$ to $(X', Y', f')$ is a pair $(\varphi, \psi)$ in $\text{Hom}_{B/b}(X, X') \times \text{Hom}_{B/a}(Y, Y')$ such that $f \varphi = (1 \otimes \psi)f'$.

The $A$-module structure over $(X, Y, f)$ is given by

$$\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \otimes (x, y) \mapsto (ux + vy, wy)$$

for $u \in B/b$, $v \in a$, $w \in B/a$, $x \in X$ and $y \in Y$.

Martinez-Villa defined a functor $H : \text{mod } B \to \text{mod } A$ by $H(X) = (aX, X/aX, \mu_X)$, where $\mu_X : a \otimes_{B/a} (X/aX) \to aX$ is induced from multiplication, and he proved that $H$ induces a stable equivalence: $\text{mod } B \to \text{mod } A$. For any $B$-module $X$, we define

$$\pi_X : \begin{pmatrix} a \\ B/b \end{pmatrix} \otimes_B X \to (aX, X/aX, \mu_X)$$

by

$$\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \otimes x \mapsto (vx, wx).$$

The map $\pi_X$ is a well-defined $A$-homomorphism and functorial in $X$. We also define an $A$-homomorphism

$$\iota_X : (aX, X/aX, \mu_X) \to \begin{pmatrix} a \\ B/b \end{pmatrix} \otimes_B X$$

by

$$(x, \bar{y}) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes y.$$ 

It is easy to check that $\iota_X \circ \pi_X = \text{id}$ and therefore $\pi_X$ is a split epimorphism for each $B$-module $X$. The above discussion shows that the induction functor $A_B \otimes_B -$ and $H$ define the same stable equivalence: $\text{mod } B \to \text{mod } A$. Martinez-Villa does not construct the inverse functor. Instead he uses information on the module categories to show that his functor is an equivalence.

Under the assumption of the Auslander–Reiten conjecture (that is, if two algebras are stably equivalent with respect to projective modules, then they have the same number of isoclasses of non-projective simple modules), we can prove that the converse of the above theorem is also true.
Proposition 4.11. Let $B$ be a radical embedding obtained from $A$ by gluing two primitive idempotents. If $A$ and $B$ are stably equivalent with respect to projective modules and if the Auslander–Reiten conjecture is true for this stable equivalence and for the induced stable equivalence between the opposite algebras $A^{\text{op}}$ and $B^{\text{op}}$, then $B$ is obtained from $A$ by gluing a simple projective vertex and a simple injective vertex.

Proof. Suppose that $1 = e_1 + e_2 + \cdots + e_n$ is a decomposition of identity into primitive orthogonal idempotents in $A$, and that $B$ is a subalgebra obtained from $A$ by gluing $e_1$ and $e_n$, that is, $B$ has primitive idempotents $f_1 = e_1 + e_n$, $f_i = e_i$ ($2 \leq i \leq n - 1$), and $B$ and $A$ have the same radical. Clearly $e_i$ is a sink (respectively, source) in $A$ if and only if $f_i$ is a sink (respectively, source) in $B$ for all $2 \leq i \leq n - 1$. Assume now that $A$ and $B$ are stably equivalent with respect to projective modules. By Auslander–Reiten conjecture, at least one of $e_1$ and $e_n$ is a simple projective vertex. Without loss of generality, we assume that $e_1$ is a simple projective vertex. Since the opposite algebras $A^{\text{op}}$ and $B^{\text{op}}$ are also stably equivalent with respect to projective modules, the same reason shows that at least one of $e_1$ and $e_n$ is a simple projective vertex in $A^{\text{op}}$, or equivalently, a simple injective vertex in $A$. $e_1$ cannot be a simple injective vertex since otherwise the algebra $A$ will contain a semisimple summand. It follows that $e_n$ is a simple injective vertex and the conclusion follows.

We are now in the position to state our main result in this section.

Theorem 4.12. Let $A = kQ/I$ and $B = k\overline{Q}/\overline{I}$ be two finite dimensional algebras such that there is a radical embedding $f : B \to A$. Consider the following conditions.

(1) $A$ and $B$ are stably equivalent with respect to projective modules;
(2) $A$ and $B$ are stably equivalent with respect to semisimple modules;
(3) $B$ is obtained from $A$ by a finite number of steps of gluing a simple projective vertex and a simple injective vertex;
(4) There exists a pair of bimodules which induce inverse stable equivalences between $\text{mod} A$ and $\text{mod} B$, that is modulo projectives.

Then (2) and (3) are equivalent to each other, each implies (4) and thus also implies (1).

Under the assumption of the Auslander–Reiten conjecture, all four conditions are equivalent. In particular, if $A$ or $B$ has finite representation type, then all four conditions are equivalent.

Proof. Use Theorems 3.8 and 4.10 and proceed as in the proof of 3.8.

Remarks. (1) To show the equivalence of all conditions, we need the Auslander–Reiten conjecture. Conversely, suppose there exists a stable equivalence modulo projectives in a situation when $A$ and $B$ are not stably equivalent modulo semisimples, that is in a situation when in some step of the construction two idempotents are glued that are not a pair of simple injective and simple projective. Then this provides a counterexample to the Auslander–Reiten conjecture.

In fact, we only need a consequence of Auslander–Reiten conjecture: Let $A$ be a basic self-injective $k$-algebra and $1 = e + f$ where $e$ and $f$ are idempotents in $A$. Then $\text{mod} A$ and $\text{mod}(e Ae)$ cannot be stably equivalent.

(2) Under the above condition (2), the induction functor $A A \otimes_B -$ induces both a stable equivalence modulo projectives and a stable equivalence modulo semisimples between $A$ and $B$. 
Suppose that \( P \) is an indecomposable projective–injective \( B \)-module, then by Lemma 3.6, it is easy to show that \( A \otimes_B P \simeq Q \oplus S \), where \( Q \) is an indecomposable projective–injective \( A \)-module and \( S \) is a semisimple projective \( A \)-module. It follows easily that \( A \otimes_B (P/\text{soc}(P)) \simeq (Q/\text{soc}(Q)) \oplus (\text{some semisimple projective } A \text{-module}) \). By [11, Proposition 3], the induction functor induces a stable equivalence modulo projectives between \( A/\text{soc}(P) \) and \( B/\text{soc}(Q) \). Clearly the induction functor also induces a stable equivalence modulo semisimples between \( A/\text{soc}(P) \) and \( B/\text{soc}(Q) \).

(3) Krause [7] proved that the stable equivalences modulo projectives induced by a pair of bimodules preserve the representation type of algebras (note that there he used the notion of stable equivalence of Morita type for a stable equivalence induced by a pair of bimodules not necessarily projective on both sides).

Acknowledgments

This paper has been written during two research visits of the second author to University of Koeln, supported by the Leverhulme funded network ‘Algebras, Representations and Applications’ and by a Humboldt fellowship. The authors wish to thank the referee for valuable suggestions.

References