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The Finite Upper Half Space and Related Hypergraphs

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A space generalizing the finite upper half plane is presented along with a projective action by the finite general linear group. A volume generalizing the pseudo-distance on the finite upper half plane is also given. Then this volume is used to create hypergraphs which are analyzed with respect to the Ramanujan property. © 2000 Academic Press

1. INTRODUCTION

The *finite upper half plane* [18] is defined as

$$H_q = \{x + y\sqrt{\delta} : x, y \in \mathbb{F}_q, y \neq 0\},$$

where q is an integral power of an odd prime number, $\mathbb{F}_q$ is the finite field with q elements, and δ is a nonsquare in $\mathbb{F}_q$. H_q is a finite analogue of the *Poincaré upper half plane*

$$H = \{x + iy : x, y \in \mathbb{R}, y > 0\}, \qquad \text{where} \quad i = \sqrt{-1}.$$

It turns out that H_q has many properties analogous to those of H. For example, one has an action by $\mathrm{GL}(2, \mathbb{F}_q)$ on H_q analogous to that of $\mathrm{GL}(2, \mathbb{R})$ on H. Then, H_q can be given a geometrical setting by assigning it a pseudo-distance, which imitates that of Stark [15] on a p-adic analogue of H and is analogous to the non-Euclidean arc-length on H. Namely, we have that for $z, w \in H_q$,

$$k(z, w) = \frac{\mathcal{N}(z - w)}{\Im(z)\,\Im(w)}, \tag{1}$$

where if $z = x + y\sqrt{\delta}$ then $\Im(z) = y$, $\bar{z} = x - y\sqrt{\delta}$, and $\mathcal{N}(z) = z\bar{z}$. This pseudo-distance is invariant under the projective action by $\mathrm{GL}(2, \mathbb{F}_q)$ on H_q.

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There are classical eigenvalue problems on H that involve the non-Euclidean Laplacian

$$\Delta = \frac{dx^2 + dy^2}{y^2}.$$

For example, there is much interest in obtaining a closed form for the eigenvalues of Δ, acting on the fundamental domain of H under the full modular group, corresponding to nonzero cuspidal Maass wave forms. This problem is still open, but a similar one can be posed in the finite setting of H_q. A finite analogue of the Laplacian is the adjacency operator of a regular graph. Then, one is interested in analyzing the eigenvalues of such operators in terms of their distribution and size. Using k as in (1), regular graphs of H_q were constructed [2, 5, 14, 16, 19] which turned out to be Ramanujan as defined by Lubotzky *et al.* [13]. That is, these graphs satisfied the property that all nontrivial eigenvalues were bound in absolute value by

$$2\sqrt{d-1},$$

where d is the corresponding degree of regularity.

In this paper we present a generalization of H_q, which we call the finite upper half space. We also give a projective action by $\mathrm{GL}(n, \mathbb{F}_q)$ and an n-point invariant on the finite upper half space, which we use to construct related hypergraphs. The adjacency operator on such hypergraphs is as defined by Feng and Li in [9]. Then, some examples are constructed, which turn out to be Ramanujan as defined by Li and Solé in [11]. We hope that eventually this work will throw some light on similar eigenvalue problems in the infinite setting using infinite fields such as $\mathbb{R}$, $\mathbb{C}$, and the p-adic number fields.

2. THE FINITE UPPER HALF SPACE

DEFINITION 1. For θ a root of an nth degree irreducible polynomial in $\mathbb{F}_q[x]$, the *finite upper half space* is

$$\mathscr{H}_q^n = \left\{ \begin{pmatrix} W_1 \\ \vdots \\ W_{n-1} \\ 1 \end{pmatrix} : \begin{array}{l} W_1, ..., W_{n-1} \in \mathbb{F}_q(\theta) \\ W_1, ..., W_{n-1}, 1 \text{ linearly independent over } \mathbb{F}_q \end{array} \right\},$$

where q is now an integral power of any prime number.

$\mathscr{H}_q^n$ is a subset of the finite projective space $\mathbb{P}^{n-1}(\mathbb{F}_q(\theta))$. We want to have an action on $\mathscr{H}_q^n$ by $\mathrm{GL}(n, \mathbb{F}_q)$ in such a way that the space is preserved.

Naturally, such an action will be a projective one and the following theorem gives it to us.

THEOREM 2. *For* $\alpha \in \mathrm{GL}(n, \mathbb{F}_q)$ *and*

$$W = \begin{pmatrix} W_1 \\ \vdots \\ W_{n-1} \\ 1 \end{pmatrix} \in \mathscr{H}^n_q, \qquad \alpha \circ W = \alpha W \left(\frac{1}{(\alpha W)_n} \right) \tag{2}$$

defines an action by $\mathrm{GL}(n, \mathbb{F}_q)$ *on* $\mathscr{H}^n_q$, *where* αW *denotes just ordinary matrix multiplication and* $(\alpha W)_n$ *is the* nth *coordinate of the vector* αW.

Proof. (i) It is clear that if $\alpha = I_n$, the $n \times n$ identity matrix, then

$$\alpha \circ W = W.$$

(ii) We now see that $\mathscr{H}^n_q$ is preserved under the map in (2); that is, we need to see that the entries of $\alpha \circ W$ are in $\mathbb{F}_q(\theta)$ and linearly independent over $\mathbb{F}_q$. By the definition of $\mathscr{H}^n_q$, we know $W_1, ..., W_{n-1}$, 1, the entries of W, are in $\mathbb{F}_q(\theta)$ and linearly independent over $\mathbb{F}_q$. Since α is in $\mathrm{GL}(n, \mathbb{F}_q)$, then its rows are linearly independent over $\mathbb{F}_q$. When we think of the rows of α as coefficients of elements in $\mathbb{F}_q(\theta)$ with respect to the basis $W_1, ..., W_{n-1}$, 1, we see that the entries of the vector αW are linearly independent over $\mathbb{F}_q$. Moreover, the last entry of αW is nonzero and therefore the entries of $\alpha \circ W$, as in (2), are in $\mathbb{F}_q(\theta)$ and linearly independent over $\mathbb{F}_q$.

(iii) We now show that if β is in $\mathrm{GL}(n, \mathbb{F}_q)$, then $\alpha \circ (\beta \circ W) = (\alpha\beta) \circ W$.

$$\begin{aligned}
\alpha \circ (\beta \circ W) &= \alpha \left(\beta W \left(\frac{1}{(\beta W)_n} \right) \right) \left(\frac{1}{\left(\alpha \left(\beta W \left(\frac{1}{(\beta W)_n} \right) \right) \right)_n} \right) \qquad \text{by (2)} \\
&= \left(\alpha\beta W \left(\frac{1}{(\beta W)_n} \right) \right) \left(\frac{1}{(\alpha\beta W)_n \left(\frac{1}{(\beta W)_n} \right)} \right) \\
&= \alpha\beta W \left(\frac{1}{(\alpha\beta W)_n} \right) \\
&= (\alpha\beta) \circ W
\end{aligned}$$

by (2). ∎

Thus, we have an action by $\mathrm{GL}(n, \mathbb{F}_q)$ on $\mathscr{H}_q^n$, which will be shown to be transitive. First, we state some preliminary facts and definitions.

Remark 3. Let $\{\theta^{n-1}, ..., \theta, 1\}$ be a basis of $\mathbb{F}_q(\theta)$. The algebra $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_q(\theta))$ is isomorphic to $\mathrm{M}_n(\mathbb{F}_q)$, the $n \times n$ matrices over $\mathbb{F}_q$, and the isomorphism ϕ: $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_q(\theta)) \to \mathrm{M}_n(\mathbb{F}_q)$ can be defined by mapping $A \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_q(\theta))$ to the matrix $\phi(A)$, whose (i, j) entries for $i, j = 1, ..., n$ are given by

$$A(\theta^{n-i}) = \sum_{j=1}^{n} (\phi(A))_{i,j}\, \theta^{n-j} \qquad \text{for} \quad i = 1, ..., n.$$

This is a standard result in linear algebra; for a reference see [4]. Note that representing $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_q(\theta))$ with $\mathrm{M}_n(\mathbb{F}_q)$, as above, would mean that the elements of $\mathbb{F}_q(\theta)$ are represented as row vectors with respect to the basis $\{\theta^{n-1}, ..., \theta, 1\}$. Thus, the elements of $\mathrm{M}_n(\mathbb{F}_q)$ would be acting on the right of such row vectors.

Now, $X \in \mathbb{F}_q(\theta)$ give rise to a linear transformation on $\mathbb{F}_q(\theta)$ by multiplication. Let M_X be the corresponding matrix of coefficients with respect to the basis $\{\theta^{n-1}, ..., \theta, 1\}$, whose (i, j) entries for $i, j = 1, ..., n$ are given by

$$X \cdot \theta^{n-i} = \sum_{j=1}^{n} (M_X)_{i,j}\, \theta^{n-j} \qquad \text{for} \quad i = 1, ..., n. \tag{3}$$

Then, we have for $X \in \mathbb{F}_q(\theta)$,

$$\det(M_X) = \mathscr{N}(X), \tag{4}$$

where M_X is as in (3) and $\mathscr{N}(X)$ (the norm of X) is defined as the product of X with all its conjugates in the extension $\mathbb{F}_q(\theta)$. To see this note that X satisfies the characteristic polynomial of M_X by the isomorphism given in Remark 3. Thus, all conjugates of X in $\mathbb{F}_q(\theta)$ also satisfy this polynomial. As a result, the eigenvalues of M_X are X and all its conjugates in $\mathbb{F}_q(\theta)$, and the desired result follows.

It will turn out that $\mathscr{H}_q^n$ is a homogeneous space under the action by $\mathrm{GL}(n, \mathbb{F}_q)$, which means that all elements of $\mathscr{H}_q^n$ look more or less the same with respect to this action. Thus, we pick one element of $\mathscr{H}_q^n$ and call it the special point. For a further reference on homogeneous spaces see [8].

DEFINITION 4. The special point or origin of $\mathscr{H}_q^n$ is defined as

$$\Theta_n = \begin{pmatrix} \theta^{n-1} \\ \vdots \\ \theta \\ 1 \end{pmatrix}. \tag{5}$$

PROPOSITION 5. *The action by* $\mathrm{GL}(n, \mathbb{F}_q)$ *on* $\mathscr{H}_q^n$, *as defined in Theorem 2, is transitive. That is, for* $W, Z \in \mathscr{H}_q^n$, *there exists* $\alpha \in \mathrm{GL}(n, \mathbb{F}_q)$ *such that* $\alpha \circ W = Z$. *In this case* $\mathscr{H}_q^n$ *is said to be a homogeneous space under the action by* $\mathrm{GL}(b, \mathbb{F}_q)$.

Proof. For

$$W = \begin{pmatrix} u_{1,1}\theta^{n-1} & + \cdots + & u_{1,n-1}\theta & + & u_{1,n} \\ & & \vdots & & \\ u_{n-1,1}\theta^{n-1} & + \cdots + & u_{n-1,n-1}\theta & + & u_{n-1,n} \\ & & 1 & & \end{pmatrix} \in \mathscr{H}_q^n,$$

define

$$\alpha_W = \begin{pmatrix} u_{1,1} & \cdots & u_{1,n-1} & u_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ u_{n-1,1} & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{6}$$

which is in $\mathrm{GL}(n, \mathbb{F}_q)$ since the entries of W are linearly independent over $\mathbb{F}_q$ by the definition of $\mathscr{H}_q^n$. Then $\alpha_W \circ \Theta_n = W$, which implies that $\Theta_n = \alpha_W^{-1} \circ W$. Thus,

$$Z = \alpha_Z \circ \Theta_n = \alpha_Z \circ (\alpha_W^{-1} \circ W) = (\alpha_Z \alpha_W^{-1}) \circ W. \quad \blacksquare$$

We now want to represent the action in Theorem 2 in another form. For $\beta \in \mathrm{GL}(n, \mathbb{F}_q)$ and $W \in \mathscr{H}_q^n$,

$$\beta \circ W = \beta W \left(\frac{1}{(\beta W)_n} \right)$$

$$= \beta(\alpha_W \circ \Theta_n) \left(\frac{1}{(\beta W)_n} \right)$$

by the definition of α_W (6);

$$= \beta(\alpha_W \Theta_n) \left(\frac{1}{(\beta W)_n} \right)$$

since $\alpha_W \circ \Theta_n = \alpha_W \Theta_n$, and

$$= (\beta \alpha_W)\, \Theta_n \left(\frac{1}{(\beta W)_n} \right).$$

The rows of $\beta\alpha_W$ can be interpreted as the coefficients of elements of $\mathbb{F}_q(\theta)$ with respect to the basis $\{\theta^{n-1}, ..., \theta, 1\}$ which are obtained by multiplying $\beta\alpha_W$ by Θ_n on the right. Then, by the isomorphism is given in Remark 3, dividing $\beta\alpha_W\Theta_n$ by $(\beta W)_n$ can be written as in the following lemma.

LEMMA 6. *For $W \in \mathscr{H}_q^n$ and $\beta \in \mathrm{GL}(n, \mathbb{F}_q)$,*

$$\beta \circ W = (\beta\alpha_W)\, M_{(\beta W)_n}^{-1} \Theta_n,$$

where $M_{(\beta W)_n}$ is as in (3), *α_W is as in* (6), *and Θ_n is as in* (5).

We now want to identity the stabilizer of Θ_n; that is, the subgroup of $\mathrm{GL}(n, \mathbb{F}_q)$ which is given by $\{\alpha \in \mathrm{GL}(n, \mathbb{F}_q) : \alpha \circ \Theta_n = \Theta_n\}$. The following theorem gives it to us.

THEOREM 7. *The stabilizer of Θ_n is given by*

$$K_n = \{M_X \colon X \in \mathbb{F}_q(\theta)^*\} \cong \mathbb{F}_q(\theta)^*,$$

where M_X is as in (3).

Proof. For $\beta \in \mathrm{GL}(n, \mathbb{F}_q)$ we have

$$\beta \circ \Theta_n = \Theta_n \qquad \text{if and only if} \quad \beta M_{(\beta\Theta_n)_n}^{-1} \Theta_n = \Theta_n,$$

by Lemma 6, since $\alpha_{\Theta_n} = I_n$, the $n \times n$ identity matrix. Now, $\{\theta^{n-1}, ..., \theta, 1\}$ linearly independent over $\mathbb{F}_q$ implies

$$\beta M_{(\beta\Theta_n)_n}^{-1} \Theta_n = \Theta_n \qquad \text{if and only if} \quad \beta M_{(\beta\Theta_n)_n}^{-1} = \mathrm{I}_n.$$

Thus, $\beta = M_{(\beta\Theta_n)_n}$, which implies $\beta \in K_n$.

Now, given $\beta = M_X$ for some $X \in \mathbb{F}_q(\theta)^*$, we have

$$\beta \circ \Theta_n = M_X M_{(\beta\Theta_n)_n}^{-1} \Theta_n$$

by Lemma 6,

$$= M_X M_X^{-1} \Theta_n.$$

The last inequality follows since the last row of $\beta = M_X$ corresponds to the coefficients of $X \cdot 1$ with respect to the basis $\{\theta^{n-1}, ..., \theta, 1\}$, as in (3). This implies that $(\beta\Theta_n)_n = X \cdot 1$ with respect to the basis $\{\theta^{n-1}, ..., \theta, 1\}$, and so $M_{(\beta\Theta_n)_n}^{-1} = M_X^{-1}$. Thus,

$$M_X \circ \Theta_n = \mathrm{I}_n \Theta_n = \Theta_n. \quad \blacksquare$$

Since $\mathscr{H}_q^n$ is a homogeneous space under the action in (2) by $\mathrm{GL}(n, \mathbb{F}_q)$, we can establish a bijection between $\mathrm{GL}(n, \mathbb{F}_q)/K_n$ and $\mathscr{H}_q^n$ that preserves such action. Before we do so we define a related subgroup of $\mathrm{GL}(n, \mathbb{F}_q)$.

Definition 8. The *affine group* is defined as

$$\mathscr{A}\mathrm{ff}_q^n = \left\{ \begin{pmatrix} u_{1,1} & \cdots & u_{1,n-1} & u_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ u_{n-1,1} & \cdots & u_{n-1,n-1} & u_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{GL}(n, \mathbb{F}_q) \right\}. \tag{7}$$

We claim that $\mathrm{GL}(n, \mathbb{F}_q) = \mathscr{A}\mathrm{ff}_q^n \cdot K_n$ and $\mathscr{A}\mathrm{ff}_q^n \cap K_n = \{\mathrm{I}_n\}$, where I_n is the $n \times n$ identity matrix. To see that $\mathscr{A}\mathrm{ff}_q^n \cap K_n = \{\mathrm{I}_n\}$, recall that any element of K_n is given by M_X for $X \in \mathbb{F}_q(\theta)^*$, M_X as in (3). Then, the last row of M_X corresponds to $X \cdot 1$ with respect to the basis $\{\theta^{n-1}, ..., \theta, 1\}$, which is $(0, ..., 0, 1)$ if and only if $X = 1$. Thus, any element $M_X \in K_n$ is in $\mathscr{A}\mathrm{ff}_q^n$ if and only if $M_X = \mathrm{I}_n$. Now, $\alpha \in \mathrm{GL}(n, \mathbb{F}_q)$ can be expressed as

$$\alpha = (\alpha M^{-1}_{(\alpha\Theta_n)_n})\, M_{(\alpha\Theta_n)_n},$$

where $(\alpha M^{-1}_{(\alpha\Theta_n)_n}) \in \mathscr{A}\mathrm{ff}_q^n$. To see this, note that $(\alpha\Theta_n)_n$ is the element of $\mathbb{F}_q(\theta)$ whose coefficients with respect to the basis $\{\theta^{n-1}, ..., \theta, 1\}$ are in the last row of α. Thus, by the isomorphism given in Remark 3, multiplying α on the right by $M^{-1}_{(\alpha\Theta_n)_n}$ is equivalent to dividing by $(\alpha\Theta_n)_n$, and the result is a matrix with its last row equal to $(0, ..., 0, 1)$.

As a result, $\mathscr{A}\mathrm{ff}_q^n$ is a complete set of coset representatives for $\mathrm{GL}(n, \mathbb{F}_q)/K_n$. We now define the following map between $\mathrm{GL}(n, \mathbb{F}_q)$ and $\mathscr{H}_q^n$,

$$\alpha \in \mathrm{GL}(n, \mathbb{F}_q) \mapsto \alpha \circ \Theta_n. \tag{8}$$

This map is clearly a bijection between $\mathrm{GL}(n, \mathbb{F}_q)/K_n$ and $\mathscr{H}_q^n$ and is well-defined since K_n is the stabilizer of Θ_n.

We are now ready to introduce the generalization of the pseudo-distance on H_q given in (1), but before we do so we give a preliminary definition and a lemma.

Definition 9. For $W \in \mathscr{H}_q^n$, define

$$\det(W) = \det(\alpha_W),$$

where α_W is as in (6).

LEMMA 10. *For* $\beta \in \mathrm{GL}(n, \mathbb{F}_q)$ *and* $W \in \mathscr{H}_q^n$,

$$\det(\beta \circ W) = \frac{\det(\beta)\det(W)}{\mathscr{N}((\beta W)_n)},$$

where $\det(\beta \circ W)$ *and* $\det(W)$ *are as in Definition* 9.

Proof. We have

$$\beta \circ W = (\beta \alpha_W M_{(\beta W)_n}^{-1})\, \Theta_n \tag{9}$$

by Lemma 6, where $(\beta \alpha_W M_{(\beta W)_n}^{-1})$ is in $\mathscr{A}\mathrm{ff}_q^n$ and α_W is as in (6). To see that $(\beta \alpha_W M_{(\beta W)_n}^{-1})$ is in $\mathscr{A}\mathrm{ff}_q^n$, note that $(\beta W)_n = (\beta \alpha_W \Theta_n)_n$, which is the element in $\mathbb{F}_q(\theta)$ whose coefficients with respect to $\{\theta^{n-1}, ..., \theta, 1\}$ are in the last row of $\beta \alpha_W$. Thus, $(\beta \alpha_W M_{(\beta W)_n}^{-1})$ has its last row equal to $(0, ..., 0, 1)$. Now, by (6),

$$\beta \circ W = \alpha_{\beta \circ W} \circ \Theta_n,$$

which together with (9) implies

$$\alpha_{\beta \circ W} \circ \Theta_n = (\beta \alpha_W M_{(\beta W)_n}^{-1})\, \Theta_n = (\beta \alpha_W M_{(\beta W)_n}^{-1}) \circ \Theta_n.$$

The last equality follows since $(\beta \alpha_W M_{(\beta W)_n}^{-1})$ is in $\mathscr{A}\mathrm{ff}_q^n$, which implies in this case that the action in (2) and ordinary matrix multiplication are the same.

Thus,

$$\alpha_{(\beta \circ W)} = (\beta \alpha_W M_{(\beta W)_n}^{-1}), \tag{10}$$

by the bijection between $\mathrm{GL}(n, \mathbb{F}_q)/K_n$ and $\mathscr{H}_q^n$ defined in (8).

Therefore,

$$\det(\beta \circ W) = \det(\beta \alpha_W M_{(\beta W)_n}^{-1})$$

by (10) and Definition 9, which implies that

$$\det(\beta \circ W) = \frac{\det(\beta)\det(W)}{\mathscr{N}((\beta W)_n)},$$

since $\det(M_{(\beta W)_n}) = \mathscr{N}((\beta W)_n)$ by the result in (4). ∎

The following definition gives an n-point invariant generalizing the pseudo-distance on H_q, as given in (1), to the finite upper half space $\mathscr{H}_q^n$.

DEFINITION 11. For $T_1, ..., T_n \in \mathcal{H}_q^n$, define

$$\mathcal{K}(T_1, ..., T_n) = \left(\frac{\mathcal{N}(\det[T_1, ..., T_n])}{\det(T_1) \cdots \det(T_n)}\right)^{((n \bmod 2)+1)}.$$

That is, when n is odd we want to square everything to make $\mathcal{K}$ symmetric under permutation of entries. $[T_1, ..., T_n]$ denotes the matrix whose ith column is T_i, $i = 1, ..., n$.

THEOREM 12. *For $\beta \in \mathrm{GL}(n, \mathbb{F}_q)$ and $T_1, ..., T_n \in \mathcal{H}_q^n$,*

$$\mathcal{K}(\beta \circ T_1, ..., \beta \circ T_n) = \mathcal{K}(T_1, ..., T_n).$$

That is, $\mathcal{K}$ is an n-point invariant of $\mathrm{GL}(n, \mathbb{F}_q)$ on $\mathcal{H}_q^n$.

Proof. We have

$$\mathcal{K}(\beta \circ T_1, ..., \beta \circ T_n) = \left(\frac{\mathcal{N}(\det[\beta \circ T_1, ..., \beta \circ T_n])}{\det(\beta \circ T_1) \cdots \det(\beta \circ T_n)}\right)^{((n \bmod 2)+1)}. \tag{11}$$

Now, by (2),

$$[\beta \circ T_1, ..., \beta \circ T_n] = \beta \left[T_1 \left(\frac{1}{(\beta T_1)_n}\right), ..., T_n \left(\frac{1}{(\beta T_n)_n}\right)\right],$$

which implies

$$\det[\beta \circ T_1, ..., \beta \circ T_n] = \frac{1}{(\beta T_1)_n \cdots (\beta T_n)_n} \det(\beta) \det([T_1, ..., T_n]). \tag{12}$$

Moreover, by Lemma 10,

$$\det(\beta \circ T_i) = \frac{\det(\beta) \det(T_i)}{\mathcal{N}((\beta T_i)_n)} \qquad \text{for} \quad i = 1, ..., n. \tag{13}$$

Thus, plugging (12) and (13) into Eq. (11), taking norms, and canceling terms, we obtain the desired result. ∎

3. HYPERGRAPHS OF $\mathcal{H}_q^n$

We give some background for hypergraphs; however, every hypergraph has an underlying ordinary graph and many of its properties depend on the properties of this graph, so we refer the reader to [18] and [3] for the graph theory background. For further hypergraph background, the reader is referred to [9] and [6].

A hypergraph X consists of a hypervertex set $V(X)$ and a hyperedge set $E(X)$ such that each element of $E(X)$ is a subset of $V(X)$. Our hypergraphs will be finite, undirected, and have no self-loops, so any given hyperedge contains distinct elements and is unordered. We will usually denote $V(X)$ and $E(X)$ just by V and E, respectively, when no ambiguity can arise.

For a hypergraph X, the finite Hilbert space on $V(X)$ is given by

$$L^2(V) = \{f\colon V \to \mathbb{R}\},$$

with the corresponding inner product

$$\langle f, g\rangle = \sum_{x\in V} f(x)\, g(x) \qquad \forall f, g \in L^2(V).$$

Then, the main operator associated with X is its adjacency operator $A(X)$, A for short. For $f \in L^2(V)$ we have

$$(Af)(x) = \sum_{\substack{e\in E\\ x\in e}} \sum_{\substack{y\in e\\ y\neq x}} f(y) \qquad \forall x \in V. \tag{14}$$

One also speaks of the *adjacency matrix* of X, which is the matrix obtained by expressing A with respect to the basis of $L^2(V)$,

$$\{\delta_x\colon x \in V\}, \qquad \text{where} \quad \delta_x(y) = \begin{cases} 1 & \text{if} \quad y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Namely, the adjacency matrix of X is parametrized by the hypervertices and we have for $v, v' \in V$,

$$A_{v, v'} = \#\{e \in E : v, v' \in e\}.$$

This is the definition given by Feng and Li in [9]. We will use A for both the adjacency operator and the adjacency matrix. Also, since our hypergraphs will be undirected and with no self-loops, the corresponding adjacency matrices are symmetric and with zeros on the diagonal.

Thus, when we speak of the spectrum of a hypergraph what we are referring to is the spectrum of the corresponding adjacency operator. Two hypergraphs (graphs) are isospectral if they have the same spectrum. Also, two hypergraphs (graphs) are isomorphic if there is a bijection between their hypervertices (vertices) which preserves adjacencies or connections. Now, for a hypergraph X, given a hypervertex $v \in V(X)$ and a hyperedge $e \in E(X)$, when we say that v is incident to e or that e is incident to v what we mean is that $v \in e$.

DEFINITION 13 (Feng and Li [9]). A hypergraph is said to be (d, r)-regular if every hypervertex is incident to d hyperedges and every hyperedge is incident to r hypervertices. Note that for an ordinary graph $r=2$.

PROPOSITION 14. *Given a finite (d, r)-regular hypergraph X, if λ is an eigenvalue of $A(X)$ then $|\lambda| \leqslant d(r-1)$. Moreover, $d(r-1)$ is an eigenvalue of $A(X)$.*

The *Laplacian* of a (d, r)-regular hypergraph X, which also operates on $L^2(V)$, is defined as

$$\Delta = d(r-1)\,\mathrm{I} - A,$$

where I is the identity operator and $A = A(X)$ is as in (14). Thus, the eigenvalues of Δ are strongly related to those of A; furthermore, they are all nonnegative since the eigenvalues of A are bounded in absolute value by $d(r-1)$, as seen in Proposition 14.

A hypergraph X is said to be connected if for any given pair of hypervertices, v and v' in $V(X)$, there is a path connecting v to v'.

PROPOSITION 15. *For a (d, r)-regular hypergraph X, the multiplicity of $d(r-1)$, as an eigenvalue of $A(X)$, is equal to the number of connected components of X.*

We have the following definition of a Ramanujan hypergraph given by Li and Solé [11], which is in the same spirit as that given by Lubotzky *et al.* [13] for an ordinary regular graph.

DEFINITION 16 (Li and Solé [11]). A finite connected (d, r)-regular hypergraph X is a Ramanujan hypergraph if every eigenvalue λ of $A(X)$, $|\lambda| \neq d(r-1)$, satisfies

$$|\lambda - (r-2)| \leqslant 2\sqrt{(d-1)(r-1)}.$$

The motivation for this definition is given by the following facts about regular hypergraphs.

THEOREM 17 (Feng and Li [9]). *Let X be a (d, r)-regular hypergraph with diameter $\geqslant 2l+2 \geqslant 4$, and set $q=(d-1)(r-1)=k-(r-1)$, where $k=d(r-1)$ is the degree of the underlying graph. Let $\lambda_2(X) \overset{\mathrm{def}}{=}$ the second largest eigenvalue of the adjacency matrix of X. Then,*

$$\lambda_2(X) > (r-2) + 2\sqrt{q} - \frac{2\sqrt{q}-1}{l}.$$

The diameter is just the maximum of the minimum of all lengths of paths between any two vertices.

We state a more general result after the following corollary.

COROLLARY 18. *Let* $\{X_m\}_{m=1}^{\infty}$ *be a family of connected* (d, r)*-regular hypergraphs with* $|V(X_m)| \to \infty$ *as* $m \to \infty$; *then*

$$\liminf_{m \to \infty} \lambda_2(X_m) \geqslant r - 2 + 2\sqrt{q},$$

where $\lambda_2(X_m)$ *is the second largest eigenvalue of* X_m.

THEOREM 19 (Feng and Li [9]). *Suppose that* X' *is a* k*-regular graph,* $k \in \mathbb{Z}^+$, *for which there exists a constant* c *such that for all pairs of adjacent vertices in* X' *there exist at least* c *vertices in* X' *adjacent to both. If the diameter,* D, *of* X' *satisfies* $D \geqslant 2l + 2 \geqslant 4$, *for some* $l \in \mathbb{Z}^+$, *then*

$$\lambda_2(X') > c + 2\sqrt{q} - \frac{2\sqrt{q} - 1}{l},$$

where $q = k - c - 1$ *and* $\lambda_2(X')$ *is the second largest eigenvalue of* X'.

For a proof see [9].

DEFINITION 20. Given a finite group G and a multiset S (which can have repeats), which is a subset of G, one has the *group graph* $\mathscr{G}(G, S)$ whose vertex set is G and we put a connection between $g \in G$ and gs for each $s \in S$.

We make this definition for an ordinary graph since the underlying graph of our hypergraphs will be a group graph. Note also that this is a generalization of what is known as a Cayley graph [3].

DEFINITION 21. For $a \in \mathbb{F}_q$, let $\mathscr{H}_q^n(a)$ be the hypergraph of $\mathscr{H}_q^n$ with vertex set $\mathscr{H}_q^n$ and a hyperedge between $T_1, ..., T_n \in \mathscr{H}_q^n$, pairwise distinct, if $\mathscr{K}(T_1, ..., T_n) = a^{((n \bmod 2) + 1)}$.

By the very definition of $\mathscr{H}_q^n(a)$, each of its hyperedges is incident to n hypervertices. Moreover, each hypervertex of $\mathscr{H}_q^n(a)$ will be incident to the same number of hyperedges. That is, $\mathscr{H}_q^n(a)$ will be (d, n)-regular as given in Definition 13, where d is the number of hyperedges incident to Θ_n, for example. This can be easily seen by the following argument: for $T_1, ..., T_{n-1} \in \mathscr{H}_q^n - \{\Theta_n\}$, pairwise distinct,

$$\{\Theta_n, T_1, ..., T_n\}$$

is a hyperedge of $\mathscr{H}_q^n(a)$ if and only if

$$\{\alpha_W \circ \Theta_n, \alpha_W \circ T_1, ..., \alpha_W \circ T_n\} \qquad \forall W \in \mathscr{H}_q^n$$

is also a hyperedge of $\mathscr{H}_q^n(a)$. This is by the invariance of $\mathscr{K}$ under $\mathrm{GL}(n, \mathbb{F}_q)$ and by the definition of $\mathscr{H}_q^n(a)$. Since $\alpha_W \circ \Theta_n = W$ and the action by $\mathrm{GL}(n, \mathbb{F}_q)$ on $\mathscr{H}_q^n$ is transitive, then each $W \in \mathscr{H}_q^n$ is incident to the same number of hyperedges that Θ_n is incident to in $\mathscr{H}_q^n(a)$.

The following theorem provides us with another way of viewing $\mathscr{H}_q^n(a)$.

THEOREM 22. *For* $a \in \mathbb{F}_q$, $\mathscr{H}_q^n(a)$ *is the group graph* $\mathscr{G}(\mathscr{A}\mathrm{ff}_q^n, S_n)$, *where* $\mathscr{H}_q^n(a)$ *is as in Definition* 21, $\mathscr{G}(\mathscr{A}\mathrm{ff}_q^n, S_n)$ *is as in Definition* 20, *and*

$$S_n = \bigcup^{\mathrm{M}}_{\substack{e \in E(\mathscr{H}_q^n(a)) \\ \Theta_n \in e}} \{\alpha_W : W \in e, W \neq \Theta_n\}.$$

Remark 23. $\bigcup^{\mathrm{M}}$ denotes a *multiunion*; that is, a union where all repeats are included.

Proof. Consider the hypergraph G obtained by matching each hypervertex of $\mathscr{H}_q^n(a)$ to its preimage in $\mathscr{A}\mathrm{ff}_q^n$ according to the bijection given in (8). Then, for $T_1, ..., T_{n-1} \in \mathscr{H}_q^n - \{\Theta_n\}$, pairwise distinct, we have the hyperedge of $\mathscr{H}_q^n(a)$

$$\{\Theta_n, T_1, ..., T_{n-1}\}$$

if and only if we also have the hyperedge of $\mathscr{H}_q^n$

$$\{\alpha_W \circ \Theta_n, \alpha_W \circ T_1, ..., \alpha_W \circ T_{n-1}\} \qquad \forall W \in \mathscr{H}_q^n,$$

which, by (6), is equal to

$$\{\alpha_W \mathrm{I}_n \Theta_n, \alpha_W \alpha_{T_1} \Theta_n, ..., \alpha_W \alpha_{T_{n-1}} \Theta_n\}. \tag{15}$$

Thus, for each hyperedge in $\mathscr{H}_q^n(a)$, we can map (15) to the hyperedge of G

$$\{\alpha_W, \alpha_W \alpha_{T_1}, ..., \alpha_W \alpha_{T_{n-1}}\} \qquad \forall W \in \mathscr{H}_q^n,$$

which is in $\mathcal{G}(\mathcal{A}\mathrm{ff}_q^n, S_n)$. That is, the connections of α_W are obtained by multiplying $\alpha_W \alpha_T$ for each

$$\alpha_T \in \bigcup^{\mathbf{M}}_{\substack{e \in E(\mathcal{H}_q^n(a)) \\ \Theta_n \in e}} \{\alpha_{T'} : T' \in e, T' \neq \Theta_n\}.$$

This is precisely the definition of $\mathcal{G}(\mathcal{A}\mathrm{ff}_q^n, S_n)$ and so $G = \mathcal{G}(\mathcal{A}\mathrm{ff}_q^n, S_n)$. ▮

We have examples of $\mathcal{H}_q^n(a)$ for $n = 3$ and $q = 2, 3$, and we have obtained Ramanujan hypergraphs. The case $n = 2$ and q odd, which corresponds to the finite upper half plane, has been worked out, see [18]. And for fixed $\delta \in \mathbb{F}_q^*$ and δ a nonsquare, all corresponding graphs turned out to be Ramanujan graphs for $a \neq 0, 4\delta$.

4. EXAMPLES

The hypergraphs to be considered will be checked for the Ramanujan property. That is, for $a \in \mathbb{F}_q$, we will check if the eigenvalues λ of $\mathcal{H}_q^n(a)$, a (d, r)-regular hypergraph with $\lambda \neq d(r-1)$, satisfy the bound

$$|\lambda - (r-2)| \leqslant 2\sqrt{(d-1)(r-1)}.$$

The eigenvalues given here were obtained by using Mathematica to obtain the adjacency matrices and Matlab to obtain the corresponding eigenvalues.

Case $q = 2$ and $n = 3$. The corresponding finite upper half space is

$$\mathcal{H}_2^3 = \left\{ \begin{pmatrix} W_1 \\ W_2 \\ 1 \end{pmatrix} : \begin{array}{l} W_1, W_2 \in \mathbb{F}_2(\theta) \\ W_1, W_2, 1 \text{ linearly independent over } \mathbb{F}_2 \end{array} \right\},$$

where θ is a root of the cubic irreducible polynomial

$$P(x) = x^3 - x - 1 \in \mathbb{F}_2[x]. \tag{16}$$

Here, the stabilizer of

$$\Theta_3 = \begin{pmatrix} \theta^2 \\ \theta \\ 1 \end{pmatrix}$$

can be easily obtained by matching θ^2, θ, and 1 to the matrices that represent them as linear transformations by multiplication on $\mathbb{F}_2(\theta)$ with respect to the basis $\{\theta^2, \theta, 1\}$. Thus, by (16),

$$\theta\theta^2 = \theta^3 \cong \theta + 1, \qquad \theta\theta = \theta^2, \qquad \text{and} \qquad \theta \cdot 1 = \theta,$$

which implies that

$$M_\theta = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_{\theta^2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad M_1 = \mathrm{I}_3,$$

where M_X, for $X \in \mathbb{F}_q(\theta)$, is as in (3). Then, by the isomorphism given in (3), for $(g, h, k) \in \mathbb{F}_2^3$ we have

$$M_{g\theta^2 + h\theta + k} = \begin{pmatrix} g+k & g+h & h \\ h & g+k & g \\ g & h & k \end{pmatrix}.$$

Thus, since K_3 consists of M_X for $X \in \mathbb{F}_2(\theta)^*$,

$$K_3 = \left\{ \begin{pmatrix} g+k & g+h & h \\ h & g+k & g \\ g & h & k \end{pmatrix} \in \mathrm{GL}(3, \mathbb{F}_2) \right\}.$$

Hypergraphs

For $a \in \mathbb{F}_2$ connect $T_1, T_2, T_3 \in \mathscr{H}_2^3$, pairwise distinct, if $\mathscr{K}(T_1, T_2, T_3) = a^2$.

Case $a^2 = 0$: $\mathscr{H}_2^3(0)$ is (d, r)-regular, where $d = 21$ and $r = 3$. The distinct eigenvalues of the corresponding adjacency matrix are $\{42, -6, 0\}$ and the Ramanujan bound is given by

$$2\sqrt{(d-1)(r-1)} = 12.65.$$

Thus, we have a Ramanujan hypergraph as all the eigenvalues λ, with $|\lambda| \neq 42$, satisfy the inequality

$$|\lambda - 1| \leqslant 12.65.$$

See Fig. 1 for a picture of $\mathscr{H}_2^3(0)$.

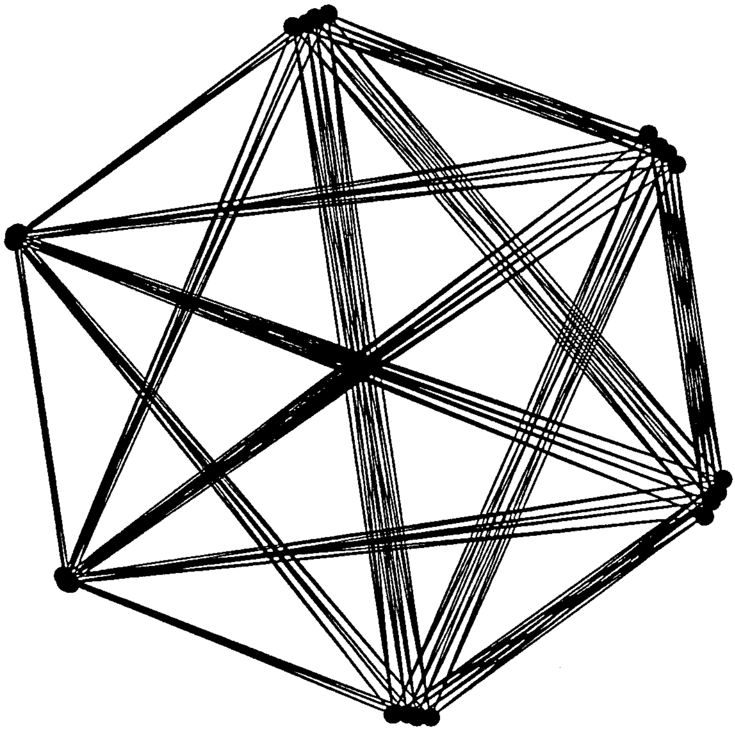

FIG. 1. $\mathscr{H}_2^3(0)$, the finite upper half space hypergraph for $n=3$, $p=2$, and $a=0$. This graph was created with Mathematica using the Combinatorica package.

Case $a^2=1$: *$\mathscr{H}_2^3(1)$ is (d, r)-regular, where $d=232$ and $r=3$.* The distinct eigenvalues of $\mathscr{H}_2^3(1)$ are $\{464, -22, -16\}$ and the Ramanujan bound is

$$2\sqrt{(d-1)(r-1)}=42.99.$$

All the nontrivial eigenvalues λ satisfy the inequality

$$|\lambda-1| \leqslant 42.99.$$

Thus, $\mathscr{H}_2^3(1)$ is a Ramanujan hypergraph.

Case $q=3$ and $n=3$. The corresponding finite upper half space is

$$\mathscr{H}_3^3=\left\{\begin{pmatrix} W_1 \\ W_2 \\ 1 \end{pmatrix} : \begin{array}{l} W_1, W_2 \in \mathbb{F}_3(\theta) \\ W_1, W_2, 1 \text{ linearly independent over } \mathbb{F}_3 \end{array}\right\},$$

where θ is a root of the cubic irreducible polynomial

$$P(x) = x^3 - x - 2. \tag{17}$$

We now obtain the stabilizer of Θ_3. We have, by (17),

$$\theta\theta^2 = \theta^3 \cong \theta + 2, \qquad \theta\theta = \theta^2, \qquad \text{and} \qquad \theta \cdot 1 = \theta.$$

This implies that

$$M_\theta = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_{\theta^2} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad M_1 = \mathrm{I}_3,$$

where M_X, for $X \in \mathbb{F}_3(\theta)$, is as in (3). Thus, by the isomorphism given in (3), for g, h, $k \in \mathbb{F}_3$

$$M_{g\theta^2 + h\theta + k} = \begin{pmatrix} g+k & 2g+h & 2h \\ h & g+k & 2g \\ g & h & k \end{pmatrix}.$$

Looking at the nonzero elements of $\mathbb{F}_3(\theta)$

$$K_3 = \left\{ \begin{pmatrix} g+k & 2g+h & 2h \\ h & g+k & 2g \\ g & h & k \end{pmatrix} \in \mathrm{GL}(3, \mathbb{F}_3) \right\}.$$

Corresponding Hypergraphs

Case $a^2 = 0$: $\mathscr{H}_3^3(0)$ *is* (d, r)*-regular, where* $d = 3133$ *and* $r = 3$. The distinct eigenvalues of $\mathscr{H}_3^3(0)$ are $\{6266, -52, 26, 2\}$ and the Ramanujan bound is

$$2\sqrt{(d-1)(r-1)} = 158.29.$$

Thus, $\mathscr{H}_3^3(0)$ is a Ramanujan hypergraph since all its nontrivial eigenvalues λ satisfy

$$|\lambda - 1| \leqslant 158.29.$$

Case $a^2 = 1$: $\mathscr{H}_3^3(1)$ *is* (d, r)*-regular, where* $d = 89532$ *and* $r = 3$. The distinct eigenvalues of $\mathscr{H}_3^3(1)$ are

$$\{179064, -456, -432, -378\}.$$

The Ramanujan bound is $2\sqrt{(d-1)(r-1)} = 846.31$ and is satisfied by all the nontrivial eigenvalues. That is,

$$|\lambda - 1| \leqslant 846.31 \qquad \forall \lambda \ni |\lambda| \neq 179064.$$

Thus, we have a Ramanujan hypergraph.

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