Rational Interpolation and State-Variable Realizations

B. D. O. Anderson*
Department of Systems Engineering
Research School of Physical Sciences, Australian National University,
Canberra, A.C.T. 2601.

and

A. C. Antoulas
Mathematical System Theory
E. T. H. Zürich
CH-8092 Zürich
and
Department of Electrical and Computer Engineering
Rice University
Houston, Texas 77251

Submitted by Joseph A. Ball

ABSTRACT

The problem is considered of passing from interpolation data for a real rational transfer-function matrix to a minimal state-variable realization of the transfer-function matrix. The tool is a Loewner matrix, which is a generalization of the standard Hankel matrix of linear system realization theory, and which possesses a decomposition into a product of generalized observability and controllability matrices.

1. INTRODUCTION

Let $W(s)$ be a real rational transfer-function matrix, with $W(\infty)$ finite. Define matrices W_i (the Markov coefficients) via

$$W(s) = W_0 + W_1 s^{-1} + W_2 s^{-2} + \cdots . \tag{1}$$

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655 Avenue of the Americas, New York, NY 10010 0024-3795/90/$3.50

The conventional realization problem of linear system theory is one of constructing from the infinite sequence $\{W_i\}$ the transfer-function matrix $W(s)$ or a state-variable realization thereof, i.e. a quadruple of real constant matrices A, B, C, D for which

$$W(s) = D + C'(sI - A)^{-1}B. \tag{2}$$

Generally, the constraint that A is of least dimension is applied. See e.g. [1, 2] for a treatment.

The study of this problem is greatly aided by the concept of Hankel, controllability, and observability matrices. An important identity is that

$$\begin{bmatrix} W_1 & W_2 & W_3 & \cdots \\ W_2 & W_3 & W_4 & \cdots \\ W_3 & W_4 & W_5 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} C' \\ C'A \\ C'A^2 \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}. \tag{3}$$

with the infinite Hankel matrix on the left possessing a finite rank, the McMillan degree of $W(s)$; for minimal dimension A, the factorization on the right is into two matrices with full column rank (the infinite observability matrix) and full row rank (the infinite controllability matrix). The factorization can be exploited in the realization problem.

Sometimes, the data are not an infinite sequence of W_i but a finite sequence. One is then faced with a *partial realization* problem, and a finite version of (3). This is discussed in [1, 3].

The transformation $s \to 1/s$ produces

$$W(s^{-1}) = W_0 + W_1 s + W_2 s^2 + \cdots, \tag{4}$$

and one can thus regard the W_i as providing interpolation data concerning $W(s^{-1})$ at $s = 0$ [i.e. the values of $W(s^{-1})$ and its derivatives at $s = 0$]. More or less equivalently, we can regard the W_i as providing interpolation data at $s = \infty$ for $W(s)$.

Now let us ask what happens when the interpolation data concerning $W(s)$ are not confined to $s = \infty$, but can be associated with arbitrary points in the complex plane. Clearly, we still have a form of (partial) realization problem. In [4], we examined this problem under two significant restrictions: first, that $W(s)$ was a scalar transfer function, and second, that we sought to construct $W(s)$ alone, rather than a state-variable realization, in the process eschewing examination of identities analogous to (3).

In this paper, our aim is to remove these restrictions. In particular, we consider matrix transfer functions, we find state-variable realizations, and we present a (finite) analogue for (3). The Hankel matrix is replaced by a Loewner matrix [4–6], and the factors on the right side are replaced by generalized observability and controllability matrices (defined below). We have to take care to distinguish proper and nonproper $W(s)$, this being more of an issue than for the conventional realization problem.

In [4], we cited a number of occurrences of the interpolation problem in linear system theory. These continue to be of relevance when we are working with transfer-function matrices rather than scalar transfer functions, and a state-variable description may often be preferred.

The paper is structured as follows: The next section is a review of key ideas from [4]. In Section 3, we deduce a number of properties of a (generalized) Loewner matrix and display the factorization analogous to (3). For clarity of exposition this section is divided into two parts, labeled 3 and 3R, for the distinct and the repeated-point case, respectively. The results are used in Section 4 to present a construction for the quadruple $\{A, B, C, D\}$. Section 5 contains some remarks on nonproper transfer functions, and Section 6 some concluding remarks. In contrast to [4], we pay no attention to the issue of recursion, i.e., taking an $\{A, B, C, D\}$ solution of an interpolation problem and then stating how to modify it when one acquires an additional interpolation datum.

2. THE LOEWNER MATRIX

In this section we review the principal results of [4]. Consider first the problem of interpolating given distinct points. Thus the data are an array $P := \{(s_i, y_i),\ i \in \underline{N}\}$, with $s_i \neq s_j$, and $s_i \in C$, $y_i \in C$. If we are interested in interpolation with real functions, then $s_i = s_j^*$ implies $y_i = y_j^*$. A rational function

$$y(s) = \frac{n(s)}{d(s)} \tag{5}$$

with n, d coprime is said to interpolate the above points iff

$$y(s_i) = y_i, \qquad i \in \underline{N}. \tag{6}$$

The rational interpolation problem is the problem of constructing one, or all, interpolating functions, sometimes with certain side conditions, such as minimality of the McMillan degree of $y(\cdot)$, denoted $\deg y(s)$, and given by $\max[\deg n(s), \deg d(s)]$.

A key tool for studying this problem is the Loewner matrix. Consider the rational function $y(s)$ defined by the identity

$$\sum_{j=1}^{r+1} c_j \frac{y(s)-y_j}{s-s_j} = 0, \qquad c_j \neq 0 \text{ but otherwise arbitrary.} \tag{7}$$

Generically, $\deg y(s) = r$. Clearly, $y(s_j) = y_j$ for $j = 1,2,\dots,r+1$, and if $r+1 = N$, then all the interpolation conditions (6) are fulfilled, with $y(s)$ of degree $N-1$ (generically). However, interpolation of N points should be possible with a $y(s)$ of degree approximately $N/2$. It turns out that if we choose $r+1 < N$ in (7) and choose the c_j in a specific way, then, subject to the satisfaction of a certain side condition given later (and satisfiability is generic), the entire N points can be interpolated. In particular, in order to interpolate the points indexed by $j = r+2,\dots,N$, the coefficients c_j have to satisfy

$$\sum_{j=1}^{r+1} c_j \frac{y_{r+1+i}-y_j}{s_{r+1+i}-s_i} = 0, \qquad i = 1,2,\dots,N-r-1, \tag{8}$$

or

$$Lc = 0, \tag{9}$$

where $c = (c_1,\dots,c_{r+1})'$ and L is a matrix of dimensions $(N-r-1)\times(r+1)$, the Loewner matrix, with

$$L_{ij} = \frac{y_{r+1+i}-y_j}{s_{r+1+i}-s_j}. \tag{10}$$

A key property of L, given in [4] and formally stated below in a comprehensive theorem, is the following: Given a rational function $y(s)$, let the pairs (s_i, y_i) be obtained by sampling $y(s)$. If L is any $p \times q$ Loewner matrix formed form these pairs with $p, q \geqslant \deg y$, there holds

$$\operatorname{rank} L = \deg y. \tag{11}$$

As a corollary, every square Loewner matrix of size $\deg y$ formed from a subset of the above pairs of points, and thus any square submatrix of L of size $\deg y$, is nonsingular.

Before reviewing the main result, we shall explain how to treat multiple points. These are points s_i at which information is available not only about the value of the function, but also about the values of a certain number of derivatives. The key is to define a *generalized* Loewner matrix, which still has the property (11). Let ν_i be the multiplicity of s_i, with $s_i \neq s_j$ for $i \neq j$. There are θ distinct s_i, and $\nu_1 + \nu_2 + \cdots + \nu_\theta = N$. The array is written as

$$P := \{(s_i; y_{i,j-1}) : (i,j) \in I\}, \qquad I := \{(i,j) : j \in \underline{\nu_i}, i \in \underline{\theta}\}, \tag{12}$$

and a rational function is said to interpolate P if

$$D^{j-1}y(s_i) = y_{i,j-1}, \qquad (i,j) \in I. \tag{13}$$

Here, D denotes differentiation with respect to s. Thus the array information is

$$P = \{s_i; y(s_i), \dots, y^{\nu_i - 1}(s_i), \dots, s_\theta; y(s_\theta), \dots, y^{\nu_\theta - 1}(s_\theta)\}. \tag{14}$$

The array has distinct points just when $\nu_i = 1$ for all i, and $y_{i,0}$ is what was earlier denoted by y_i.

Let Q denote the set of s_i, with each listed ν_i times. Partition Q arbitrarily into two nonempty sets R, T, called the *row set* and *column set* respectively. The sum of the number of occurrences of s_i in R and T is ν_i. The elements of R are ordered and denoted by r_i, $i = 1, 2, \dots, |R|$, and those of T, also ordered, by t_j, $j = 1, 2, \dots, |T|$. Assume that $|T| = r + 1$. Thus

$$R = \{r_i := s_k' \text{ for some } k \in \underline{\theta}, i \in \underline{N - r - 1}\},$$

$$T = \{t_j := s_l' \text{ for some } l \in \underline{\theta}, j \in \underline{r + 1}\}.$$

To each such partitioning of Q, we associate an $(N - r - 1) \times (r + 1)$ matrix L, referred to as a Loewner or generalized Loewner matrix according as $\nu_i = 1$ for all i or $\nu_i > 1$ for some i. To determine L_{ij} we need to know how many times the value assumed by r_i occurs in the subset $\{r_1, \dots, r_{i-1}\}$ of R

and how many times the value assumed by t_j occurs in the subset $\{t_1, \dots, t_{j-1}\}$ of T. Let these two nonnegative integers be k, l respectively. Then

$$L_{ij} := D_r^k D_t^l \left\{ \frac{y(r) - y(t)}{r - t} \right\}_{r = r_i,\, t = t_j} \qquad \text{if} \quad r_i \neq t_j \tag{15}$$

and

$$L_{ij} = \frac{k!\,l!}{(k + l + 1)!} D_t^{(k+l+1)} y(t)\Big|_{t = t_j} \qquad \text{if} \quad r_i = t_j. \tag{16}$$

EXAMPLE. Suppose $P = \{(s_1; y_{10}), (s_2; y_{20}, y_{21}, y_{22}, y_{23}), (s_3; y_{30})\}$. Take $R = \{r_1, r_2, r_3\} = \{s_3, s_2, s_2\}$ and $T = \{t_1, t_2, t_3\} = \{s_1, s_2, s_2\}$. Then

$$L = \begin{bmatrix} \dfrac{y_{30} - y_{10}}{s_3 - s_1} & \dfrac{y_{30} - y_{20}}{s_3 - s_2} & \dfrac{\partial}{\partial t}\left(\dfrac{y_3 - y(t)}{s_3 - t} \right)_{t = s_2} \\[2ex] \dfrac{y_{20} - y_{10}}{s_2 - s_1} & \dfrac{d}{dt} y(t)\Big|_{t = s_2} & \dfrac{1}{2!}\dfrac{d^2}{dt^2}[y(t)]\Big|_{t = s_2} \\[2ex] \dfrac{\partial}{\partial r}\left(\dfrac{y(r) - y_{10}}{r - s_1} \right)_{r = s_2} & \dfrac{1}{2!}\dfrac{d^2}{dt^2}[y(t)]\Big|_{t = s_2} & \dfrac{1}{3!}\dfrac{d^3}{dt^3}[y(t)]\Big|_{t = s_2} \end{bmatrix}.$$

Note that any submatrix of a Loewner matrix is again a Loewner matrix, while only certain submatrices of a generalized Loewner matrix are generalized Loewner matrices. For example, the submatrix formed from rows 1, 2 and columns 1, 2 in the example above is a generalized Loewner matrix, but the submatrix formed from rows 1, 3 and columns 1, 2 or columns 1, 3 is not.

The definition of the generalized Loewner matrix is, not surprisingly, such that the result (11) continues to hold; see the main theorem below. For use in the main theorem, we also need to define the generalized Loewner matrix L^* which is constructed from L by rearranging the row and column sets (through reassignment of the last element of the column set to be the last element of the row set); thus

$$R^* = R \cup \{t_{r+1}\} = \{r_1, r_2, \dots, r_{N-r-1}, t_{r+1}\},$$

$$T^* = T - \{t_{r+1}\} = \{t_1, t_2, \dots, t_r\}.$$

The main result of [4] is:

THEOREM 2.1. *Given the N pairs of points* $(s_i, y_{i,j-1})$, $j \in \underline{\nu}_i$, $i \in \underline{\theta}$, *let L be a square or almost square generalized Loewner matrix, so that L is* $r \times r$ *or* $r \times (r+1)$ *with* $r =$ *(integer part of* $N/2$*), according as N is even or odd. Let* rank $L = q$, *and suppose that if N is even,* $q < r$. *Then*

(a) *If all* $q \times q$ *submatrices of L and* L^* *are nonsingular, the minimal-McMillan-degree rational function* $y^{\min}(s)$ *interpolating the given points satisfies*

$$\deg y^{\min} = q, \tag{17}$$

and in this case, $y^{\min}(s)$ *is the unique interpolating function of degree* q, *and the degrees of all possible interpolating functions are* $q, N-q, N-q+1, \dots$.

(b) *If the condition in (a) is not satisfied, then*

$$\deg y^{\min} = N - q, \tag{18}$$

and $y^{\min}(s)$ *is not unique. The degrees of all possible interpolating functions are* $N-q, N-q+1, \dots$.

REMARKS.

(a) When all s_i are distinct, the formulae (7) and (9) can be used for the construction of $y(s)$ in case $\deg y^{\min} = q$. Generalization is possible for the case of repeated s_i. In case $\deg y^{\min} > q$ and/or one seeks interpolating functions of degree at most $N - q + \pi - 1$ for $\pi = 1, 2, \dots, q$, one proceeds as follows. Let L_π denote a Loewner matrix of size $(q-\pi) \times (N-q+\pi)$, obtained via reassignment of some of the row set defining L to to the column set for L_π. Let $\mathscr{C}_\pi$ be the set of column vectors $c_\pi = (c_1, c_2, \dots, c_{N-q+\pi})'$ satisfying

$$L_\pi c_\pi = 0 \tag{19}$$

and such that with

$$d_{L_\pi}(s) = \left(\sum_{j=1}^{N-q+\pi} \frac{c_j}{s - t_j} \right) \left(\prod_{j=1}^{N-q+\pi} (s - t_j) \right) \tag{20}$$

there holds

$$d_{L_\pi}(s_i) \neq 0, \qquad i = 1, 2, \dots, N. \tag{21}$$

Then (7) with $r+1 = N - q + \pi$ yields $y(s)$. Again, generalization is possible when there are repeated points. The family of all interpolating functions of degree at most $N - q + \pi - 1$ is parametrizable in terms of $N - 2q + 2\pi - 1$ parameters, since the normalized c_π are parametrized in terms of $N - 2q + 2\pi - 1$ parameters. [The effect of (21) is inessential on this conclusion.]

(b) There is a simple condition for the interpolating function to be proper, viz.

$$\sum_{j=1}^{q+1} c_j \neq 0 \qquad \text{when} \quad \deg y^{\min} = q, \tag{22}$$

$$\sum_{j=1}^{N-q+\pi} c_j \neq 0 \qquad \text{when} \quad \deg y^{\min} = N - q, \quad \deg y = (N - q + \pi - 1). \tag{23}$$

Note that in case $\deg y^{\min} = q$ the c_j are unique (to within scaling) and it may be that the unique $y^{\min}$ is improper. In this case, or when $\deg y^{\min} > q$, there always exists a proper interpolating $y(s)$ with McMillan degree $N - q$.

(c) Realization data at $s = \infty$, or Markov coefficients (such as arise in the usual linear system-theory problems) can be accommodated. If Markov coefficients are known, i.e. coefficients in a power-series expansion of $f(s)$ in powers of s^{-1}, then one can work with $g(s) = f(s^{-1})$, in which case the Markov coefficients of $f(s)$ become equivalent to $g(0), g'(0), \dots$. More generally, one can work with

$$g(s) = f\left(\frac{as+b}{cs+d}\right) \qquad \text{with} \quad ad - bc \neq 0.$$

The conventional Hankel matrix of realization theory becomes a generalized Loewner matrix.

(d) Recursive solutions to the interpolation problem are also available.

(e) The dichotomy that either $\deg y^{\min} = q$ or $\deg y^{\min} = N - q$ with nothing in between—i.e., $\deg y^{\min} = q + 1$ is excluded (unless $N - q =$

$q+1)$—is explained in [4], but in a fairly technical way. We can offer more insight in an alternative way, provided we appeal to the remark stated immediately after (11). Suppose that L has rank q, has a singular $q \times q$ submatrix, and is of dimension $m \times (m+1)$; thus $N = 2m+1$. Then $\deg y^{\min} = q$ is excluded by this singularity [see remark following (11)]. Suppose that $\deg y^{\min} = g$ say. Let us take this $y^{\min}(s)$ and evaluate it at a whole further collection of points additional to those used in the construction of L, so that in all, $2g+1$ points are involved. Let us then set up a Loewner matrix of dimension $g \times (g+1)$, call it L_e; the row set and column set used for L_e include the row set and column set used for L. Thus we have

$$L_e = \begin{array}{cc} & \begin{array}{cc} m+1 & g-m \end{array} \\ \begin{array}{c} m \\ g-m \end{array} & \begin{bmatrix} L & L_a \\ L_b & L_e \end{bmatrix} \end{array}$$

with every $g \times g$ submatrix of L_e nonsingular, by virtue of (11) and the following remark, and with L possessing rank q (and a singular $q \times q$ submatrix). A necessary consequence is that $g \geqslant N - q$. To see this, let $\bar{L}_a, \bar{L}_c$ denote L_a, L_c without their last columns, and observe that by row and column operations the following hold:

$$g = \operatorname{rank}\begin{bmatrix} L & \bar{L}_a \\ L_b & \bar{L}_c \end{bmatrix}$$

$$\Leftrightarrow \quad g = \operatorname{rank}\left(\begin{array}{cc} & \begin{array}{ccc} \phantom{I_{q\times q}} & m-q+1 & g-m+1 \end{array} \\ \begin{array}{c} \\ m-q \\ g-m \end{array} & \begin{bmatrix} I_{q\times q} & 0 & \bar{L}_{a_1} \\ 0 & 0 & \bar{L}_{a_2} \\ L_{b_1} & L_{b_2} & \bar{L}_c \end{bmatrix} \end{array} \right)$$

$$\Leftrightarrow \quad g = \operatorname{rank}\left(\begin{array}{cc} & \begin{array}{ccc} \phantom{I_{q\times q}} & m-q+1 & g-m-1 \end{array} \\ \begin{array}{c} \\ m-q \\ g-m \end{array} & \begin{bmatrix} I_{q\times q} & 0 & 0 \\ 0 & 0 & \bar{L}_{a_2} \\ 0 & L_{b_2} & \bar{L}_c \end{bmatrix} \end{array} \right)$$

$$\Leftrightarrow \quad g - q = \operatorname{rank}\left(\begin{array}{cc} & \begin{array}{cc} m-q+1 & g-m-1 \end{array} \\ \begin{array}{c} m-q \\ g-m \end{array} & \begin{bmatrix} 0 & \bar{L}_{a_2} \\ L_{b_2} & \bar{L}_c \end{bmatrix} \end{array} \right).$$

Since this last matrix has to be of full row rank, it is necessary that $\bar{L}_{a_2}$ be of full row rank, which can only be so if it has at least as many columns as rows, i.e., $g - m - 1 \geqslant m - q$ or $g \geqslant N - q$. (Note that this argument only establishes that if $\deg y^{\min} \neq q$, then necessarily $\deg y^{\min} \geqslant N - q$; it does not show that $\deg y^{\min} = N - q$.)

3. STATE-VARIABLE REALIZATIONS AND BLOCK LOEWNER MATRICES

In the previous section, we reviewed a number of results on Loewner matrices associated with the interpolation of scalar transfer functions. In this section, we shall establish new results applicable to the interpolation of real rational matrix transfer functions. We shall begin with the supposition that such a matrix transfer function exists, and derive properties of the associated (block) Loewner matrix. In the next section, we shall reverse the procedure, by showing how we can start with a (block) Loewner matrix possessing various properties, and construct therefrom a state-variable realization of an interpolating matrix transfer function.

The repeated-point versions of the results given in this section are collected in Section 3R. This is done in order to avoid clouding the main issues with unnecessary complications.

Suppose there is given a real rational transfer-function matrix $Y(s)$ of dimensions $\alpha \times \beta$, proper, and possessing a minimal state-variable realization $\{A, B, C, D\}$, i.e.

$$Y(s) = D + C'(sI - A)^{-1}B. \tag{24}$$

Now observe that

$$\begin{aligned}\frac{Y(r) - Y(t)}{r - t} &= \frac{C'(rI - A)^{-1}B - C'(tI - A)^{-1}B}{r - t}\\ &= \frac{C'(rI - A)^{-1}[(tI - A) - (rI - A)](tI - A)^{-1}B}{r - t}\\ &= -C'(rI - A)^{-1}(tI - A)^{-1}B. \end{aligned}\tag{25}$$

Now let us define a block Loewner matrix L associated with the transfer-function matrix $Y(s)$, using the obvious generalizations of (10) and (15),(16). Suppose the row set is

$$R = \{r_1, r_2, \dots, r_\gamma\} \tag{26}$$

with $r_i \neq r_j$ for $i \neq j$, and the column set is

$$T = \{t_1, t_2, \dots, t_\delta\} \tag{27}$$

with $t_i \neq t_j$ for $i \neq j$; and allow the possibility that $R \cap T \neq \varnothing$. Then (25) implies that

$$-L = \begin{bmatrix} C'(r_1 I - A)^{-1} \\ C'(r_2 I - A)^{-1} \\ \vdots \\ C'(r_\gamma I - A)^{-1} \end{bmatrix} \left[(t_1 I - A)^{-1} B, (t_2 I - A)^{-1} B, \dots, (t_\delta I - A)^{-1} B \right]. \tag{28}$$

[The generalization of (16) to the matrix case with $k = l = 0$ is needed in case $r_i = t_j$.]

The matrices appearing on the right side of (28) can be thought of as *generalized controllability* and *observability matrices*. The key property of such matrices is as follows:

LEMMA 3.1. *Let (A, B) be a controllable pair with A of dimension $q \times q$. Let t_i, $i = 1, \dots, \delta$, be distinct points with $\delta \geqslant q$, none of which is an eigenvalue of A. Then*

$$\operatorname{rank}\left[(t_1 I - A)^{-1} B \cdots (t_\delta I - A)^{-1} B \right] = q. \tag{29}$$

The proof of this result is provided in Section 3R following the statement of the multiple-point version.

An immediate consequence of the lemma and the decomposition of (28) and the later (28R) is the following theorem, which is almost the same in statement for the nonrepeated- and repeated-point cases.

THEOREM 3.1. *Let $Y(s)$ be a proper transfer-function matrix with minimal state-variable realization $\{A, B, C, D\}$, and A of dimension $q \times q$. Suppose interpolation data $P := \{s_i; Y(s_i), Y'(s_i), \dots Y^{\nu_i - 1}(s_i),\ i = 1, \dots, \theta\}$ are given. Make an arbitrary partition of the s_i into row sets R and T as in* (26), (26R) *and in* (27), (27R), *and let the generalized block Loewner matrix be constructed using* (10) *and* (15), (16). *Assume that $|R| \geqslant q$, $|T| \geqslant q$. Then* $\operatorname{rank} L = q$. *If $|T| \geqslant q+1$, and if the last element of the column set is reassigned as the last element of the row set and a new Loewner matrix L^* is constructed, then* $\operatorname{rank} L^* = q$. *Further, any generalized block Loewner matrix which is a submatrix of L or L^* with at least q block columns and q block rows also has rank q.*

In this section, we have worked with Loewner matrices derived from proper $Y(s)$. As a result of the properness, the Loewner matrix inherits a further property. It is tied to the property given in the last section, to the effect that the sum of the entries of a right null vector of the Loewner matrix must be nonzero, but is far richer in its statement.

We shall state and prove the next lemma (which establishes the property) first for the case when there are no repeated points. Partition the generalized controllability matrix

$$N = \left[(t_1 I - A)^{-1} B, (t_2 I - A)^{-1} B, \dots, (t_\delta I - A)^{-1} B \right] \tag{30}$$

as

$$N = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \tag{31}$$

with

$$N_1 = (t_1 I - A)^{-1} B. \tag{32}$$

Define also

$$\begin{aligned} \overline{N} &= N_2 - [N_1, N_1, \dots, N_1] \\ &= N \begin{bmatrix} -I & -I & \cdots & -I \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} \end{aligned} \tag{33}$$

and

$$\tilde{N} = N_2 \operatorname{diag}[t_2 I, t_3 I, \dots, t_\delta I] - t_1[N_1, N_1, \dots, N_1]$$

$$= N \begin{bmatrix} -t_1 I & -t_1 I & \cdots & -t_1 I \\ t_2 I & 0 & \cdots & 0 \\ 0 & t_3 I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_\delta I \end{bmatrix}. \tag{34}$$

LEMMA 3.2. *Using the above notation, and assuming that $\delta \geqslant q$, $\bar{N}$ has rank q, and*

$$A\bar{N} = \tilde{N}. \tag{35}$$

Proof. Observe that

$$(t_j I - A)^{-1}B - (t_1 I - A)^{-1}B = (t_1 I - A)^{-1}(t_j I - A)^{-1}B(t_1 - t_j).$$

Hence

$$\bar{N} = (t_1 I - A)^{-1}\left[(t_2 I - A)^{-1}B, \dots, (t_\delta I - A)^{-1}B\right]$$
$$\times \operatorname{diag}[(t_1 - t_2)I, \dots, (t_1 - t_\delta)I].$$

The first and last matrices in the product on the right are nonsingular, while the middle matrix has rank q, by Lemma 3.1. This proves the first claim of the lemma. For the second claim, observe using the past equality above that

$$A\bar{N} = -(t_1 I - A)\bar{N} + t_1 \bar{N}$$
$$= -N_2 \operatorname{diag}[(t_1 - t_2)I, \dots, (t_1 - t_\delta)I] + t_1 N_2 - t_1[N_1, N_1, \dots, N_1]$$
$$= N_2 \operatorname{diag}[t_2 I, \dots, t_\delta I] - t_1[N_1, N_1, \dots, N_1]$$
$$= \tilde{N}.$$

■

The formulae (28) and (28R) relate L to N through premultiplication by a generalized observability matrix, of full column rank in case the row set is big enough. It is accordingly immediate to translate the conclusions of Lemma 3.2 to block Loewner matrices and generalized block Loewner matrices.

For this purpose we define

$$L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}, \tag{36}$$

where L_1 is the first block column of L and define

$$Q = L_2 - [L_1, L_1, \dots, L_1] = LJ, \tag{37}$$

where

$$J = \begin{bmatrix} -I & -I & \cdots & -I \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} \tag{38}$$

and

$$R = L_2 \operatorname{diag}[t_2 I, t_3 I, \dots, t_\delta I] - t_1[L_1, L_1, \dots, L_1] = LJ_t, \tag{39}$$

where

$$J_t = \begin{bmatrix} -t_1 I & -t_1 I & \cdots & -t_1 I \\ t_2 I & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t_\delta I \end{bmatrix}. \tag{40}$$

Theorem 3.2. *Adopt the same hypotheses as Theorem* 3.1, *save that* $\nu_i = 1$ *for all* i, *with the notation introduced above, and assume that* $\delta > q$. *Then* Q *has rank* q, *and* $Qx = 0$ *for some* $x \neq 0$ *implies* $Rx = 0$.

Proof. Let M denote the generalized observability matrix

$$M = \begin{bmatrix} C'(r_1 I - A)^{-1} \\ \vdots \\ C'(r_\gamma I - A)^{-1} \end{bmatrix}, \tag{41}$$

which has rank q. Then $Q = M\bar{N}$, $R = M\tilde{N}$. Because $M, \bar{N}$ have full column and row rank respectively, viz. q, we have $\operatorname{rank} Q = q$. Also $M'M$ is nonsingular. Hence $Qx = 0$ implies $(M'M)^{-1}M'Qx = 0$, or $\bar{N}x = 0$. The rest is trivial. ■

Let us observe a certain connection between Theorem 3.2 and the condition, applicable in the scalar-transfer-function case, that the interpolating function is proper. In case $\delta = q + 1$, the matrix Q is $q \times q$. The theorem states that under certain hypotheses, including properness of the underlying transfer function, Q has rank q. Violation of this would imply that $Q[\beta_2, \beta_3, \dots, \beta_{q+1}]' = 0$ for some β_i, which in the light of (37) would imply

$$L\left[-\left(\sum_{i=2}^{q+1} \beta_i \right), \beta_2, \beta_3, \dots, \beta_{q+1} \right]' = 0,$$

and this is a violation of (9) and (22). Conversely, if $Lc = 0$ with $\sum_{i=1}^{q+1} c_i = 0$, there follows $Q[c_2, c_3, \dots c_{q+1}]' = 0$, which shows that $\operatorname{rank} Q = q$ is false.

Of course, Theorem 3.2 encompasses much more that the properness issue. Through its tie with Lemma 3.2, it will prove the basis for solving the construction problem in the next section.

3R. STATE-VARIABLE REALIZATIONS AND BLOCK LOEWNER MATRICES: THE REPEATED-POINT CASE

In this section, for the sake of clarity of exposition, we collect the repeated-point versions of the results of Section 3.

The first is concerned with (25):

$$D_r^k D_t^l \left\{ \frac{Y(r) - Y(t)}{r - t} \right\}$$

$$= -(-1)^k (-1)^l k!\,l!\,C'(rI - A)^{-(k+1)}(tI - A)^{-(l+1)}B \tag{25aR}$$

and

$$\frac{k!l!}{(k+l+1)!}D_t^{(k+l+1)}Y(t) = (-1)^{k+l+1}k!l!C'(tI-A)^{-(k+l+2)}B. \tag{25bR}$$

The row and column sets are

$$R = \Big\{\underbrace{r_1,\dots,r_1}_{\lambda_1 \text{ times}}, \underbrace{r_2,\dots,r_2}_{\lambda_2 \text{ times}},\dots, \underbrace{r_\gamma,\dots,r_\gamma}_{\lambda_\gamma \text{ times}}\Big\}, \tag{26R}$$

$$T = \Big\{\underbrace{t_1,\dots,t_1}_{\mu_1 \text{ times}}, \underbrace{t_2,\dots,t_2}_{\mu_2 \text{ times}},\dots, \underbrace{t_\delta,\dots,t_\delta}_{\mu_2 \text{ times}}\Big\}. \tag{27R}$$

The formula (28) is varied if one or more of the λ_i, μ_i exceed 1; we have a generalized block Loewner matrix:

$$-L = \begin{bmatrix} C'(r_1I-A)^{-1} \\ -C'(r_1I-A)^{-2} \\ \vdots \\ (-1)^{\lambda_1-1}(\lambda_1-1)!C'(r_1I-A)^{-\lambda_1} \\ C'(r_2I-A)^{-1} \\ \vdots \\ (-1)^{\lambda_2-1}(\lambda_2-1)!C'(r_2I-A)^{-\lambda_2} \\ \vdots \end{bmatrix}$$

$$\times\Big[(t_1I-A)^{-1}B,\dots,(-1)^{\mu_1-1}(\mu_1-1)!(t_1I-A)^{-\mu_1}B,\dots,$$

$$(-1)^{\mu_\delta-1}(\mu_\delta-1)!(t_\delta I-A)^{\mu_\delta}B\Big]. \tag{28R}$$

Note that, in contrast to the formulation of Section 2, (28) and (28R) do not change in case $r_i = t_j$ for some i, j pair. Furthermore we have

LEMMA 3.1R. *Let $[A, B]$ be a controllable pair with A of dimension $q\times q$. Let t_i for $i = 1,2,\dots,\delta$ be distinct points none of which is an eigen-*

value of A, *and let* μ_i *for* $i = 1, 2, \ldots, \delta$ *be positive integers with* $\Sigma\mu_i \geqslant q$. *Then*

$$\operatorname{rank}\Big[(t_1 I - A)^{-1}B, (t_1 I - A)^{-2}B, \ldots, (t_1 I - A)^{-\mu_1}B, \ldots, (t_\delta I - A)^{-\mu_\delta}B\Big] = q. \qquad \text{(29R)}$$

Proof. Suppose (29R) fails. Then there is a nonzero row vector ω' in the left nullspace of the generalized controllability matrix. It follows that the transfer-function matrix (actually a row vector of transfer functions) $\omega'(sI - A)^{-1}B$ has μ_1 zeros at t_1, μ_2 at $t_2, \ldots,$ μ_δ at t_δ. In particular, each of the transfer functions $\omega'(sI - A)^{-1}Be_j$, $j = 1, \ldots, \beta$, with e_j a unit vector, has $\Sigma\mu_i \geqslant q = \dim A$ zeros. The numerator of $\omega'(sI - A)^{-1}Be_j$ has degree at most $q - 1$, and so must be identically zero. Thus $\omega'(sI - A)^{-1}Be_j \equiv 0$ for all j, i.e. $\omega'(sI - A)^{-1}B \equiv 0$, or $\omega' e^{At}B \equiv 0$. This violates the requirement that $[A, B]$ is controllable.

Remark. It is trivial to extend the above lemma to cope with matrices such as occur as the right member in the product of (28R), differing from the matrix in (29R) by inessential column scaling. Extension is also trivial to matrices

$$\Big[B, AB, \ldots, A^{\mu_0 - 1}B, (t_1 I - A)^{-1}B, \ldots, (t_\delta I - A)^{-\mu_\delta}B\Big],$$

where $\mu_0 + \mu_1 + \cdots + \mu_\delta \geqslant q$. Such matrices arise when we mix finite interpolating points and Markov-parameter data, which is akin to having data at the interpolating point $s = \infty$.

Following we shall state, without proof, the version of the Lemma 3.2 applying with repeated points. The proof is of course similar to the case when there are no repeated points, but involves much more algebraic manipulation. Even the lemma statement is much more involved. For the case when there are repeated points, we define

$$N = \Big[(t_1 I - A)^{-1}B, \ldots, (-1)^{\mu_1 - 1}(\mu_1 - 1)!(t_1 I - A)^{-\mu_1}B, \ldots, (-1)^{\mu_\delta - 1}(\mu_\delta - 1)!(t_\delta I - A)^{-\mu_\delta}B\Big]. \qquad \text{(30R)}$$

Recall that

$$N = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \tag{31}$$

with N_1 as in (32). Define

$$\bar{N} = N_2 - \Big[\overbrace{0,\dots,0}^{\mu_1 - 1}, N_1, \overbrace{0,\dots,0}^{\mu_2 - 1}, N_1, 0, \dots, 0, N_1\ \overbrace{0,\dots,0}^{\mu_\delta - 1}\Big], \tag{33R}$$

where each zero block is $q \times \beta$ (the same dimensions as a block column of $\bar{N}$). Define also

$$\tilde{N} = -[N \text{ with block column } \mu_1 \text{ missing}]Z + t_1\bar{N}, \tag{34R}$$

where

$$Z = \operatorname{diag}[Z_1, Z_2, \dots, Z_\delta]$$

and

$$Z_1 = \operatorname{diag}[-I, -2I, \dots, -(\mu_1 - 1)I],$$

$$Z_2 = \begin{bmatrix} (t_1 - t_2)I & 0 & & & \\ -I & (t_1 - t_2)I & & & \\ 0 & -2I & \ddots & & \\ & & \ddots & \ddots & \\ & & & -(\mu_2 - 1)I & (t_1 - t_2)I \end{bmatrix},$$

$Z_3, \dots, Z_\delta$ being constructed similarly to Z_2.

LEMMA 3.2R. *Assume that $\Sigma\mu_i \geqslant q$. Then $\bar{N}$ has rank q, and*

$$A\bar{N} = \tilde{N}. \tag{35}$$

We remark that, as for Lemma 3.2, the second claim of the lemma is established by a straightforward algebraic verification.

The corresponding result for repeated interpolation points is as follows:

THEOREM 3.2R. *Adopt the same hypotheses as Theorem* 3.1 *with also the assumption* $\Sigma\mu_i > q$. *Write L as in* (36), *and define*

$$Q = L_2 - \left[\overbrace{0,\ldots,0}^{\mu_1-1}, L_1, \overbrace{0,\ldots,0}^{\mu_2-1}, L_1, 0,\ldots,0, L_1, \overbrace{0,\ldots,0}^{\mu_\delta-1}\right] = LJ^R, \quad (37\text{R})$$

$$R = -[L \text{ with block column } \mu_1 \text{ missing}]\, Z + t_1 Q = LJ_t^R, \quad (39\text{R})$$

where Z, J^R, J_t^R *are appropriately defined. Then* Q *has rank* q, *and* $Qx = 0$ *for some* $x \neq 0$ *implies* $Rx = 0$.

The proof parallels that of Theorem 3.2, using now of course Lemmas 3.1R and 3.2R.

4. CONSTRUCTION OF A STATE-VARIABLE REALIZATION

In the last section, we have stated two theorems that describe the properties inherited by a Loewner matrix or generalized Loewner matrix obtained from a rational transfer-function matrix. In this section, we shall reverse these ideas, i.e., we shall take as the data a (generalized) Loewner matrix with certain properties, and from it, show how a minimal state-variable realization of a rational transfer-function matrix may be constructed.

To keep the ideas simple, we shall assume when providing proofs that there are no repeated points in this section. However, we set out the construction procedure when there are repeated points.

In this section, we make two key assumptions, motivated by the results of the last section. Interpolation data $\{s_i; Y(s_i), T'(s_i), \ldots, Y^{\nu_i-1}(s_i),\ i = 1,2,\ldots,\theta\}$ are given, with the s_i partitioned into row and column sets $R = \{r_1,\ldots,r_1, r_2,\ldots,r_2,\ldots,r_\gamma,\ldots,r_\gamma\}$ and $T = \{t_1,\ldots,t_1, t_2,\ldots,t_2,\ldots,t_\delta,\ldots,t_\delta\}$, there being λ_i and μ_j occurrences of r_i and t_j respectively. There holds $r_i \neq r_j$ for $i \neq j$ and $t_i \neq t_j$ for $i \neq j$. The associated generalized Loewner matrix contains $\rho = \Sigma_{i=1}^{\gamma}\lambda_i$ block rows and $\tau = \Sigma_{j=1}^{\delta}\mu_j$ block columns.

Assumption 4.1. If

$$\operatorname{rank} L = q, \quad (42)$$

then $q \leqslant \rho$, $q < \tau$. Further, all $q \times q$ block submatrices of L, L^* (the latter being constructed as defined in Section 2 by reassignment of the last column-set element as the last row-set element) also have rank q.

For the second assumption, we partition

$$L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}, \tag{43}$$

where L_1 is the first block column of L. Recall the definition of J, J_t given in (38), (40) for the non-repeated-point case and the definition of J^R and J_t^R in Theorem 3.2R for the repeated-point case as set out in Theorem 3.2R.

We define

$$Q = L_2 - [L_1, L_1, \ldots, L_1] = LJ \tag{44}$$

and

$$R = L_2 \operatorname{diag}[t_2 I, t_3 I, \ldots, t_\delta I] - t_1 [L_1, L_1, \ldots, L_1] = LJ_t \tag{45}$$

for the non-repeated-point case, with the obvious modification in the repeated-point case as set out in Theorem 3.2R.

Assumption 4.2. One has $\operatorname{rank} Q = \operatorname{rank} L = q$, and $Qx = 0$ implies $Rx = 0$ for $x \neq 0$.

Assumption 4.1 guarantees that the underlying rational function has McMillan degree q. In other words, the realization constructed will necessarily be controllable and observable.

If our data do not satisfy this condition, we need to add interpolation data until the condition becomes satisfied. In the scalar case, dealt with in [4] and summarized in Section 2, the way this can be done is set out, and is rather complicated. For the matrix case, some developments can be found in [8], and the situation is even more complicated. Of course, the added data will necessarily drive up the degree of the interpolating transfer-function matrix; that data can be found so that the increase in degree is finite is a nontrivial fact, and was proved for the scalar case in [4], where the admissible degrees of solutions to the interpolation problem are identified. (In effect, [8] gives the theory behind the determination of the minimal McMillan degree and all admissible degrees, while this paper gives the theory behind the construction, in state-space terms, of the solution of admissible degree.) The situation is actually very analogous to the matrix partial-realization problem; in particular, one is faced with either having first enough data to generate a unique transfer-function matrix consistent with the truncated series of Markov coefficients, or else having to add specially chosen but nonunique data to the given data to obtain a transfer-function matrix consistent with the originally given data and the added nonunique data. The degree of the transfer-function matrix in this second case is finite, but driven up by the extra data. See [9] for a discussion.

Assumption 4.2 is needed to secure properness of the interpolating function. As shown in Section 5, it can be eliminated by means of an appropriate bilinear transformation.

Notice that the properties demanded by Assumptions 4.1, 4.2 necessarily hold if $Y(s)$ is defined by a causal transfer-function matrix with minimal state-variable dimension q. This is a consequence of the results of Section 3, and justifies adoption of the assumptions.

The first step in the constructive procedure is to factor L into a product of two matrices with column and row rank q respectively. Thus we shall assume that

$$-L = MN, \tag{46}$$

where M has q columns, and N has q rows. Of course N is unique up to left multiplication by a nonsingular matrix T. As it turns out, two different factorizations M_1N_1 and M_2N_2 will give rise to two different state-variable realizations $\{A_i, B_i, C_i, D_i\}$ with $i = 1, 2$ for $Y(s)$. They are related by a nonsingular coordinate transformation, i.e. $A_2 = TA_1T^{-1}$ etc.

REMARK. An equivalent way of expressing Assumption 4.2 in terms of the above factorization of L is the following:

$$\operatorname{rank} NJ = q.$$

The main strategy now is to find A, B, C such that M, N are the generalized observability and controllability matrices associated with A, B, C; see (28) and (28R). Once A, B, C have been found, the identification of D is immediate from a single interpolation datum, viz.

$$D = Y(s_1) - C'(s_1I - A)^{-1}B.$$

We shall describe first the construction of A, B, C; then we shall prove for the non-repeated-point case that this construction results in no t_i or r_i being an eigenvalue of A and that

$$N = \left[(t_1 I - A)^{-1}B, (t_2 I - A)^{-1}B, \ldots, (t_\delta I - A)^{-1}B\right], \qquad (47)$$

$$M = \begin{bmatrix} C'(r_1 I - A)^{-1} \\ C'(r_2 I - A)^{-1} \\ \vdots \\ C'(r_\gamma I - A)^{-1} \end{bmatrix}. \qquad (48)$$

(Extensions to the case of repeated points would be messy, but straightforward.) Finally, we shall show that with appropriate choice of D, the transfer-function matrix $D + C(sI - A)^{-1}B$ correctly interpolates the data. In the last two steps, we are evidently checking the validity of the construction procedure.

We summarize the result we are establishing as follows.

THEOREM 4.1. *Suppose interpolation data* $P = \{s_i; Y(s_i), Y'(s_i), \ldots, Y^{\nu_i - 1}(s_i),\ i = 1, \ldots, \theta\}$ *are given, with the* s_i *partitioned into row and column sets* $R = \{r_1, \ldots, r_1, r_2, \ldots, r_2, \ldots, r_\gamma, \ldots, r_\gamma\}$ *and* $T = \{t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_\delta, \ldots, t_\delta\}$, *there being* λ_i *and* μ_j *occurrences of* r_i *and* t_j *respectively, and with* $r_i \neq r_j$ *for* $i \neq j$ *and* $t_i \neq t_j$ *for* $i \neq j$. *Let* L *be the associated generalized Loewner matrix with* $\rho = \sum_{i=1}^{\gamma} \gamma_i$ *block rows and* $z = \sum_{j=1}^{\delta} \mu_j$ *block columns. Let Assumptions* 4.1 *and* 4.2 *hold for the case of no repeated points, define*

$$\bar{N} = NJ, \qquad (49)$$

$$\tilde{N} = NJ_t, \qquad (50)$$

and when there are repeated points, let J *be replaced by* J^R *and* J_t *by* J_t^R *(with definition as implied in the statement of Theorem* 3.2*R*). *Define the matrix* A *as*

$$A = \tilde{N}\bar{N}'(\bar{N}\bar{N}')^{-1} \qquad (51)$$

(*with the inverse existing because* $\bar{N}$ *has full row rank*). *Define further*

$$B = (t_1 I - A) N \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{52}$$

there being δ *block entries in the rightmost member of the product on the right side of* (52), *and*

$$C = [I, 0, \ldots, 0] M (r_1 I - A) \tag{53}$$

with the leftmost matrix on the right of (53) *possessing* γ *block entries. Then the matrices* $(t_j I - A), (r_i I - A)$ *are nonsingular, and the formulae* (47) *and* (48) *hold. Further, the definition*

$$D = Y(r_1) - C(r_1 I - A)^{-1} B \tag{54}$$

ensures that the transfer-function matrix $D + C(sI - A)^{-1}B$ *interpolates the data, has least degree among interpolating transfer-function matrices, and is the only transfer-function matrix with this degree.*

The remainder of the section is devoted to proving this theorem.

Verification of (47), (48)

Observe from (44), (46) and (49) that $Q = M\bar{N}$, and from (45), (46) and (50) that $R = M\tilde{N}$. Because M has full row rank, and because $Qx = 0$ implies $Rx = 0$, it follows that $\bar{N}x = 0$ implies $\tilde{N}x = 0$. Hence there exists a unique $\mathscr{A}$ such $\mathscr{A}\bar{N} = \tilde{N}$. Since Q has rank q, $\bar{N}$ has full row rank, and so with A as in (51), we necessarily have $A = \mathscr{A}$. Hence

$$A\bar{N} = \tilde{N}. \tag{55}$$

Consider the jth block column on each side of (55) for $j \geqslant 2$, and let N_1, N_j denote the first and jth block columns of N. Evidently,

$$A(-N_1 + N_j) = (-t_1 N_1 + t_j N_j),$$

or

$$(t_1 I - A)N_1 = (t_j I - A)N_j.$$

By (52),

$$B = (t_j I - A)N_j, \qquad j \geqslant 2. \tag{56}$$

Hence if $t_j I - A$ is nonsingular for all j, then (47) is verified, by (52) and (56).

We now demonstrate for $j = 1, 2, \dots, \delta$ that $t_j I - A$ is nonsingular. Using (55) and the definitions (49), (50) of $\overline{N}$ and $\tilde{N}$, we have

$$(t_j I - A)\overline{N} = N \begin{bmatrix} (t_1 - t_j)I & (t_1 - t_j)I & \cdots & (t_1 - t_j)I \\ (t_j - t_2)I & & & \\ & (t_j - t_3)I & & \\ & & \ddots & \\ & & & (t_j - t_\delta)I \end{bmatrix}. \tag{57}$$

The second matrix in the product on the right has one zero block row (the jth), so that provided that block columns $1, 2, \dots, j-1, j+1, \dots, \delta$ of N (call this matrix $\mathcal{N}_j$) have rank q, the matrix on the right has rank q; however, the desired rank-q property is a consequence of Assumption 4.1, which states that any block $q \times q$ submatrix of L is of rank q, and so in particular $M\mathcal{N}_j$, which includes such a submatrix, has rank q. Now since the matrix on the right of (56) has rank q, $t_j I - A$ is nonsingular.

In order to verify that our definitions lead to (48), we first require two lemmata:

LEMMA 4.1. *Let $E_i' L, LE_k$ denote the ith block row and kth block column of L. (Thus $E_j' = [0, \dots, 0, I, 0, \dots, 0]$ with I in the jth block entry.) Then*

$$(r_i E_i' - r_j E_j')L(E_k - E_l) = (E_i' - E_j')L(t_k E_k - t_l E_l) \tag{58}$$

for all $1 \leqslant i, j \leqslant \gamma$ and $1 \leqslant k, l \leqslant \delta$.

Proof. Observe that

$$(r_i E_i' - r_j E_j')L(E_k - E_l) = r_i L_{ik} - r_j L_{jk} - r_i L_{il} + r_j L_{jl}$$

and

$$(E_i' - E_j')L(t_k E_k - t_l E_l) = t_k L_{ik} - t_k L_{jk} - t_l L_{il} + t_l L_{jl}.$$

Subtracting and using equalities such as $(r_i - t_k)L_{ik} = Y(r_i) - Y(t_k)$ leads to the result. ■

Now let M_j' denote the jth block row of M, with N_j (as before) the jth block column of N. Then (58) implies

$$\begin{aligned}
&(r_i M_i' - r_j M_j')(N_k - N_l) \\
&\quad = (M_i' - M_j')(t_k N_k - t_l N_l) \\
&\quad = (M_i' - M_j')\left[(t_k I - A)N_k - (t_l I - A)N_l\right] + (M_i' - M_j')A(N_k - N_l) \\
&\quad = (M_i' - M_j')A(N_k - N_l)
\end{aligned}$$

on using (56). It follows that

$$\left[M_i'(r_i I - A) - M_j'(r_j I - A)\right](N_k - N_l) = 0.$$

Fixing $k = 1$ and letting l range from 2 to δ yields

$$\left[M_i'(r_i I - A) - M_j'(r_j I - A)\right]\overline{N} = 0,$$

and because $\overline{N}$ has full row rank,

$$M_i'(r_i I - A) - M_j'(r_j I - A) = 0. \tag{59}$$

By (53), $C' = M_1'(r_1 I - A)$, and (48) is then immediate provided that $r_j I - A$ is nonsingular for all j.

This nonsingularity is proved in the following way. Recall from Assumption 4.1 that if the last column-set element is reassigned to be the last row-set element, the new generalized Loewner matrix L^* has all block $q \times q$

(generalized) Loewner matrices of rank q. If L has a factorization (46), then clearly

$$L^* = -\begin{bmatrix} M \\ M'_{\gamma+1} \end{bmatrix} [N_1, N_2, \dots, N_{\delta-1}]$$

with $r_{\gamma+1} = t_\delta$, and (59) holds with $1 \leqslant i, j \leqslant \gamma+1$ and $1 \leqslant k, l \leqslant \delta - 1$.
Now let us state

LEMMA 4.2. *With quantities defined as above, there holds for* $j = 1, 2, \dots, \delta - 1$

$$(t_\delta - t_j) M'_{\gamma+1} N_j = (r_1 - t_j) M'_1 N_j - (r_1 - t_\delta) M'_1 N_\delta. \tag{60}$$

This is a consequence of identities such as $(t_\delta - t_j) M'_{\gamma+1} N_j = -Y(t_\delta) + Y(t_j)$. From it, we obtain

$$\begin{aligned} M'_{\gamma+1}(t_\delta I - A) N_j - M'_{\gamma+1}(t_j I - A) N_j &= M'_1(r_1 I - A)(N_j - N_\delta) \\ &\quad - M'_1(t_j I - A) N_j + M'_1(t_\delta I - A) N_\delta, \end{aligned}$$

or

$$M'_{\gamma+1}(t_\delta I - A)(N_j - N_\delta) = M'_1(r_1 I - A)(N_j - N_\delta), \tag{61}$$

on using the fact that $(t_j I - A) N_j = (t_\delta I - A) N_\delta$. By using the equality (61) for all j, we obtain

$$M'_{\gamma+1}(t_\delta I - A)\overline{N} = M'_1(r_1 I - A)\overline{N},$$

and so (59) holds for $i, j = 1, 2, \dots, \gamma+1$ when $r_{\gamma+1}$ is identified with t_δ. Now suppose (to obtain a contradiction) that $(r_i I - A)x = 0$ for some $x \neq 0$. Then (59) yields

$$0 = M'_i(r_i I - A)x = M'_j(r_j I - A)x = (r_j - r_i) M'_j x.$$

So

$$M'_j x = 0, \qquad j = 1, 2, \dots, i-1, i+1, \dots, \gamma+1. \tag{62}$$

Now L^* with its ith block row eliminated, call the result L_i^*, has rank q. Since

$$L_i^* = \begin{bmatrix} M_1' \\ \vdots \\ M_{i-1}' \\ M_{i+1}' \\ \vdots \\ M_{\gamma+1}' \end{bmatrix} [N_1, \ldots, N_{\delta-1}],$$

it follows that each of the matrices on the right side of (61) has rank q. Then (62) with $x \neq 0$ is a contradiction. Consequently $r_i I - A$ is nonsingular.

Correct Interpolation of the Data

Observe that for all j,

$$\begin{aligned} Y(r_1) - Y(t_j) &= (r_1 - t_j) E_1' L E_j \\ &= -(r_1 - t_j) M_1' N_j \\ &= -(r_1 - t_j) C'(r_1 I - A)^{-1}(t_j I - A)^{-1} B \\ &= C'(r_1 I - A)^{-1} B - C'(t_j I - A)^{-1} B, \end{aligned}$$

whence, using (54),

$$Y(t_j) = D + C'(t_j I - A)^{-1} B.$$

Similarly, we may prove that $D + C'(sI - A)^{-1}B$ interpolates correctly at $r_2, r_3, \ldots, r_\gamma$.

Minimality of Degree

If there were an interpolating transfer-function matrix of degree $q' < q$, Theorem 3.1 would yield rank $L = q'$, a contradiction.

Uniqueness of Interpolating Functions

Let $D_1 + C_1(sI - A)^{-1}B_1$ and $D_2 + C_2(sI - A_2)^{-1}B_2$ be two interpolating functions of McMillan degree q. Then these generate two factorizations

$L = M_1 N_1 = M_2 N_2$, with M_i, N_i being defined by trivial variations on the formulae (47) and (48). Since rank N_1 = rank $N_2 = q$, there exists nonsingular square T with $TN_1 = N_2$. Let matrices $\bar{N}_i, \tilde{N}_i$ be formed from N_i in the standard way; see e.g. (49) and (50). Then $T\bar{N}_1 = \bar{N}_2, T\tilde{N}_1 = \tilde{N}_2$. By Lemma 3.2 (and 3.2R), we have $A_2 \bar{N}_2 = \tilde{N}_2$ and $A_1 T^{-1} \bar{N}_2$ or $TA_1 T^{-1} \bar{N}_2 = \tilde{N}_2$. Since $\bar{N}_2$ has rank q, $TA_1T^{-1} = A_2$. The rest of the argument is trivial.

5. NONCAUSAL TRANSFER-FUNCTION MATRICES

In Section 3, we described properties of a Loewner or generalized Loewner matrix associated with a proper transfer-function matrix, and in Section 4 we showed how a realization of an interpolating transfer function could be constructed from a Loewner matrix having these properties. In this section, our aim is to shed the properness assumption. We shall restrict attention to the problem with distinct interpolating points.

Let $F(s)$ be a nonproper transfer-function matrix, and suppose, to begin, that $r_1, \dots, r_\gamma$ and $t_1, \dots, t_\delta$ define distinct points at which interpolating values of $F(s)$ are known. Suppose that the McMillan degree of $F(s)$ is q, and that $q \leqslant \gamma$, $q < \delta$. The $i-j$ block entry of the associated Loewner matrix is

$$(L_F)_{ij} = \frac{F(r_i) - F(t_j)}{r_i - t_j}.$$

Now observe that for almost all σ_1, σ_2, and σ_3, the transfer-function matrix

$$G(s) = F\left(\frac{\sigma_2 s + \sigma_1}{s + \sigma_3} \right) \tag{63}$$

is proper, with the same McMillan degree as F. Observe also that when

$$\frac{\sigma_2 s + \sigma_1}{s + \sigma_3} = r_i$$

then

$$s = \frac{\sigma_3 r_i - \sigma_1}{\sigma_2 - r_i}.$$

Thus at the point $(\sigma_3 r_i - \sigma_1)(\sigma_2 - r_i)^{-1}$, which is finite for almost all $\sigma_1, \sigma_2, \sigma_3$, the transfer-function matrix $G(s)$ assumes the value $F(r_i)$. Consequently, the $i-j$ block entry of the Loewner matrix for $G(s)$ is

$$(L_G)_{ij} = \frac{F(r_i) - F(t_j)}{\dfrac{\sigma_3 r_i - \sigma_1}{\sigma_2 - r_i} - \dfrac{\sigma_3 t_j - \sigma_1}{\sigma_2 - t_j}}$$

$$= \frac{\sigma_2 - r_i}{\sigma_2\sigma_3 - \sigma_1} \cdot \frac{F(r_i) - F(t_j)}{r_i - t_j}(\sigma_2 - t_j),$$

which implies that

$$L_G = \frac{1}{\sigma_2\sigma_3 - 1}\operatorname{diag}[(\sigma_2 - r_i)I]L_F \operatorname{diag}[(\sigma_2 - t_j)I]. \tag{64}$$

For almost all choices of σ_1, σ_2, and σ_3, this implies that L_G and L_F have the property that submatrices composed of the same rows and columns have the same rank; in particular, L_G, L_F have the same rank. This is true when

$$\sigma_2 \neq r_i, \qquad \sigma_2 \neq t_j, \qquad \sigma_1\sigma_2 \neq \sigma_3. \tag{65}$$

When repeated points are allowed, the replacement for (64) is much more complicated.

The theory of Section 4 shows how to construct a minimal state variable realization of $G(s)$, say

$$G(s) = D_G + C_G'(sI - A_G)^{-1}B_G. \tag{66}$$

Because $F(s)$ becomes infinite when $s \to \infty$, $G(s)$ has a pole at $s = -\sigma_3$. It is straightforward to change the coordinate basis in (66) so that

$$A_G = A_{G1} + A_{G2}, \tag{67}$$

where all eigenvalues of A_{G1} are at $-\sigma_3$ with A_{G1} in Jordan form, and no eigenvalues of A_{G2} are at $-\sigma_3$. Write, in obvious notation,

$$G(s) = C_{G1}'(sI - A_{G1})^{-1}B_{G1} + \left[D_G + C_{G2}'(sI - A_{G2})^{-1}B_{G2}\right].$$

Let J_λ denote a certain Jordan form with λ on the diagonal; J_μ denotes the same form μ replacing λ. Then

$$G(s) = C'_{G1}(sI - J_{-\sigma_3})^{-1}B_{G1} + \left[D_G + C'_{G2}(sI - A_{G2})^{-1}B_{G2}\right]$$
$$= -C'_{G1}\left[J_{-(s+\sigma_3)}\right]^{-1}B_{G1} + \left[D_G + C'_{G2}(sI - A_{G2})^{-1}B_{G2}\right]$$

with $\{J_{-\sigma_3}, B_{G1}, C_{G1}\}$ and $\{A_{G2}, B_{G2}, C_{G2}, D_G\}$ both minimal. Now (63) implies and is implied by

$$F(s) = G\left(\frac{\sigma_1 - \sigma_3 s}{s - \sigma_2}\right).$$

It is straightforward to deduce then that

$$F(s) = -C'_{G1}\left[J_{(\sigma_2\sigma_3 - \sigma_1)(s-\sigma_2)^{-1}}\right]^{-1}B_{G1}$$
$$+ \left[D_G - C'_{G2}(\sigma_3 I + A_{G2})^{-1}B_{G2}\right]$$
$$+ C'_{G2}\left[\sigma_2 I - (\sigma_3 I + A_{G2})^{-1}(\sigma_1 I + \sigma_2 A_{G2})\right]$$
$$\times\left[sI - (\sigma_3 I + A_{G2})^{-1}(\sigma_1 I + \sigma_2 A_{G2})\right]^{-1}(\sigma_3 I + A_{G2})^{-1}B_{G2}.$$

The first two terms define the nonstrictly proper part of $F(s)$, and the last term a (minimal) state-variable realization of the strictly proper part of $F(s)$.

As far as construction is concerned, the initial data is organized into L_F; one observes that L_F fulfills Assumption 4.1, but not 4.2; one forms L_G via (64), and checks that L_G satisfies Assumption 4.2; then one constructs the state-variable realization of $G(s)$ and then a generalized state-variable realization for $F(s)$.

The nonproper case is of course not of great interest; with repeated points, it is extremely complex. The inclusion of the parameter σ_i is a further complication. Accordingly, we refrain from presenting indigestible formulae to the reader. The whole point is simply to rely on the bilinear transformation.

6. CONCLUSIONS

In this paper, we have set out a theory paralleling that known for the so-called realization problem of linear system theory, which allows construction of a minimal state-variable realization from interpolation data. Deficiencies of the theory include the absence of a tidy parametrization of solutions when the original data have to be added to, in order to guarantee satisfaction of Assumptions 4.1 and 4.2 (the case $\deg y^{\min} = N - q$ in the scalar situation), and the absence of recursive formulae for allowing update of a realization when one more interpolation datum becomes available.

We can also state that we have not addressed the tangent problem at all [where interpolation data are available at point s_i not for the whole matrix $Y(s_i)$ but part of it, e.g., one has α_i and β_i for which $Y(s_i)\alpha_i = \beta_i$, and this is all one knows about $Y(s_i)$]. It would also be interesting to continue the development of connections between Nevanlinna-Pick and Loewner matrices, as set out for scalar functions in for example [7].

REFERENCES

1 R. E. Kalman, P. L. Falb, and M. A. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1969.
2 T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1978.
3 R. E. Kalman, On partial realization, transfer-functions and canonical forms, *Acta Polytech. Scand. Math.* 31:9–32 (1979).
4 A. C. Antoulas and B. D. O. Anderson, On the scalar rational interpolation problem, in *Proceedings of the 7th International Conference on Analysis and Optimization of Systems*, Antibes, France, June 1986; *IMA J. Math. Control Inform.* 3(2, 3):61–88 (1986).
5 V. Belevitch, Interpolation matrices, *Philips Res. Rep.* 25:337–369 (1970).
6 M. Fiedler, Hankel and Loewner matrices, *Linear Algebra Appl.* 58:75–95 (1984).
7 A. C. Antoulas and B. D. O. Anderson, On the stable rational interpolation problem, *Linear Algebra Appl.*, 122–124:301–329 (1989).
8 A. C. Antoulas, J. A. Ball, J. Kang, and J. C. Willems, On the solution of the minimal rational interpolation problem, *Linear Algebra Appl.*, to appear.
9 R. E. Kalman, On minimal partial realizations of a linear input/output map, in *Aspects of Network and System Theory* (R. E. Kalman and N. De Claris, Eds.), Holt, Reinhart and Winston, 1971.

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