Discrete Mathematics 53 (1985) 63–78
North-Holland

COMPLEMENTS AND QUASICOMPLEMENTS IN THE LATTICE OF SUBALGEBRAS OF $P(\omega)$

Ivo DÜNTSCH
Department of Mathematics, Bayero University, Kano, Nigeria

Sabine KOPPELBERG
2. Mathematisches Institut der FU Berlin, 1000 *Berlin 33, Fed. Rep. Germany*

In the lattice of subalgebras of a Boolean algebra D call B a complement of A if $A \cap B = \{0, 1\}$ and $A \cup B$ generates D. B is called a quasicomplement of A if it is maximal w.r.t. the property $A \cap B = \{0, 1\}$. We characterize those countable subalgebras of $P(\omega)$ which have a complement, and, assuming Martin's Axiom, describe the isomorphism types of some quasicomplements of the finite-cofinite subalgebra of $P(\omega)$.

Dans le treillis des sous-algèbres d'une algèbre booléenne D, la sous-algèbre B est un complément de A si $A \cap B = \{0, 1\}$ et si D est engendrée par $A \cup B$. La sous-algèbre B est un quasicomplément de A si B est maximale parmi les algèbres C satisfaisant $A \cap C = \{0, 1\}$. On caractérise les sous-algèbres dénombrables de $P(\omega)$ qui possèdent un complément et, en admettant l'axiome de Martin, on decrit les types d'isomorphisme de quelques quasicompléments de $FC(\omega)$, la sous-algèbre de $P(\omega)$ des parties finies et cofinies de ω.

1. Introduction

For a Boolean algebra D, the set Sub(D) of subalgebras of D is a complete lattice under set inclusion with least element $\mathbf{2} = \{0, 1\}$ and greatest element D; we write $A \leqslant D$, if A is a subalgebra of D. For $A, B \leqslant D$, the infimum of A and B in Sub(D) is just $A \cap B$, and their supremum $A \vee B$ is the subalgebra of D generated by $A \cup B$. Call B

(i) a complement of A, if $A \cap B = \mathbf{2}$ and $A \vee B = D$;

(ii) a quasicomplement of A, if B is maximal w.r.t. the property $A \cap B = \mathbf{2}$.

An arbitrary $A \leqslant D$ need not have a complement, but, by Zorn's Lemma it certainly has a quasicomplement; neither complements nor quasicomplements are, in general, uniquely determined.

Quasicomplements have been considered by Remmel [4], where they are called complements. The question for which algebras D the lattice Sub(D) is complemented has been studied first by Rao and Rao [3] and later by Todorčević [8]; the following are the general facts known about this problem:

Fact 1. *If D is a subalgebra of an interval algebra, then* Sub(D) *is complemented (cf.* [6] *and* [8]).

0012-365X/85/$3.30

Fact 2. *If* Sub(D) *is complemented, then D is retractive in the terminology of* [5], *cf. also* [3].

Rubin proved in [6] that each subalgebra of an interval algebra is retractive; exactly the same proof gives Fact 1. A special case of Fact 1, namely that Sub(D) is complemented for countable D, was proved by Remmel [4] and later by Jech [1].

For the example D constructed under $\diamond$ in [6] of a retractive algebra which is not embeddable into an interval algebra, Sub(D) is not complemented, so the converse of Fact 2 does not hold. It is not known whether the converse of Fact 1 holds.

The proof of Fact 2 given in [3] shows that, in particular, $FC(\omega)$, the algebra of finite or cofinite subsets of ω does not have a complement in $P(\omega)$, the power set algebra of the set ω of non-negative integers – in fact, such a complement must be isomorphic to $P(\omega)$ modulo the ideal of finite sets, but this algebra, having an uncountable disjoint subset, is not embeddable into $P(\omega)$. Nevertheless, it will turn out that many countable subalgebras of $P(\omega)$ have complements; whether such an A has a complement depends solely on the way A is embedded into $P(\omega)$, and not on the isomorphism type of A.

We shall deal with quasicomplements – mostly in Sub($P(\omega)$) – in Section 2, and with complements in Sub($P(\omega)$) in Section 3. Moreover, a problem raised in [4, p. 62] is solved at the end of this section. The example given also shows that any Boolean algebra which is embeddable into $P(\omega)$ can be embedded into $P(\omega)$ so that it has a complement.

We shall use the following notation: $f[X]$, resp. $f^{-1}[X]$, is the image, resp. the preimage of a set X under a function f. The finitary operations of a Boolean algebra A are denoted by $+$, $\cdot$, $-$, 0, 1; we shall also use this notation for $A \leqslant P(\omega)$. Infinitary joins are denoted by Σ. $a = a_1 \dot{+} \cdots \dot{+} a_n$ means that $a = a_1 + \cdots + a_n$ and the a_i are pairwise disjoint. A^+ is $A \setminus \{0_A\}$, and At(A) is the set of atoms of A. For $a \in A$, $A \upharpoonright a$ is the relative algebra $\{x \in A \mid x \leqslant a\}$. $[M]$ denotes the subalgebra of A generated by M. Let $A \leqslant D$, and $u \in D$. Then $A(u) = [A \cup \{u\}]$ is called a simple extension of A. The elements of $A(u)$ can be written in the form

$$a_1 \cdot u + a_2 \cdot -u,$$

where $a_1, a_2 \in A$, or in the form

$$a_1 \cdot u \dot{+} a_2 \cdot -u \dot{+} a_3$$

for some quadruple $(a_1, \ldots, a_4)$ of elements of A satisfying $a_1 \dot{+} a_2 \dot{+} a_3 \dot{+} a_4 = 1$. u is independent of A if, for $a \in A^+$, $a \cdot u$ and $a \cdot -u$ are nonzero. For the terminology on partial orders and Martin's Axiom, see [7]: if $(P, \leqslant)$ is a partially ordered set, call $D \subseteq P$ dense for each $p \in P$ there is some $q \in D$ such that $p \leqslant q$. If

$\mathfrak{D}$ is a family of dense subsets of P, $G \subseteq P$ is said to be generic for $\mathfrak{D}$ if

(1) $p \in G$, $q \in P$, $q \leqslant p$ implies $q \in G$;

(2) for $p, q \in G$ there exists $r \in G$ such that $p, q \leqslant r$;

(3) $G \cap D \neq \emptyset$ for every $D \in \mathfrak{D}$.

$p, q \in P$ are compatible if $p, q \leqslant r$ for some $r \in P$, otherwise incompatible; $(P, \leqslant)$ satisfies the countable chain condition (ccc) if each set of pairwise incompatible elements of P is countable. We denote by (MA_κ) the assertion that for each family $\mathfrak{D}$ of dense subsets of a ccc partial order $(P, \leqslant)$ satisfying $|\mathfrak{D}| \leqslant \kappa$, there exists a subset G of P which is generic for the family $\mathfrak{D}$. (MA) is the assertion that (MA_κ) for each $\kappa < 2^\omega$. Note that (MA_ω) is a theorem of ZFC, thus, all our theorems are provable within ZFC+CH. Routine details in applications of Martin's Axiom are sometimes omitted.

2. Quasicomplements

First we consider the question whether in a 'large' algebra D there can be 'small' subalgebras A, B such that B is a quasicomplement of A. We then concentrate on $D = P(\omega)$. There is a general answer to the question:

Proposition 1. *If B is a quasicomplement of A in* Sub(D), *then $A \vee B$ is dense in D. Hence $|D| \leqslant 2^{\max(|A|, |B|)}$.*

Proof. Assume that $A \vee B$ is not dense in D. Then there exists a $z \in D^+$ such that for no $c \in A \vee B, 0 < c \leqslant z$; in particular, $z \notin B$. We prove that $A \cap B(z) = \mathbf{2}$, contradicting the maximality of B. Let $a \in A \cap B(z)$, e.g.,

$$a = b_1 \cdot z + b_2 \cdot -z + b_3, \tag{1}$$

where $b_1 \dot{+} b_2 \dot{+} b_3 \dot{+} b_4 = 1$. Now, $b_1 \cdot a = b_1 \cdot z \leqslant z$. Since $b_1 \cdot a \in A \vee B$ and by our assumption, $b_1 \cdot a = 0$. By

$$-a = b_2 \cdot z + b_1 \cdot -z + b_4$$

we get $b_2 \cdot -a = 0$, hence $b_2 \leqslant a$ by the same reasoning. By (1) again, $b_2 \cdot a = b_2 \cdot -z$; thus $a = b_2 \cdot a + b_3 = b_2 + b_3 \in B$. This implies $a \in \mathbf{2}$. For the rest note that, if C is a dense subalgebra of D, then each element of D is the join of all elements of C below it. □

Example 1 (MA). *There is a Boolean algebra D of power 2^ω and $A, B \leqslant D$ such that $|A| = |B| = \omega$ and A, B are quasicomplements of each other.*

Proof. Construct a chain $(D_\alpha)_{\alpha<2^\omega}$ of atomless Boolean algebras and $A, B \leqslant D_\alpha$, s.t.

(1) $D_\lambda = \bigcup \{D_\alpha \mid \alpha < \lambda\}$ for limit ordinals $\lambda < 2^\omega$.

(2) $D_{\alpha+1}$ is a simple extension of D_α, constructed by the following Lemma 1.

(3) $|A| = |B| = \omega$, A and B are quasicomplements of each other in D_0, and A, B are dense in every D_α.

Then set $D = \bigcup \{D_\alpha \mid \alpha < 2^\omega\}$. □

Let D_0 be the interval algebra of the set $\mathbb{Q}$ of rationals. We sketch how to find A and B: Let M be a subset of $\mathbb{Q}$ such that both M and $\mathbb{Q}\backslash M$ are dense in $\mathbb{Q}$. Let $A_0(B_0)$ be the set of elements of D_0 having endpoints only in $M(\mathbb{Q}\backslash M)$. Since $A_0 \cap B_0 = \mathbf{2}$, we may then find A and B by enlarging A_0 and B_0.

Lemma 1 (MA_κ). *Let E be an atomless Boolean algebra and $A, B \leqslant E$ such that $|E| \leqslant \kappa$, $|A| = |B| = \omega$, A and B are quasicomplements of each other, and A and B are both dense in E. Then there is a simple extension $E(u)$ of E such that E (and hence A and B) is dense in $E(u)$, and A and B are quasicomplements of each other in $E(u)$.*

Proof. We are going to construct u in the completion of E. Put $C = A \vee B$, and let

$$P = \{(x, y) \in C \times C \mid x \cdot y = 0 \text{ and } x + y < 1\}.$$

For $(x, y), (x', y') \in P$ let $(x, y) \leqslant (x', y')$ iff $x \leqslant x'$ and $y \leqslant y'$. Since C is countable, so is P, and thus trivially satisfies the ccc.

For $e \in E$,

$$D_e = \{(x', y') \in P \mid x' \cdot -e \neq 0 \text{ or } y' \cdot e \neq 0\}$$

is dense in P.

Motivated by the representation of elements in $E(u)$ given in Section 1, we define Qu_E to be the set of quadruples $(e_1, e_2, e_3, e_4) \in E^4$ such that $e_1 \dot{+} e_2 \dot{+} e_3 \dot{+} e_4 = 1$; Qu_A and Qu_B are defined similarly. For $x, y \in E$ and $\bar{e} = (e_1, \ldots, e_4) \in \mathrm{Qu}_E$, let

$$f_{\bar{e}}(x, y) = e_1 x + e_2 y + e_3,$$

for $x, y, \bar{e}$ as above and $\bar{a} = (a_1, \ldots, a_4) \in \mathrm{Qu}_A$, let

$$f_{\bar{e}\bar{a}}(x, y) = (a_1 e_1 + a_2 e_2)x + (a_1 e_2 + a_2 e_1)y + (a_1 e_3 + a_2 e_4) + a_3.$$

We claim that for $\bar{e} \in \mathrm{Qu}_E$,

$$D^A_{\bar{e}} = \{(x', y') \in P \mid e_1 + e_2 \leqslant x' + y' \text{ or there is some } \bar{a} \in \mathrm{Qu}_A \text{ such that } a_1 + a_2 \leqslant x' + y' \text{ and } f_{\bar{e}\bar{a}}(x', y') \notin A\}$$

is a dense subset of P: let $(x, y) \in P$ be given. If $e_1 + e_2 \leqslant x + y$, put $(x', y') = (x, y)$.

So, let $e_1+e_2 \leqslant x+y$. Since A is atomless and dense in E, there is some $a \in A$ such that

$$0<a \leqslant e_1+e_2, \qquad a \cdot (x+y)=0, \qquad a+x+y<1.$$

Let w.l.o.g. $0<a \leqslant e_1$, otherwise, we may assume $a \leqslant e_2$. Pick $a_1, a_2 \in A^+$ such that $a=a_1 \dot{+} a_2$. Since B is dense in E, pick $\beta, \delta \in B^+$ such that $a_1=\beta \dot{+} \gamma$, $a_2=\delta \dot{+} \varepsilon$ for some $\gamma, \varepsilon \in C^+$. Then define

$$s=\beta+\delta, \qquad t=\gamma+\varepsilon,$$

$$x'=x+s, \qquad y'=y+t.$$

We shall see that $(x', y') \in D^A_{\bar{e}}$. Note that δ, β, γ, ε, s, t and hence x' and y' are elements of C. Moreover,

$$x'+y'=x+s+y+t=x+y+a<1,$$

and $x' \cdot y'=0$ since $(x+y) \cdot a=0$ and β, γ, δ, ε are in $C \restriction a$ and pairwise disjoint. Let $\bar{a}=(a_1, a_2, 0, -a) \in \mathrm{Qu}_A$, so $a_1+a_2 \leqslant x'+y'$. Also,

$$\begin{aligned} f_{\bar{e}\bar{a}}(x', y') &= (a_1+0) \cdot x'+(0+a_2) \cdot y'+(0+0)+0 \\ &= a_1x'+a_2y' \\ &= a_1(x+s)+a_2(y+t)=\beta+\varepsilon \notin A, \end{aligned}$$

since otherwise $a_1 \cdot f_{\bar{e}\bar{a}}(x', y')=\beta \in A$, but $\beta \in B \setminus \mathbf{2}$ and $A \cap B=\mathbf{2}$.

For $\bar{e} \in \mathrm{Qu}_E$, we may define a dense subset $D^B_{\bar{e}}$ of P w.r.t. B instead of A in a similar way.

By (MA_κ) and $|E| \leqslant \kappa$, there is a subset G of P generic for these families of dense sets.

Let, in the completion $\bar{E}$ of E,

$$u=\Sigma^{\bar{E}}\{x \mid (x, y) \in G \text{ for some } y \in C\}$$

and note that, for $(x, y) \in G$, $x \leqslant u$ and $y \leqslant -u$. We have $E \leqslant E(u) \leqslant \bar{E}$, hence E is dense in $E(u)$. To prove that A is a quasicomplement of B in $\mathrm{Sub}(E(u))$, take $t \in E(u) \setminus A$. We show that $A(t) \cap B \neq \mathbf{2}$. There is some $\bar{e} \in \mathrm{Qu}_E$ such that $t=f_{\bar{e}}(u, -u)$; pick $(x', y') \in G \cap D^A_{\bar{e}}$; so $x' \leqslant u$ and $y' \leqslant -u$.

If $e_1+e_2 \leqslant x'+y'$, then $e_1 \cdot u=e_1 \cdot x'$ and $e_2 \cdot -u=e_2y'$, so $t=e_1x'+e_2y'+e_3 \in E \setminus A$, since $x', y' \in C$, $e_1, e_2, e_3 \in E$; since A is a quasicomplement of B in $\mathrm{Sub}(E)$, $A(t) \cap B \neq \mathbf{2}$. If $e_1+e_2 \not\leqslant x'+y'$, then by definition of $D^A_{\bar{e}}$, there is some $\bar{a} \in \mathrm{Qu}_A$ such that $a_1+a_2 \leqslant x'+y'$ and $f_{\bar{e}\bar{a}}(x', y') \notin A$. By $a_1+a_2 \leqslant x'+y'$, we have $a_1 \cdot u=a_1 \cdot x'$, $a_2 \cdot u=a_2x'$, $a_1 \cdot -u=a_1y'$ and $a_2 \cdot -u=a_2y'$. Put $s=f_{\bar{a}}(t, -t)$; then

$$\begin{aligned} s &= f_{\bar{a}}(f_{\bar{e}}(u, -u), -f_{\bar{e}}(u, -u)) \\ &= f_{\bar{e}\bar{a}}(u, -u) \\ &= (a_1e_1+a_2e_2) \cdot u+(a_1e_2+a_2e_1) \cdot -u+(a_1e_3+a_2e_4)+a_3 \\ &= (a_1e_1+a_2e_2) \cdot x'+(a_1e_2+a_2e_1) \cdot y'+(a_1e_3+a_2e_4)+a_3 \\ &= f_{\bar{e}\bar{a}}(x', y') \in E \setminus A. \end{aligned}$$

Since A is a quasicomplement of B in E, $A(s)\cap B\neq\mathbf{2}$; but $s\in A(t)$ by our definition of s, so $A(s)\leqslant A(t)$ and $A(t)\cap B\neq\mathbf{2}$. □

The situation described in Example 1 cannot occur for $D=P(\omega)$:

Proposition 2 (MA_κ). *Let* $A, B\leqslant P(\omega)$ *such that* $|A|, |B|\leqslant\kappa$ *and* $A\cap B=\mathbf{2}$. *Then* B *is not a quasicomplement of* A.

Proof. Let

$$P=\{p:\omega\to 2 \mid \mathrm{dom}(p) \text{ finite and } |p[b]|\leqslant 1 \text{ for each finite atom } b \text{ of } B\}$$

be partially ordered by set inclusion. The following subsets of P are dense in P:

(1) $D_n=\{q\in P\mid n\in\mathrm{dom}(q)\}$ for $n\in\omega$.

(2) $D_{ab}=\{q\in P\mid \text{for some } n\in\mathrm{dom}(q)\cap b,\ n\in a \text{ iff } q(n)=1\}$ for each $a\in A$ and each infinite $b\in B$: if $p\in P$, choose $n\in b\backslash(\mathrm{dom}(p)\cup\bigcup\{b_0\in\mathrm{At}(B)\mid b_0 \text{ finite}, b_0\cap\mathrm{dom}(p)\neq\emptyset\})$, and define $q=p\cup\{(n,\varepsilon)\}$, where $\varepsilon=1$ iff $n\in a$. Then $q\in D_{ab}$ and $p\leqslant q$.

(3) $D_b=\{q\in P\mid \text{for some } n\in\mathrm{dom}(q),\ q(n)=1 \text{ iff } n\in b\}$ for $b\in B$. This is seen as in (2), since b or $-b$ is infinite.

By (MA_κ) and $|A|, |B|\leqslant\kappa$, there is some $G\subseteq P$ generic for the union of these families of dense sets. By $G\cap D_n\neq\emptyset$ for $n\in\omega$, $f=\bigcup G$ is a function from ω to 2; let $u=f^{-1}(0)$. Since $G\cap D_b\neq\emptyset$ for $b\in B$, $u\notin B$. We prove $A\cap B(u)=\mathbf{2}$ by assuming there is some $a\in A\cap B(u)$, $a\neq 0, 1$.

Choose $b_1,\ldots,b_4$ in B such that $b_1\dot{+}b_2\dot{+}b_3\dot{+}b_4=1$ and

$$a=b_1\cdot u+b_2\cdot -u+b_3.$$

Now, b_1 is finite, for otherwise, pick $q\in G\cap D_{ab_1}$. There is some $n\in b_1\cap\mathrm{dom}(a)$ such that $n\in a$ iff $q(n)=1$, hence, $n\in a$ iff $n\notin u$, and $b_1\cdot a\neq b_1\cdot u$, contradicting the definition of D_{ab_1}. The same argument shows that b_2 is finite by considering $-a=b_2u+b_1\cdot -u+b_4$.

If b is a finite atom of B, then by definition of P, $b\leqslant u$ or $b\leqslant -u$, hence $b\cdot u=b\in B$ or $b\cdot u=0\in B$ and also $b\cdot -u\in B$. Now, b_1 and b_2 are finite unions of finite atoms of B, so $b_1\cdot u, b_2\cdot -u\in B$. This gives $a\in B$, a contradiction to $a\in A\backslash\mathbf{2}$. □

For the rest of this section we try to describe the structure of quasicomplements of $FC(\omega)$ in $P(\omega)$; note that Proposition 3(a), without assuming (MA) as in Proposition 2, guarantees that these quasicomplements have power 2^ω.

Lemma 2 (MA_κ). *Let* $B\leqslant P(\omega)$ *such that* $|B|\leqslant\kappa$ *and each* $b\in B^+$ *is infinite. Then there is some* $u\subseteq\omega$ *such that* $b\cdot u$ *and* $b\cdot -u$ *are infinite for each* $b\in B^+$. *In particular,* u *is independent from* B.

Proof. Consider $(P, \subseteq)$, where

$$P=\{p:\omega \to 2 \mid \mathrm{dom}(p) \text{ is finite}\},$$

and let for $n \in \omega$ $D_n \subseteq P$ be as in the proof of Proposition 2, and for $k \in \omega$ and $b \in B^+$,

$$D_{bk}=\{q \in P \mid \text{there are } e, f \subseteq \mathrm{dom}(q) \cap b \text{ such that } |e|=|f|=k \text{ and } q(n)=0 \text{ for } n \in e, q(n)=1 \text{ for } n \in f\}.$$

Every D_{bk} is dense, since each b is infinite. If $G \subseteq P$ is generic for these dense subsets of P, let $f=\bigcup G$; then $u=f^{-1}(0)$ has the desired properties. □

For a Boolean algebra B, let $\pi(B)$ the least possible cardinal of some dense subset of B.

Proposition 3. *Let B be a quasicomplement of $FC(\omega)$ in $P(\omega)$.*
(a) *B is an atomless complete Boolean algebra.*
(b) (MA_κ). *$\pi(B)>\kappa$.*

Proof. (a) Assume b is an atom of B; since b is infinite, pick infinite subsets b_1, b_2 of b such that $b_1 \dot{+} b_2=b$. Clearly, B is a proper subalgebra of $B(b_1)$ and each $c \in B(b_1)^+$ is infinite. Next assume that B is not complete, hence a proper subalgebra of its completion $\bar{B}$. By the Sikorski extension theorem, there is a homomorphism $e:\bar{B} \to P(\omega)$ extending the identity map on B. Since B is dense in $\bar{B}$, e is one-to-one, so w.l.o.g. assume $B \leqslant \bar{B} \leqslant P(\omega)$. Again by density of B in $\bar{B}$, each $c \in \bar{B}^+$ is infinite.

(b) Assume that B_0 is a dense subalgebra of B of power at most κ. For B_0, choose $u \subseteq \omega$ as in Lemma 2. B_0 is dense in B, thus, $b \cdot u$ and $b \cdot -u$ are infinite for all $b \in B^+$. So, $B(u)$ is a proper extension of B, and each $c \in B(u)^+$, having the form $b \cdot u+b' \cdot -u$ for some $b, b' \in B$, is infinite. □

Example 2 will be based on the following improvement of Lemma 2. Call $u \subseteq \omega$ compatible with $B \leqslant P(\omega)$, if each $c \in B(u)^+$ is infinite, otherwise incompatible.

Lemma 3 (MA_κ). *Let $B \leqslant P(\omega)$ such that B is complete, $\pi(B) \leqslant \kappa$, and each $c \in B^+$ is infinite; let $x \in P(\omega)\backslash B$. Then there is some $u \subseteq \omega$ such that u is compatible with B, independent from B, and x is incompatible with $B(u)$.*

Proof. If x is incompatible with B, choose u as in the proof of Proposition 3(b), so assume that x is compatible with B. Put

$$I=\{b \in B \mid b \leqslant x\}, \qquad J=\{b \in B \mid b \leqslant -x\},$$
$$\alpha=\Sigma^B I, \qquad \beta=\Sigma^B J,$$

so $\alpha \cdot \beta = 0$. It is impossible that both $x \leq \alpha$ and $-x \leq \beta$ since this would imply $x = \alpha \in B$, so assume $x \not\leq \alpha$ and choose some $n_0 \in x \cdot -\alpha$.

Let

$$P = \{p : \omega \to 2 \mid \text{dom}(p) \text{ finite, } n_0 \in \text{dom } p, \text{ and } p(n_0) = 0, \\ p(n) = 1 \text{ for every } n \in (\text{dom } p \cap x) \setminus (\alpha \cup \{n_0\})\}.$$

Let B_0 be a fixed dense subalgebra of B of power at most κ. Define the subsets D_n for $n \in \omega$ and D_{bk} for $b \in B_0^+$, $k \in \omega$ as in the proof of Lemma 2. We check that D_{bk} is still dense in P. This follows easily if we know that $e = b \setminus (x \setminus \alpha)$ is infinite. Assume e is finite. Now x is compatible with B and $e \in B(x)$; so $e = 0, b \leq x \cdot -\alpha \leq x$, $b \in I$ and $b \leq -\alpha$, a contradiction.

Let G, f, u be as in the proof of Lemma 2. x is incompatible with $B(u)$, since

$$(u \cap x) \setminus \alpha = \{n_0\}:$$

$n_0 \in x \setminus \alpha$ by our choice of n_0, and $n_0 \in u$ by our choice of P. If $n \in x \setminus \alpha$ such that $n \neq n_0$, then $n \notin u$ follows from the definition of P. □

Example 2 (MA). *$FC(\omega)$ has a quasicomplement B in* Sub($P(\omega)$) *which is the completion of the free Boolean algebra on 2^ω generators.*

Proof. For a cardinal μ denote by F_μ the free Boolean algebra on μ generators. Let $\{x_\alpha \mid \alpha < 2^\omega\}$ be an enumeration of $P(\omega)$.

Construct by induction a chain $(B_\alpha)_{\alpha < 2^\omega}$ of subalgebras of $P(\omega)$ such that $FC(\omega) \cap B_\alpha = \mathbf{2}$ and $B_\alpha \cong \overline{F_{|\alpha|}}$: let $B_0 = \mathbf{2}$; for a limit ordinal $\lambda < 2^\omega$, let B_λ be the completion of $\bigcup_{\alpha<\lambda} B_\alpha$, embedded in $P(\omega)$ over $\bigcup_{\alpha<\lambda} B_\alpha$ as in the proof of Proposition 3(a). If B_α has been constructed, let $B_{\alpha+1} = B_\alpha(u_\alpha)$ where u_α is chosen by Lemma 3 such that $x_\alpha \in B_\alpha$ or x_α is incompatible with $B_\alpha(u_\alpha)$; this is possible by $\pi(B_\alpha) = \pi(\overline{F_{|\alpha|}}) = |\alpha| < 2^\omega$.

Put $B = \bigcup_{\alpha<2^\omega} B_\alpha$, so $FC(\omega) \cap B = \mathbf{2}$ and $B \cong \overline{F_{2^\omega}}$. B is a quasicomplement of $FC(\omega)$: if $x \in P(\omega) \setminus B$, w.l.o.g. $x = x_\alpha$; then by construction of $B_{\alpha+1}$ and $x \notin B_\alpha$, x_α is incompatible with $B_{\alpha+1}$, hence with B. □

We need some preparation for the construction of a quasicomplement of $FC(\omega)$ in Sub($P(\omega)$) very different from Example 2. Recall that $C \leq B$ is a regular subalgebra of B if the inclusion map from C to B preserves all meets and joins existing in C. If I is an ideal in a Boolean algebra A, let $I^* = \{a \in A \mid a \cdot i = 0$ for all $i \in I\}$. I^* is the pseudocomplement of I in the lattice of ideals of A; clearly, $I \subseteq I^{**}$. Call I regular if $I = I^{**}$ – this means that the open subset corresponding to I in the Stone space St(A) of A is regular open. So, a proper dense ideal of A is never regular.

If $A \leq B$ call $u \in B$ regular over A if the ideal $\{x \in A \mid x \leq u\}$ of A is regular. If A is a dense subalgebra of C, then each $u \in C$ is regular over A, since u is essentially an element of the completion $\bar{A}$ of A, and elements of $\bar{A}$ correspond to regular open subsets of St(A). If $A \leqslant C \leqslant B$ where A is dense in C and C is a complete regular subalgebra of B, then each $u \in B$ is regular over A, since $\{x \in A \mid x \leqslant u\} = \{x \in A \mid x \leqslant c\}$ where $c = \Sigma^C\{y \in C \mid y \leqslant u\}$.

Lemma 4 (MA$_\kappa$). *Let $A \leqslant B \leqslant P(\omega)$ where A is atomless, $B \cap FC(\omega) = \mathbf{2}$ and $|B| \leqslant \kappa$. Then there is a $u \subseteq \omega$ such that u is compatible with B, and $\{x \in A \mid x \leqslant u\}$ is a proper dense ideal of A. So, u is not regular over A.*

Proof. Let

$$P = \{(p, i) \mid p : \omega \to 2, \mathrm{dom}(p) \text{ finite}, \ i \in A, i < 1, p[i] \subseteq \{0\}\},$$

and $(p, i) \leqslant (q, j)$ if $p \subseteq q$ and $i \leqslant j$. We check that $(P, \leqslant)$ satisfies the ccc: (p, i) and $(q, j) \in P$ are compatible in P iff $p \cup q$ is a function, $i + j < 1$, and $p[j] \cup q[i] \subseteq \{0\}$. Let $(p_\alpha, i_\alpha) \in P$ for $\alpha < \omega_1$; w.l.o.g. let $p_\alpha = p$ for each $\alpha < \omega_1$. So,

$$p_\alpha[i_\beta] \cup p_\beta[i_\alpha] = p_\alpha[i_\alpha] \cup p_\beta[i_\beta] \subseteq \{0\} \quad \text{for } \alpha, \beta < \omega_1.$$

Also, since A satisfies the ccc, there are $\alpha < \beta < \omega_1$ such that $i_\alpha + i_\beta < 1$.

The following subsets of P are dense in P:

(1) $D_n = \{(q, j) \in P \mid n \in \mathrm{dom}(q)\}$ for $n \in \omega$.

(2) $D^+_{kb} = \{(q, j) \in P \mid \text{there is } e \subseteq b \cap \mathrm{dom}(q) \text{ such that } |e| = k \text{ and } p[e] \subseteq \{0\}\}$ for $k \in \omega, b \in B^+$.

(3) $D^-_{kb} = \{(q, j) \in P \mid b \leqslant j \text{ or there is } e \subseteq b \cap \mathrm{dom}(q) \text{ such that } |e| = k \text{ and } q[e] \subseteq \{1\}\}$ for $k \in \omega, b \in B^+$: let $(p, i) \in P$, if $b \leqslant i$, then put $(q, j) = (p, i)$; otherwise, $b \backslash i$ is infinite, being an element of B^+; choose $e \subseteq b \backslash (i \cup \mathrm{dom}(p))$ such that $|e| = k$ and set $j = i$, and $q = p \cup \{(n, 1) \mid n \in e\}$.

(4) $D_a = \{(q, j) \in P \mid a \cdot j > 0\}$ for $a \in A^+$: let $(p, i) \in P$; if $a \leqslant i$, put $(q, j) = (p, i)$. Otherwise, since A is atomless, choose $c \in A^+$ such that $0 < c < a \cdot -i$ and $c \cap \mathrm{dom}\, p = \emptyset$; then set $(q, j) = (p, i + c)$.

Again for some $G \subseteq P$ generic for these dense sets, let $f = \bigcup G$ and $u = f^{-1}(0)$, then

$$I = \{i \in A \mid i \leqslant j \text{ for some } (p, j) \in G\}$$

clearly is an ideal of A; it will turn out that $I = \{x \in A \mid x \leqslant u\}$. Let J be the ideal of B generated by I.

First, I is a proper ideal of A, and $i \in I$ implies $i \leqslant u$ by definition of G. By $G \cap D_a \neq \emptyset$ for $a \in A^+$, I is a dense ideal. For $b \in B^+$, $b \cdot u$ and $b \cdot -u$ are infinite or empty, so $B(u) \cap FC(\omega) = \mathbf{2}$: $b \cdot u$ is infinite by $G \cap D^+_{bk} \neq \emptyset$. For $b \in J$, we have $b \leqslant u$ by definition of f, so $b \cdot -u = 0$. If $b \notin J$, $b \cdot -u$ is infinite (which also establishes $\{x \in A \mid x \leqslant u\} \subseteq I$): for $k \in \omega$ choose $(q, j) \in G \cap D^-_{bk}$. So, $j \in I$, and since $b \notin J$, $b \nleqslant j$, and therefore $|b \cap -u| \geqslant k$. □

Example 3 (MA). *$FC(\omega)$ has a quasicomplement B in* Sub($P(\omega)$) *such that if*

$C \leqslant B$ is a complete regular subalgebra of B, then $\pi(C) = 2^\omega$. In particular, no $\overline{F_\kappa}$ is a regular subalgebra of B for $\kappa \leqslant 2^\omega$.

Proof. Let $\{x_\alpha \mid \alpha < 2^\omega\}$ be an enumeration of $P(\omega)$, and, since (MA) implies that $2^\kappa = 2^\omega$ for $\kappa < 2^\omega$, $\{A_\alpha \mid \alpha < 2^\omega\}$ be an enumeration of

$$\mathfrak{A} = \{A \leqslant P(\omega) \mid A \text{ is atomless and } |A| < 2^\omega\}.$$

We may assume that each $A \in \mathfrak{A}$ is listed 2^ω times in this enumeration.

We construct a chain $(B_\alpha)_{\alpha<2^\omega}$ of subalgebras of $P(\omega)$ such that $B_\alpha \cap FC(\omega) = \mathbf{2}$, and $|B_\alpha| < 2^\omega$. Then we set $B = \bigcup_{\alpha<2^\omega} B_\alpha$. Let $B_0 = \mathbf{2}$ and $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ for limit ordinals. If B_α has been constructed, let $B'_\alpha = B_\alpha(x_\alpha)$ if x_α is compatible with B_α, and $B'_\alpha = B_\alpha$ otherwise. If $A_\alpha \leqslant B'_\alpha$, let $B_{\alpha+1} = B'_\alpha(u_\alpha)$, where u_α is chosen by Lemma 4 to be compatible with B'_α, and u_α is not regular over A_α; otherwise set $B_{\alpha+1} = B'_\alpha$.

Clearly B is a quasicomplement of $FC(\omega)$. Suppose that C is a complete atomless regular subalgebra of B, and $\pi(C) < 2^\omega$. Let A be a dense subalgebra of C such that $|A| < 2^\omega$; A also is atomless; pick $\alpha < 2^\omega$ such that $A = A_\alpha \leqslant B_\alpha$. Then u_α is an element of B which is not regular over A, contradicting the remark preceding Lemma 4. □

3. Complements in Sub($P(\omega)$)

To abbreviate the statement and proof of the following theorem, we give some definitions. Let $A \leqslant P(X)$ (X will be a subset of ω later on). If α is a finite atom of A, call α a proper atom if $|\alpha| > 1$, and an improper atom if $|\alpha| = 1$. Let for $b \in A$

$$d(b) = b \setminus \bigcup \mathrm{At}(A)$$

be the 'defect of b'. Note that for $b \leqslant c$ in A, $d(b) = d(c) \cap b$. Call $a \in A$ bad (w.r.t. A) if $A \upharpoonright a$ is atomic, $d(a)$ is finite, each atom of $A \upharpoonright a$ is finite, and only finitely many atoms of $A \upharpoonright a$ are proper. So in particular a is bad, if $FC(a) \leqslant A \upharpoonright a$. Call a good if it is not bad. The set of bad elements of A is an ideal Bd of A containing each finite element of A.

The proof of the theorem will split into five cases which are handled in Lemmas 6, 8, 9, 10, 12. Note that every finite subalgebra of $P(\omega)$ has a complement, hence, we shall concentrate on the case of countable $A \leqslant P(\omega)$, which means $|A| = \omega$. There are four 'positive' cases (Lemmas 8, 9, 10, 12) in which A has a complement; the proofs of these cases can also be carried out for $|A| = \kappa < 2^\omega$, assuming (MA_κ). The only negative case (Lemma 6) relies on Lemma 5(a) which has a partial analogue under (MA_κ) in Lemma 5(b); we have, however, not been able to prove Lemma 6 assuming $|A| = \kappa < 2^\omega$ and (MA_κ).

Theorem 1. *Let A be a countable subalgebra of $P(\omega)$. A does not have a complement in* $\mathrm{Sub}(P(\omega))$ *iff*

(1) *A is atomic.*

(2) *Each atom of A is finite.*

(3) $|A/\mathrm{Bd}| \leqslant 2$.

In particular, A does not have a complement if $FC(\omega) \leqslant A$ or $A \leqslant FC(\omega)$.

Proof. If A satisfies (1), (2) and (3), then A has no complement by Lemma 6. Thus, suppose A does not satisfy (1), (2) or (3). If A does not satisfy (1) or (2), then there is some $a \in A^+$ such that each $b \in (A \upharpoonright a)^+$ is infinite; then A has a complement by Lemma 8. So let A satisfy (1) and (2), and suppose there are $a_1, \ldots, a_n \in A$ such that $|a_i| = \omega$, $a_1 \dot{+} \cdots \dot{+} a_n = 1$ and w.l.o.g. a_1 and a_2 are good. Then $a_2 + \cdots + a_n$ is also good, so assume $n = 2$ and $a_1 = a$, $a_2 = -a$ are both good. Now, $d(a)$ is infinite or $A \upharpoonright a$ has infinitely many proper atoms and the same holds for $-a$. Then A has a complement by Lemmas 9, 10, 12. □

Lemma 5. *Let $A \leqslant P(\omega)$ and B a complement of A in* $\mathrm{Sub}(P(\omega))$.

(a) *If $|A| = \omega$ then A has a finite subalgebra A' such that $A' \vee B = P(\omega)$.*

(b) (MA_κ). *If $\omega \leqslant |A| = \kappa < 2^\omega$, then A has a proper subalgebra A' such that $A' \vee B = P(\omega)$.*

Proof. Both assertions follow from the fact that $\mathrm{cf}\, A = \omega$ in the terminology of [2], i.e. that $A = \bigcup_{n \in \omega} A_n$ for a strictly ascending chain $(A_n)_{n \in \omega}$ of subalgebras of A; if $|A| = \omega$, each A_n can be chosen to be finite. For $\omega \leqslant |A| = \kappa < 2^\omega$, $\mathrm{cf}(A) = \omega$ is proved in [2, Proposition 5]; in fact, the proof given there and the remark following it show that A has a homomorphic image isomorphic to $FC(\omega)$. If $A \vee B = P(\omega)$, then $P(\omega) = \bigcup_{n \in \omega} D_n$ where $D_n = A_n \vee B$. Since it is shown in [2] that $\mathrm{cf}(P(\omega)) = \omega_1$, $P(\omega) = D_n$ for some $n < \omega$. □

Lemma 6. *If a countable subalgebra A of $P(\omega)$ satisfies* (1) *to* (3) *of Theorem* 1, *then A has no complement.*

Proof. Assume that B is a complement of A. By Lemma 5(a), there are finitely many elements of A, say $e_1, \ldots, e_k, a_1, \ldots, a_n, a$ such that

$$P(\omega) = B(e_1, \ldots, e_k, a_1, \ldots, a_n, a).$$

We may assume that $e_1 \dot{+} \cdots \dot{+} e_k \dot{+} a_1 \dot{+} \cdots \dot{+} a_n \dot{+} a = 1$, the e_j are finite, the a_i are infinite and bad, and that a is infinite and good or bad. It is possible that $n = 0$, but if $n \geqslant 1$, by increasing the number of e_j's we may assume that each atom of each $A \upharpoonright a_i$ is improper. Each $u \in P(\omega)$ can be written as

$$u = \beta_1 e_1 \dot{+} \cdots \dot{+} \beta_k e_k \dot{+} b_1 a_1 \dot{+} \cdots \dot{+} b_n a_n \dot{+} ba,$$

where $\beta_1, \dots, b_1, \dots, b \in B$. So we have epimorphisms

$$p_i : B \to P(a_i), \qquad p : B \to P(a),$$

with $p_i(b) = b \cdot a_i$, $p(b) = b \cdot a$.

Call $b \in B$ selective if $|b \cap a| \leqslant 1$ and $|b \cap a_i| \leqslant 1$ for each i. The selective elements of B form a dense subset of B: let $b \in B^+$, and by induction construct $b \geqslant b_a \geqslant b_1 \geqslant \cdots \geqslant b_n$ in B^+ such that $|b_a \cap a| \leqslant 1$, $|b_1 \cap a_1| \leqslant 1, \dots, |b_n \cap a_n| \leqslant 1$; b_n will then be selective. Construct b_a as follows: if $b \cap a = \emptyset$, $b_a = b$; otherwise, pick some $x \in b \cap a$ and choose $b_a \leqslant b$ such that $p(b_a) = \{x\}$ – this is possible since p is an epimorphism. By the same argument, choose $b_1 \leqslant b_a$ such that $b_1 > 0$ and $|b_1 \cap a_1| \leqslant 1$, etc.

Put

$$e = e_1 \cup \cdots \cup e_k \cup d(a_1) \cup \cdots \cup d(a_n).$$

Since $a_1, \dots, a_n$ are bad, e is finite. For $x \in a$, pick $b_x \in B$ such that $p(b_x) = \{x\}$. Since p is homomorphism and $|a| = \omega$, the b_x may be chosen pairwise disjoint. Since e is finite, there is some $M \subseteq a$ such that $a \backslash M$ is finite and $b_x \cap e = \emptyset$ for $x \in M$. There is an atom α of $A \upharpoonright a$ such that $\alpha \subseteq M$, for the atoms of a are finite. For $x \in \alpha$, let $b'_x \in B$ be selective and $0 < b'_x \leqslant b_x$. If $x \notin b'_x$ for some $x \in \alpha$, then b'_x is a non-empty selective subset of $(a_1 \backslash d(a_1)) \cup \cdots \cup (a_n \backslash d(a_n))$. By definition of e, $b'_x \cap a_i$ is an atom of A for each i; otherwise, $b'_x \cap d(a_i) \neq \emptyset$ and thus $b'_x \cap e \neq \emptyset$. This establishes $b'_x \in A \cap B$, a contradiction.

If $x \in b'_x$ for every $x \in \alpha$, the same argument shows that $b'_x \backslash \{x\} \in A$ for each $x \in \alpha$. Then $\bigcup_{x \in \alpha} b'_x = \alpha \cup \bigcup_{x \in \alpha} (b'_x \backslash \{x\})$ is an element of $(A \cap B) \backslash \mathbf{2}$. □

Complements of $A \leqslant P(\omega)$ will be constructed in Lemmas 8, 9, 10, 12 by the following method:

Lemma 7. *Let D be an arbitrary Boolean algebra, $A \leqslant D$ and $a \in A$. Suppose φ is an epimorphism from $D \upharpoonright a$ onto $D \upharpoonright -a$. Then $B = \{x + \varphi(x) \mid x \in D \upharpoonright a\}$ is a subalgebra of D and $A \vee B = D$. Moreover, $A \cap B = \mathbf{2}$ if $\varphi(\alpha) \notin A$ for each $\alpha \in A \upharpoonright a$ satisfying $0 < \alpha < a$.*

Proof. Clearly, B is a subalgebra of D. $A \cup B$ generates D: let $d \in D$. Put $x = d \cdot a$, and choose $x' \in D \upharpoonright a$ such that $\varphi(x') = d \cdot -a$. Then $b = x + \varphi(x)$ and $b' = x' + \varphi(x')$ are both in B, and

$$d = d \cdot a + d \cdot -a = x + \varphi(x') = b \cdot a + b' \cdot -a.$$

Now suppose $b = x + \varphi(x) \in (A \cap B) \backslash \mathbf{2}$ where $x \in D \upharpoonright a$. If $x = 0$, then $\varphi(x) = 0$, hence $b = 0$, a contradiction; similarly, $b = 1$ if $x = a$, so $0 < x < a$; furthermore, $x = b \cdot a$ and $\varphi(x) = b \cdot -a \in A$, which proves the rest of the lemma. □

The epimorphism φ, or, later on, a certain function f defining φ, can be constructed in all cases by an induction argument since A is countable or by a forcing argument; the latter also works for $|A| = \kappa < 2^{\omega}$ under (MA_{κ}). We omit the details in the easier cases.

Lemma 8. *Suppose $A \leqslant P(\omega)$ is countable, and there is some $a \in A^{+}$ such that each $b \in (A \upharpoonright a)^{+}$ is infinite. Then A has a complement.*

Proof. We may assume that $-a$ also is infinite: a is an atom of A, then $A \upharpoonright -a$, and hence $-a$, are infinite. Otherwise, pick $\alpha \in (A \upharpoonright a)^{+}$ such that $0 < \alpha < a$ and consider $\alpha, -\alpha$ instead of $a, -a$. Now construct a bijection $f: a \to -a$ such that the isomorphism $\varphi: P(\omega) \upharpoonright a \to P(\omega) \upharpoonright -a$ given by $\varphi(x) = f[x]$ satisfies the requirements of Lemma 7. □

Lemma 9. *Suppose $A \leqslant P(\omega)$ is countable and atomic, that all atoms of A are finite, and that for some $a \in A$, both $d(a)$ and $d(-a)$ are infinite. Then A has a complement.*

Proof. It follows from $|d(a)| = \omega$ that $a \backslash d(a)$ is infinite, too, since otherwise a is the supremum of a finite set of atoms in A and $d(a) = \emptyset$; similarly, $-a \backslash d(-a)$ is infinite. Now construct a bijection $f: a \to -a$ which maps $a \backslash d(a)$ onto $d(-a)$, and $d(a)$ onto $-a \backslash d(-a)$ such that φ given by $\varphi(x) = f[x]$ satisfies the requirements of Lemma 7. □

Lemma 10. *Suppose $A \leqslant P(\omega)$ is countable and atomic, that all atoms of A are finite, and that, for some $a \in A$, $d(a)$ is infinite and $-a$ contains infinitely many proper atoms. Then A has a complement.*

Proof. Denote by PrAt the set of proper atoms of A contained in $-a$. Let $X \subseteq \bigcup \mathrm{prAt}$ such that $|X \cap \alpha| = 1$ for each $\alpha \in \mathrm{PrAt}$. Let $Y = -a \backslash X$. Let $T = d(a)$ and $S = a \backslash d(a)$. We construct a bijection $f: a \to -a$ such that $f[S] = X$, $f[T] = Y$, and such that φ given by $\varphi(x) = f[x]$ satisfies the requirements of Lemma 7. Let $P = \{p: a \to -a \mid \operatorname{dom} p \text{ is finite, } p \text{ is one-one, } p[S] \subseteq X, p[T] \subseteq Y\}$ be partially ordered by set inclusion. The following subsets of P are dense in P:

(1) $D_x = \{q \in P \mid x \in \operatorname{dom} q\}$ for $x \in a$.

(2) $D_y = \{q \in P \mid y \in \operatorname{rge} q\}$ for $y \in -a$.

(3) $D_{\alpha\beta} = \{q \in P \mid \text{there exists some } x \in \alpha \cap \operatorname{dom} q \text{ such that } q(x) \notin \beta\}$ for $\alpha, \beta \in A$ satisfying $0 \leqslant \alpha \leqslant a$, $0 \leqslant \beta \leqslant -a$, and

$$\omega = |d(\alpha)| = |a \backslash \alpha| = |\beta| = |-a \backslash \beta|:$$

let $p \in P$ and put $q = p \cup \{(x, y)\}$ where x and y are chosen as follows. Since $d(\alpha)$ is infinite, pick $x \in d(\alpha) \backslash \operatorname{dom} p$. Put $\gamma = -a \cdot -\beta$ and pick $y \in (\gamma \cap Y) \backslash \operatorname{rge} p$. The choice of y is possible by the following argument: γ is an infinite element of $A \upharpoonright -a$; let δ be an atom of A such that $\delta \leqslant \gamma$ and $\delta \cap \operatorname{rge} p = \emptyset$. Let z be an element of δ. If δ is an improper atom, then $z \notin X$, so, $z \in Y$ and we let $y = z$. If δ is a proper atom and $z \notin X$, again let $y = z$. Otherwise, let $y \in \delta$ such that $y \neq z$, so $y \in Y$.

Let $f = \bigcup G$ where $G \subseteq P$ is generic for the above family of dense sets. We check that for no $\alpha \in (A \restriction a) \backslash \{0, a\}$, $\beta = f[\alpha]$ is an element of A: assume the contrary; if α and hence β are finite, it follows that $d(\alpha) = \emptyset$, $\alpha \subseteq S$ and so $\beta \subseteq X$. Pick a proper atom δ such that $\delta \cap \beta \neq \emptyset$. Now δ, being an atom of A, is contained in β, contradicting $\beta \subseteq X$. The same argument applies if $a \backslash \alpha$ and hence $-a \backslash \beta$ are finite. In the remaining case, the following sets are infinite: a, $\alpha \cap S$, $\beta \cap X$ (apply f!), $\beta \cap Y$ (as in the previous case), $\alpha \cap T$ (apply f^{-1}!) $= d(\alpha)$. Thus $D_{\alpha\beta}$ is defined, and $D_{\alpha\beta} \cap G \neq \emptyset$ yields $f[\alpha] \neq \beta$. $\square$

Lemma 11. *Let X, Y be sets with partitions P, resp. Q, such that $|P| = |Q| = \omega$, all $p \in P$, $q \in Q$ are finite and at most one $r \in P \cup Q$ has cardinality 1. Let $x \sim_X x'$ if x, x' belong to the same element of P, and define $y \sim_Y y'$ similarly. If $f: X \to Y$ is one-one, let $\sim_f$ be the least equivalence relation on $X \cup Y$ including f. Then there exists an injective $f: X \to Y$ such that each subset of $X \cup Y$ is closed w.r.t. $\sim_X$, $\sim_Y$, and $\sim_f$ is empty or equals $X \cup Y$.*

Proof. Consider the case that each $p \in P$ and $q \in Q$ has at least two elements. Let

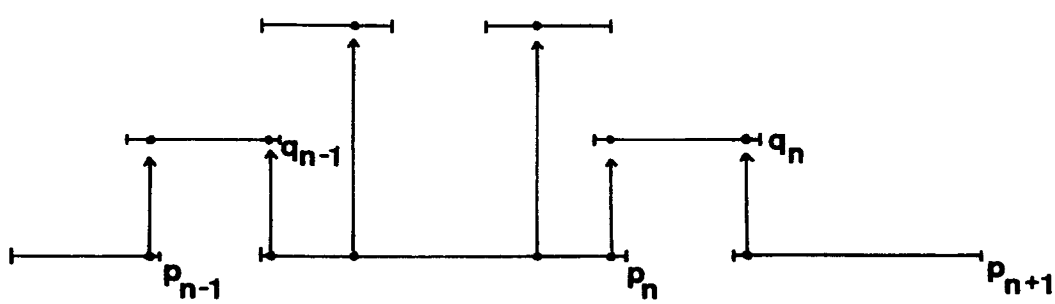
$$P = \{p_n \mid n \in \mathbb{Z}\},$$

and fix different elements $x_n, x'_n \in p_n$; then put

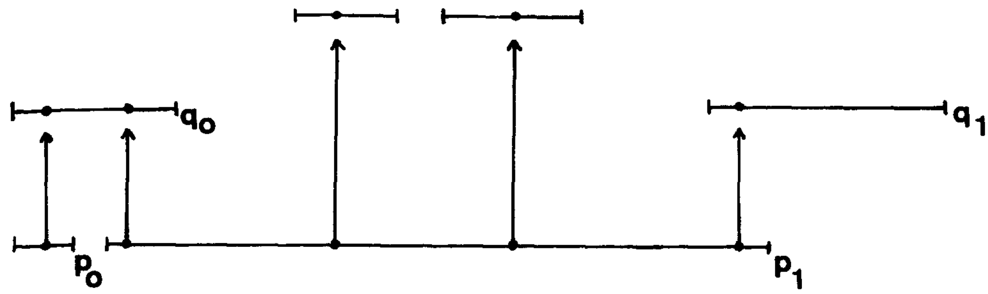
$$X' = X \backslash (\{x_n \mid n \in \mathbb{Z}\} \cup \{x'_n \mid n \in \mathbb{Z}\}).$$

Choose $Q' \subseteq Q$ such that $|Q'| = |X'|$ and $Q \backslash Q' = \{q_n \mid n \in \mathbb{Z}\}$; fix different elements $y_n, y'_n \in q_n$ and a bijection $g: X' \to Q'$. Then define $f: X \to Y$ by

$$f(x_n) = y'_{n-1}, \qquad f(x'_n) = y_n,$$
$$f(x) = \text{some element of } g(x) \quad \text{for } x \in X'.$$



If $|p| = 1$ for some $p \in P$, let $P = \{p_n \mid n \in \omega\}$ where $p_0 = p$ and proceed as shown in the following diagram:



□

Lemma 12. *Suppose $A \leq P(\omega)$ is countable and atomic, that all atoms of A are finite and that, for some $a \in A$, both a and $-a$ contain infinitely many proper atoms. Then A has a complement.*

Proof. Let

$$S = \mathrm{At}(A \restriction -a) \cup \{\{x\} \mid x \in d(-a)\},$$
$$T = \mathrm{At}(a \restriction a) \cup \{\{y\} \mid y \in d(a)\}.$$

Then S (T) partitions $-a$ (a). Construct a partition of $-a$ (a) into infinitely many sets s_n (t_n), $n \in \omega$, such that:

(1) Each s_n (t_n) is the union of infinitely many elements of S (T).

(2) $s_0, s_2, s_4, \ldots$ ($t_1, t_3, t_5, \ldots$) are unions of proper atoms of A.

(3) $s_1, s_3, s_5, \ldots$ ($t_0, t_2, t_4, \ldots$) contain at most one singleton from S (T).

(4) If $c \in A \restriction -a$ includes infinitely many proper atoms of A, then $c \cap s_n \neq \emptyset$ for every n.

For each n, let by (2), (3) and Lemma 11, $f_n : s_n \to t_n$ be a one-one function w.r.t. the partitions $\{s \in S \mid s \subseteq s_n\}$ and $\{t \in T \mid t \subseteq t_n\}$ of s_n and t_n. Then $f = \bigcup_{n \in \omega} f_n$ is a one-one function from $-a$ to a and $\varphi : P(a) \to P(-a)$ defined by $\varphi(\alpha) = f^{-1}[\alpha]$ is an epimorphism. We check that φ satisfies the condition of Lemma 7: assume that $\alpha \in A \restriction a$ and $\varphi(\alpha) = f^{-1}[\alpha] \in A$. Then

$$M = \alpha \cup f^{-1}[\alpha] \in A \cap B,$$

where B is defined by φ as in Lemma 7. For $n \in \omega$ let

$$M_n = M \cap (s_n \cup t_n).$$

We claim that each M_n is empty or equals $s_n \cup t_n$. This follows from the choice of f_n, since M_n is closed w.r.t. $\sim_{s_n}$, $\sim_{t_n}$, $\sim_{f_n}$, defined as in Lemma 11: M is an element of A (resp. B) and the non-singleton equivalence classes of $\sim_{s_n}$, $\sim_{t_n}$ (resp. $\sim_{f_n}$) are atoms of A (resp. B).

If $M \neq \emptyset$ and $M \neq \omega$, there are $k, l \in \omega$ such that $s_k \cup t_k \subseteq M$ and $(s_l \cup t_l) \cap M = \emptyset$. Now, $s_k \subseteq M \cap -a = c$, so c is an element of $A \restriction -a$ containing infinitely many proper atoms. But then $c \cap s_l \neq \emptyset$, a contradiction. □

The following example answers a question raised in [4, p. 62].

Example 4. *There are subalgebras A, B of $P(\omega)$ such that B is both a complement and a quasicomplement of A, but A is not a quasicomplement of B.*

Proof. Let $\{a_n \mid n \in \omega\}$ be a partition of ω such that $|a_n| = \omega$ for each n. Let A, resp. A^* be the subalgebra of $P(\omega)$ generated, resp. completely generated, by the a_n. Choose a partition $\{m_i \mid i \in \omega \setminus \{0\}\}$ of a_0 such that $|m_i| = \omega$ for each $i \neq 0$. Put $b_i = m_i \cup a_i$, $i \neq 0$, and let B_0 be the subalgebra of $P(\omega)$ completely generated by the b_i. Let e be a subset of ω intersecting each m_i and each a_i, $i \neq 0$, in exactly one point. Then let

$$B_1 = \{b \subseteq \omega \mid b \cap e = b_0 \cap e \text{ for some } b_0 \in B_0\}.$$

Now, $B_1(a_0) = P(\omega)$ and $A^* \cap B_1 = \mathbf{2}$ as is easily checked. Since B_1 is a complement of A^*, choose a quasicomplement B of A^* containing B_1. We claim that $A \cap B_0(a) \neq \mathbf{2}$ for $a \in A^* \setminus \mathbf{2}$: let $a = \bigcup_{i \in I} a_i$ be an element of $A^* \setminus \mathbf{2}$ where $I \subseteq \omega$. W.l.o.g. $0 \notin I$, otherwise consider $-a$. Thus, there exists $j \in I$, $j \neq 0$; now,

$$\gamma = b_j \cdot a = b_j \cdot a_j = a_j$$

is an element of $(A \cap B_0(a)) \setminus \mathbf{2}$.

B is a quasicomplement of A: let $B \leqslant B' \leqslant P(\omega)$ and $B \neq B'$. By maximality of B, let $a \in (A^* \cap B') \setminus \mathbf{2}$. By the above claim, $\mathbf{2} \neq A \cap B_0(a) \leqslant A \cap B'$. Since $A^* \cap B = \mathbf{2}$ and A is a proper subalgebra of A^*, A is not a quasicomplement of B. □

Since $B_1(a_0) = P(\omega)$ and $A^* \cap B_1 = \mathbf{2}$, the example also shows that each Boolean algebra C having at least four elements and embeddable into $P(\omega)$ is embeddable into $P(\omega)$ such that it has a complement in $P(\omega)$ – simply embed C into A^*.

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