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Global existence of solutions for fuzzy second-order differential equations under generalized H-differentiability[☆]

Dongkai Zhang[a], Wenli Feng[a], Yongqiang Zhao[a], Jiqing Qiu[b,*]

[a] Department of Mathematics, Shijiazhuang University, He Bei, 050035, PR China

[b] College of Science, Hebei University of Science and Technology, He Bei, 050018, PR China

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ABSTRACT

In this paper, we study the global existence of solutions for second-order fuzzy differential equations with initial conditions under generalized H-differentiability. Second derivative of the H-difference of two functions under generalized H-differentiability is obtained. Two theorems which assure global existence of solutions for second-order fuzzy differential equations are given and proved. Some examples are given to illustrate these results.

1. Introduction

Recently, the study of first-order and higher-order fuzzy differential equations (FDEs) has gained much attention. The most popular method is using the Hukuhara differentiability, or the Seikkala derivative for fuzzy number valued functions. Gnana Bhaskar et al. revisited the original formulation and provided examples to show a variety of properties of solutions to the FDE in [1]. The local existence and uniqueness theorems were given in [2]. Song et al. pointed out a variety of results which assured global existence of solutions to fuzzy differential equations in [3]. Wu and Song obtained the existence theorems under compactness-type conditions in [4]. Song and Wu studied the existence and uniqueness of fuzzy differential equations under dissipative conditions based on the existence theorem of approximate solutions to the Cauchy problem in [5]. Wu and Gong defined and discussed the (FH) integral for fuzzy number valued functions in [6]. Gong and Shao studied the global existence, uniqueness and the continuous dependence of a solution on fuzzy differential equations under the dissipative-type conditions using the properties of a differential and integral calculus for fuzzy set valued mappings and completeness of metric space of fuzzy numbers in [7]. Park et al. proved the existence of solutions to fuzzy integral equations in Banach spaces on $[t_0, t_0 + d]$ in [8]. Papaschinopoulos et al. studied the existence, the uniqueness and the asymptotic behavior of the solutions to the fuzzy differential equation in [9]. Lupulescu proved several theorems stating the existence, uniqueness and boundedness of solutions to fuzzy differential equations with the concept of inner product on the fuzzy space in [10]. The existence, the uniqueness and the asymptotic behavior of the solutions to fuzzy differential equation were studied in [9]. In [11], Park et al. proved the existence and uniqueness theorem of a solution to fuzzy Volterra integral equation on $[t_0, t_0 + a]$. Georgiou et al. considered nth-order fuzzy differential equations with initial value conditions and proved the local existence and uniqueness of solution for nonlinearities satisfying a Lipschitz condition on $[t_0, T]$ in [12]. Bede et al. introduced generalized concepts of differentiability and studied the existence of the solutions to fuzzy differential

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* Corresponding author.
E-mail addresses: zdkmailhot@126.com (D. Zhang), qiujiqing@263.net (J. Qiu).

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equations involving generalized differentiability in [13,14]. Rosana Rodríguez-López developed the monotone iterative technique to approximate the extremal solutions for the initial value problem relative to a fuzzy differential equation in a fuzzy functional interval [15]. Based on the results in [12], Allahviranloo et al. proved the local existence and uniqueness of solutions for second-order fuzzy differential equations with initial conditions under generalized H-differentiability on $[t_0, T]$ [16].

In this paper, the global existence of solutions for second-order fuzzy differential equations with initial conditions under generalized H-differentiability is studied. In Section 2, the preliminaries for fuzzy number and the FDEs are introduced. In Section 3, second derivative of the H-difference of two functions is discussed. Finally, two theorems for global existence of solutions, which extend the results in [16,12], are given and proved on $[t_0, \infty]$. We also give some examples to illustrate these results.

2. Preliminaries

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, which satisfy the following requirement.

(i) $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$.
(ii) $\overline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$.
(iii) $\underline{u}(r) \le \overline{u}(r)$, $0 < r \le 1$ (see e.g. [17]).

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of subsets of the real axis $u : \mathbb{R} \longrightarrow [0, 1]$, satisfying the following properties:

(i) u is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$,
(ii) u is a convex fuzzy set (i.e. $u(tx + (1 - t)y) \ge \min\{u(x), u(y)\}$, $\forall t \in [0, 1]$, $x, y \in \mathbb{R}$,
(iii) u is upper semicontinuous on $\mathbb{R}$,
(iv) $\overline{\{x \in \mathbb{R};\ u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A,

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers (see e.g. [18]).

The following properties are well known (see e.g. [18,6]).

Let $\mathbb{R}_{\mathcal{F}}$ be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level intervals. This means that if $v \in \mathbb{R}_{\mathcal{F}}$ then the r-level set

$$[v]^r = \{s | v(s) \ge r\}, \quad 0 < r \le 1$$

is a closed bounded interval which is denoted by

$$[v]^r = [\underline{v}(r), \overline{v}(r)].$$

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and $k \ge 0$, addition and multiplication by k are defined as follows:

$$\underline{(u + v)}(r) = \underline{u}(r) + \underline{v}(r),$$
$$\overline{(u + v)}(r) = \overline{u}(r) + \overline{v}(r),$$
$$(\underline{ku})(r) = k\underline{u}(r),\ (\overline{ku})(r) = k\overline{u}(r).$$

Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}_+ \bigcup \{0\}$, $D(u, v) = \sup_{\gamma \in [0,1]} \max\left\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\right\}$, be the Hausdorff distance between fuzzy numbers, where $[u]^r = [\underline{u}(r), \overline{u}(r)]$, $[v]^r = [\underline{v}(r), \overline{v}(r)]$. The following properties are well known:

$$D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}},$$
$$D(k \cdot u, k \cdot v) = |k| D(u, v), \quad \forall k \in \mathbb{R}, u, v \in \mathbb{R}_{\mathcal{F}},$$

$D(u + v, w + e) \le D(u, w) + D(v, e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$ and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

The following results and concepts are also known.

Definition 2.1 (*[19]*). Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the Hukuhara difference of x and y and it is denoted by $x \Theta y$.

Definition 2.2 (*[16]*). We define the nth-order differential of f as follows: $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable of the nth order at x_0. If there exists an element $f^{(s)}(x_0) \in E$, $\forall s = 1, \ldots, n$, such that $\forall s = 1, \ldots, n$

(i) for all $h > 0$ sufficiently small, $\exists f^{(s-1)}(x_0 + h) \Theta f^{(s-1)}(x_0)$, $f^{(s-1)}(x_0) \Theta f^{(s-1)}(x_0 - h)$ and the following limits hold (in the metric d_∞):

$$\lim_{h \to 0} \frac{f^{(s-1)}(x_0 + h) \Theta f^{(s-1)}(x_0)}{h} = \lim_{h \to 0} \frac{f^{(s-1)}(x_0) \Theta f^{(s-1)}(x_0 - h)}{h} = f^{(s)}(x_0),$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f^{(s-1)}(x_0)\Theta f^{(s-1)}(x_0+h),\ f^{(s-1)}(x_0-h)\Theta f^{(s-1)}(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0}\frac{f^{(s-1)}(x_0)\Theta f^{(s-1)}(x_0+h)}{-h}=\lim_{h\to 0}\frac{f^{(s-1)}(x_0-h)\Theta f^{(s-1)}(x_0)}{-h}=f^{(s)}(x_0),$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f^{(s-1)}(x_0+h)\Theta f^{(s-1)}(x_0),\ f^{(s-1)}(x_0-h)\Theta f^{(s-1)}(x_0)$ and the following limits hold (in the metric d_∞):

$$\lim_{h\to 0}\frac{f^{(s-1)}(x_0+h)\Theta f^{(s-1)}(x_0)}{h}=\lim_{h\to 0}\frac{f^{(s-1)}(x_0-h)\Theta f^{(s-1)}(x_0)}{h}=f^{(s)}(x_0),$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f^{(s-1)}(x_0)\Theta f^{(s-1)}(x_0+h), f^{(s-1)}(x_0)\Theta f^{(s-1)}(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0}\frac{f^{(s-1)}(x_0)\Theta f^{(s-1)}(x_0+h)}{-h}=\lim_{h\to 0}\frac{f^{(s-1)}(x_0)\Theta f^{(s-1)}(x_0-h)}{-h}=f^{(s)}(x_0).$$

Lemma 2.1 (*[16]*). *For arbitrary $(u,v)\in\mathbb{R}_{\mathscr{F}}\times\mathbb{R}_{\mathscr{F}}$, we have*

$$D(u\Theta w, u\Theta v)=D(w,v),\quad \forall u,v,w\in\mathbb{R}_{\mathscr{F}}.$$

Lemma 2.2 (*[15]*). *For arbitrary $(u,v)\in\mathbb{R}_{\mathscr{F}}\times\mathbb{R}_{\mathscr{F}}$ we have*

$$D(u\Theta v, w\Theta e)\le D(u,w)+D(v,e),\quad \forall u,v,w,e\in\mathbb{R}_{\mathscr{F}}.$$

Lemma 2.3 (*[14]*). *Let $F:(a,b)\to\mathbb{R}_{\mathscr{F}}$ be Hukuhara differentiable and denote $[F(t)]^r=[F_-^r(t),F_+^r(t)]$. Then the boundary functions $F_-^r(t)$ and $F_+^r(t)$ are differentiable and $[F'(t)]^r=[(F_-^r(t))',(F_+^r(t))'],\ r\in[0,1]$.*

The space of continuous function $x:[t_0,T]\to\mathbb{R}_{\mathscr{F}}$ by $C([t_0,T],\mathbb{R}_{\mathscr{F}})$ is denoted in [12]. $C([t_0,T],\mathbb{R}_{\mathscr{F}})$ is a complete metric space with the distance

$$H(x,y)=\sup_{t\in[t_0,T]}\{D(x(t),y(t))\mathrm{e}^{-\rho t}\}$$

where $\rho\in\mathbb{R}$. Also, by $C^1([t_0,T],\mathbb{R}_{\mathscr{F}})$, we denote the set of continuous functions $x:[t_0,T]\to\mathbb{R}_{\mathscr{F}}$ whose derivative $x':[t_0,T]\to\mathbb{R}_{\mathscr{F}}$ exists as a continuous function. For $x,y\in C^1([t_0,T],\mathbb{R}_{\mathscr{F}})$, we consider the following distance:

$$H_1=H(x,y)+H(x',y').$$

Lemma 2.4 (*[12]*). *$(C^1(I,E),H_1)$ is a complete metric space.*

Theorem 2.1 (*[16]*). *Let $t_0\in[a,b]$, and assume that $f:[a,b]\times\mathbb{R}_{\mathscr{F}}\to\mathbb{R}_{\mathscr{F}}$ is continuous. A mapping $x:[a,b]\to\mathbb{R}_{\mathscr{F}}$ is a solution to the initial value problem $x''=f(t,x,x'),x(t_0)=k_1,x'(t_0)=k_2$, if and only if x and x' are continuous and satisfy one of the following conditions:*

$$\text{(a)}\quad x(t)=k_2(t-t_0)+\int_{t_0}^{t}\left(\int_{t_0}^{t}f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}s+k_1$$

where x and x' are (i)*-differentials, or*

$$\text{(b)}\quad x(t)=k_1\Theta(-1)\left(k_2(t-t_0)\Theta(-1)\int_{t_0}^{t}\left(\int_{t_0}^{t}f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}s\right)$$

where x and x' are (ii)*-differentials.*

$$\text{(c)}\quad x(t)=\Theta(-1)\left(k_2(t-t_0)+\int_{t_0}^{t}\left(\int_{t_0}^{t}f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}s\right)+k_1$$

where x' is the (i)*-differential and x'' is* (ii)*-differential, or*

$$\text{(d)}\quad x(t)=k_2(t-t_0)\Theta(-1)\int_{t_0}^{t}\left(\int_{t_0}^{t}f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}s+k_1$$

where x' is the (ii)*-differential and x'' is* (i)*-differential.*

Theorem 2.2 ([16]). *Let $f : [t_0, T] \times \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to E^n$ be continuous, and suppose that there exist $M_1, M_2 > 0$ such that*

$$d(f(t, x_1, x_2), f(t, y_1, y_2)) \le M_1 d(x_1, x_2) + M_2 d(y_1, y_2)$$

for all $t \in [t_0, T], x_1, x_2, y_1, y_2 \in \mathbb{R}_{\mathscr{F}}$. Then the initial value problem $x'' = f(t, x, x'), x(t_0) = k_1, x'(t_0) = k_2$ has a unique solution on $[t_0, T]$ for each case.

Theorem 2.3 ([14]). *Let $f, g : (a, b) \to \mathbb{R}_{\mathscr{F}}$ be strongly generalized differentiable such that f is (i)-differentiable and g is (ii)-differentiable or f is (ii)-differentiable and g is (i)-differentiable on an interval (α, β). If the H-difference $f(x) \ominus g(x)$ exists for $x \in (\alpha, \beta)$ then $f(x) \ominus g(x)$ is strongly generalized differentiable and*

$$(f(x) \ominus g(x))' = f'(x) + (-1)g'(x)$$

for all $x \in (\alpha, \beta)$.

3. Main result

Let us consider the second-order fuzzy initial value problem (FIVP)

$$\begin{cases} x'' = f(t, x, x') \\ x(t_0) = k_1 \\ x'(t_0) = k_2 \end{cases} \tag{1}$$

where we assume that $f \in C(J \times \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}}, \mathbb{R}_{\mathscr{F}}), J = [t_0, \infty)$.

Theorem 3.1. *Let $f, g : (a, b) \to \mathbb{R}_{\mathscr{F}}$ be strongly generalized differentiable such that*

(a) *If f, f' are (i)-differentiable and g, g' are (ii)-differentiable on an interval (α, β). If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist for $x \in (\alpha, \beta)$ then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and*

$$(f(x) \ominus g(x))'' = f''(x) \ominus g''(x)$$

for all $x \in (\alpha, \beta)$.

(b) *If f, g' are (i)-differentiable and f', g are (ii)-differentiable on an interval (α, β). If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist for $x \in (\alpha, \beta)$ then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and*

$$(f(x) \ominus g(x))'' = g''(x) \ominus (-1)f''(x)$$

for all $x \in (\alpha, \beta)$.

(c) *If f is (i)-differentiable and f', g, g' are (ii)-differentiable on an interval (α, β). If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist for $x \in (\alpha, \beta)$ then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and*

$$(f(x) \ominus g(x))'' = (-1)f''(x) + g''(x)$$

for all $x \in (\alpha, \beta)$.

(d) *If f, f', g' are (i)-differentiable and g is (ii)-differentiable on an interval (α, β). If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist for $x \in (\alpha, \beta)$ then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and*

$$(f(x) \ominus g(x))'' = f''(x) + (-1)g''(x)$$

for all $x \in (\alpha, \beta)$.

Proof. From Theorem 2.3 we have

$$(f(x) \ominus g(x))' = f'(x) + (-1)g'(x).$$

(a) Since f' is (i)-differentiable it follows that $f'(x+h) \ominus f'(x)$ exists i.e. there exists $u_1'(x, h)$ such that

$$f'(x+h) = f'(x) + u_1'(x, h).$$

Analogously since g' is (ii)-differentiable there exists $v_1'(x, h)$ such that

$$g'(x) = g'(x+h) + v_1'(x, h)$$

and we get

$$f'(x+h) + (-1)g'(x) = f'(x) + u_1'(x, h) + (-1)g'(x+h) + (-1)v_1'(x, h).$$

Since the H-difference $(f'(x+h) + (-1)g'(x+h)) \ominus (f'(x) + (-1)g'(x))$ exists for $h > 0$ such that $x + h \in (\alpha, \beta)$, we get

$$(f'(x+h) + (-1)g'(x+h)) \ominus (f'(x) + (-1)g'(x)) = u_1'(x, h) \ominus (-1)v_1'(x, h).$$

By similar reasoning we get that there exist $u_2'(x,h)$ and $v_2'(x,h)$ such that

$$f'(x) = f'(x-h) + u_2'(x,h),$$
$$g'(x-h) = g'(x) + v_2'(x,h) \quad \text{and so}$$
$$(f'(x) + (-1)g'(x)) \ominus (f'(x-h) + (-1)g'(x-h)) = u_2'(x,h) \ominus (-1)v_2'(x,h).$$

We observe that

$$\lim_{h\to 0} \frac{u_1'(x,h)}{h} = \lim_{h\to 0} \frac{u_2'(x,h)}{h} = f''(x)$$

and

$$\lim_{h\to 0} \frac{v_1'(x,h)}{h} = \lim_{h\to 0} \frac{v_2'(x,h)}{h} = (-1)g''(x).$$

So we have

$$(f(x) \ominus g(x))'' = f''(x) \ominus g''(x)$$

and

$(f(x) \ominus g(x))'$ is (i)-differentiable.

(b) The proof of (b) is similar to (a) and $(f(x) \ominus g(x))'$ is (i)-differentiable.

(c) Since f' is (ii)-differentiable it follows that $f'(x) \ominus f'(x+h)$ exists i.e. there exists $u_1'(x,h)$ such that

$$f'(x) = f'(x+h) + u_1'(x,h).$$

Analogously since g' is (ii)-differentiable there exists $v_1'(x,h)$ such that

$$g'(x) = g'(x+h) + v_1'(x,h)$$

and we get

$$f'(x) + (-1)g'(x) = f'(x+h) + u_1'(x,h) + (-1)g'(x+h) + (-1)v_1'(x,h).$$

Since the H-difference $(f'(x) + (-1)g'(x)) \ominus (f'(x+h) + (-1)g'(x+h))$ exists for $h > 0$ such that $x + h \in (\alpha, \beta)$, we get

$$(f'(x) + (-1)g'(x)) \ominus (f'(x+h) + (-1)g'(x+h)) = u_1'(x,h) + (-1)v_1'(x,h).$$

By similar reasoning we get that there exist $u_2'(x,h)$ and $v_2'(x,h)$ such that

$$f'(x-h) = f'(x) + u_2'(x,h),$$
$$g'(x-h) = g'(x) + v_2'(x,h) \quad \text{and so}$$
$$(f'(x-h) + (-1)g'(x-h)) \ominus (f'(x) + (-1)g'(x)) = u_2'(x,h) \ominus (-1)v_2'(x,h).$$

We observe that

$$\lim_{h\to 0} \frac{u_1'(x,h)}{-h} = \lim_{h\to 0} \frac{u_2'(x,h)}{-h} = (-1)f''(x)$$

and

$$\lim_{h\to 0} \frac{v_1'(x,h)}{-h} = \lim_{h\to 0} \frac{v_2'(x,h)}{-h} = g''(x).$$

So we have

$$(f(x) \ominus g(x))'' = (-1)f''(x) + g''(x)$$

and

$(f(x) \ominus g(x))'$ is (ii)-differentiable.

(d) The proof of (d) is similar to (c), but $(f(x) \ominus g(x))'$ is (i)-differentiable. $\square$

Example 3.1. Let $f, g : \left(\frac{\pi}{6}, \frac{\pi}{4}\right) \to \mathbb{R}_{\mathscr{F}}$ be strongly generalized differentiable and $[f(x)]^r = [-3 + r, -1 - r]x^2$, $[g(x)]^r = [3 + r, 5 - r]\sin x$. Then we have

(a) If f, f' are (i)-differentiable and g, g' are (ii)-differentiable. If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist, then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and

$$(f(x) \ominus g(x))'' = [-6 + 2r, -2 - 2r] \ominus (-1)[3 + r, 5 - r]\sin x, \quad \text{for all } x \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right).$$

(b) If f, g' are (i)-differentiable and f', g are (ii)-differentiable. If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist, then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and

$$(f(x) \ominus g(x))'' = (-1)[3+r, 5-r]\sin x \ominus (-1)[-6+2r, -2-2r], \quad \text{for all } x \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right).$$

(c) If f is (i)-differentiable and f', g, g' are (ii)-differentiable. If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist, then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and

$$(f(x) \ominus g(x))'' = (-1)[-6+2r, -2-2r] + (-1)[3+r, 5-r]\sin x, \quad \text{for all } x \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right).$$

(d) If f, f', g' are (i)-differentiable and g is (ii)-differentiable. If the H-difference $f(x) \ominus g(x)$ and $f'(x) \ominus g'(x)$ exist, then $f(x) \ominus g(x)$ is second-order strongly generalized differentiable and

$$(f(x) \ominus g(x))'' = [-6+2r, -2-2r] + [3+r, 5-r]\sin x, \quad \text{for all } x \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right).$$

In the following we only show that the result (a) is correct, since the procedure is similar for all four cases.
In fact, we show that $f(x)$ is (i)-differentiable.
Let $f(x+h) = f(x) + u(x), f(x) = f(x-h) + v(x)$, then $[f(x+h)]^r = [f(x)]^r + [u(x)]^r, [f(x)]^r = [f(x-h)]^r + [v(x)]^r$. So

$$\begin{cases} \underline{u}(r) = (-3+r)(x+h)^2 - (-3+r)x^2 = (-3+r)(2hx+h^2) \\ \overline{u}(r) = (-1-r)(x+h)^2 - (-1-r)x^2 = (-1-r)(2hx+h^2) \end{cases}$$

and

$$\begin{cases} \underline{v}(r) = (-3+r)x^2 - (-3+r)(x-h)^2 = (-3+r)(2hx-h^2) \\ \overline{v}(r) = (-1-r)x^2 - (-1-r)(x-h)^2 = (-1-r)(2hx-h^2) \end{cases}$$

that is

$$\begin{cases} \lim\limits_{h\to 0} \dfrac{\underline{u}}{h} = 2x(-3+r) \\ \lim\limits_{h\to 0} \dfrac{\overline{u}}{h} = 2x(-1-r) \end{cases}$$
$$\begin{cases} \lim\limits_{h\to 0} \dfrac{\underline{v}}{h} = 2x(-3+r) \\ \lim\limits_{h\to 0} \dfrac{\overline{v}}{h} = 2x(-1-r). \end{cases}$$

So $\lim_{h\to 0} \frac{f(x+h) \ominus f(x)}{h} = \lim_{h\to 0} \frac{f(x) \ominus f(x-h)}{h} = 2[-3+r, -1-r]x$,
Secondly, with the same method, we have $f'(x)$ is (i)-differentiable and $g(x), g'(x)$ is (ii)-differentiable.
From Theorem 3.1(i), we have $(f \ominus g)'' = [-6+2r, -2-2r] \ominus (-1)[3+r, 5-r]\sin x$. □

Theorem 3.2. *Assume that*

(i) *$f(t, x, x')$ is locally Lipschitzian in x, x' for $(t, x, x') \in J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$;*
(ii) *$g \in C[J \times [0, \infty) \times [0, \infty), [0, \infty)]$, $g(t, u, v)$ is nondecreasing in $u, v \geq 0$ for each $t \in J$, and the maximal solution $r(t, t_0, u_0, v_0)$ of the scalar initial value problem*

$$u'' = g(t, u, u'), \qquad u(t_0) = u_0, \qquad u'(t_0) = u_1 \tag{2}$$

exists throughout J.
(iii) *$D(f(t, x, x'), \hat{0}) \leq g(t, D(x, \hat{0}), D(x', \hat{0}))$, $\forall (t, x, x') \in J \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$;*
(iv) *$D(x(t, t_0, x_0, x_0'), \widehat{0}) \leq r(t, t_0, x_0, x_0')$, $D(x_0, \hat{0}) \leq u_0$ and $D(x_0', \hat{0}) \leq u_1$.*

Then the largest interval of existence of any solution $x(t, t_0, u_0, v_0)$ of (1) with $D(x, \hat{0}) \leq u_0$ is J. In addition, If $r(t, t_0, u_0, u_0')$ is bounded on J, then $\lim_{t\to\infty} x(t, t_0, x_0, x_0')$ exists in $(\mathbb{R}_{\mathcal{F}}, D)$.

Proof. By hypothesis (i), there exists a $T > t_0$ such that the uniqueness solution of problem (1) exists on $[t_0, T]$. Let

$$V = \{x(t) | x(t) \text{is defined on} [t_0, \beta_x) \text{and is the solution to (1)}\}.$$

Then $V \neq \phi$. Taking $\beta = \sup\{\beta_x | x(t) \in V\}$, clearly, there exists a uniqueness solution of (1) which is defined on $[t_0, \beta)$ with $D(x_0, \hat{0}) \leq u_0$ and $D(x_0', \hat{0}) \leq u_1$. If $\beta = +\infty$, obviously, the theorem is proved. Hence we suppose $\beta < +\infty$ and define

$$m(t) = D(x(t, t_0, x_0, x_0'), \hat{0}), \quad t_0 \leq t < \beta. \tag{3}$$

Then we have

$$\begin{aligned}
D_+m'(t) &= \liminf_{h\to+0} \frac{D(x'(t+h,t_0,x_0,x_0'),\hat{0}) - D(x'(t,t_0,x_0,x_0'))}{h} \\
&\le \liminf_{h\to+0} \frac{D(x'(t+h,t_0,x_0,x_0'),x'(t,t_0,x_0,x_0'))}{h} \\
&= \liminf_{h\to+0} D\left(\frac{x'(t+h,t_0,x_0,x_0') - x'(t,t_0,x_0,x_0')}{h},\hat{0}\right) \\
&= D(x''(t,t_0,x_0,x_0'),\hat{0}) \\
&= D(f(t,x(t,t_0,x_0,x_0'),x'(t,t_0,x_0,x_0')),\hat{0}) \\
&\le g(t,m(t),m'(t)), \quad t_0 \le t < \beta
\end{aligned}$$

and $m(t_0) = D(x_0,\hat{0}) \le u_0$, further, by (iv), it follows that

$$m(t) \le r(t,t_0,u_0,v_0), \quad t_0 \le t < \beta.$$

Next we deduce that $\lim_{t\to\beta-0} x(t,t_0,x_0,x_0')$ exists in $(\mathbb{R}_{\mathcal{F}},D)$. In fact, for any t_1,t_2 such that $t_0 \le t_1 < t_2 < \beta$, we have

$$\begin{aligned}
D(x(t_1,t_0,x_0,x_0'),x(t_2,t_0,x_0,x_0')) &= D\Bigg(k_1\Theta(-1)\left(k_2(t_1-t_0)\Theta(-1)\int_{t_0}^{t_1}\left(\int_{t_0}^{z} f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}z\right), \\
&\qquad k_1\Theta(-1)\left(k_2(t_2-t_0)\Theta(-1)\int_{t_0}^{t_2}\left(\int_{t_0}^{z} f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}z\right)\Bigg) \\
&= D\Bigg((-1)\left(k_2(t_1-t_0)\Theta(-1)\int_{t_0}^{t_1}\left(\int_{t_0}^{z} f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}z\right), \\
&\qquad (-1)\left(k_2(t_2-t_0)\Theta(-1)\int_{t_0}^{t_2}\left(\int_{t_0}^{z} f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}z\right)\Bigg) \\
&\le D\left(\int_{t_0}^{t_1}\left(\int_{t_0}^{z} f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}z, \int_{t_0}^{t_2}\left(\int_{t_0}^{z} f(s,x(s),x'(s))\mathrm{d}s\right)\mathrm{d}z\right) \\
&\le \int_{t_1}^{t_2}\int_{t_0}^{z} D(f(s,x(s),x'(s)),\hat{0})\mathrm{d}s\mathrm{d}z \\
&\le r(t_2) - r(t_1) - k_2|t_2-t_1|. \qquad (4)
\end{aligned}$$

Since $\lim_{t\to\beta-0} r(t)$ exists and is finite, taking the limits as $t_1,t_2 \to \beta-0$, and using the completeness of $(\mathbb{R}_{\mathcal{F}},D)$, it follows from (4) that $\lim_{t\to\beta-0} x(t)$ exists in $(\mathbb{R}_{\mathcal{F}},D)$. Now we define $x(\beta) = \lim_{t\to\beta-0} x(t)$ and consider the initial value problem

$$x'' = f(t,x,x'), \qquad x(\beta) = \lim_{t\to\beta-0} x(t).$$

By assumption (i) and Theorem 2.2, It follows that $x(t)$ can be extended beyond β, this is contradicting with our assumption.

In addition, since $r(t,t_0,u_0,u_0')$ is bounded and nondecreasing on J, it follows that $\lim_{t\to\infty} x(t,t_0,x_0,x_0')$ exists and is finite. This and (3) and (4) and Lemma 2.2 yield the last part of the theorem. □

Example 3.2. Consider the second-order fuzzy differential equations

$$\begin{cases} x'' = a(t)x' + b(t)x \\ x(t_0) = k_1 \\ x'(t_0) = k_2 \end{cases} \qquad (5)$$

where we assume that $a(t),b(t) : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, then the solutions of (5) are on $[t_0,+\infty)$.

In fact, $a(t)x' + b(t)x$ is locally Lipschitzian. If we let $g(t,u,u') = a(t)u' + b(t)u$ then $u(t) \equiv 0$ is only solution of

$$\begin{cases} u'' = g(t,u,u') \\ u(t_0) = 0 \\ u'(t_0) = 0 \end{cases}$$

on $[t_0, \infty)$. Moreover

$$D(a(t)x' + b(t)x, \hat{0}) \leq a(t)D(x', \hat{0}) + b(t)D(x, \hat{0}) = g(t, D(x, \hat{0}), D(x', \hat{0})).$$

Therefore, the solutions of problem (5) are on $[t_0, \infty)$.

Theorem 3.3. *Assume that*

(i) $f \in C[J \times \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}}]$, *$f$ is bounded on bounded sets and for any point in $J \times \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}}$, there exists a local solution for problem* (1)*;*

(ii) $V \in C[J \times \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}}, [0, \infty)]$, *$V$ is locally Lipschitzian in x, x', $V(t, x, x') \to \infty$ as $\max\{D(x, \hat{0}), D(x', \hat{0})\} \to \infty$ uniformly for $t \in [t_0, T]$, for every $T > 0$ and for $(t, x, x') \in J \times \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}}$. We have*

$$\liminf_{h \to +0} \frac{1}{h}[V(t + h, x + hf(t, x, x')) - V(t, x, x')] \leq g(t, V(t, x), V'(t, x)),$$

where $g \in C[J \times [0, +\infty) \times [0, +\infty), (-\infty, +\infty)]$;

(iii) *the maximal solution $r(t) = r(t, t_0, u_0)$ of problem* (1) *exists on $[t_0, \infty)$ and is positive if $u_0 > 0$;*

(iv) $V(t, x, x') \leq r(t, t_0, x_0, x'_0)$.

Then for every $x_0 \in E^n$ and $x'_0 \in E^n$ such that $V(t_0, x_0, x'_0) \leq \max\{u_0, u'_0\}$, problem (1) *has a solution $x(t)$ on $[t_0, \infty)$ which satisfies the estimate*

$$V(t, x(t), x'(t)) \leq r(t), \quad t \geq t_0.$$

Proof. Let S denote the set of all functions x defined on $I_x = [t_0, c_x)$ with values in E^n such that $x(t)$ is a solution of (1) on I_X. We define a partial order $\leq$ on S as follows: the relation $x \leq y$ implies that $I_x \subseteq I_y$ and $y(t) \equiv x(t)$ on I_x. We shall firstly show that S is nonempty. By assumption (i), there exists a solution $x(t)$ of (1) defined on $I_x = [t_0, c_x)$. Setting $m(t) = V(t, x(t), x'(t))$ for $t \in I_x$ and using assumption (ii), it is easy to obtain the differential inequality

$$D_+ m(t) \leq g(t, m(t), m'(t)), \quad t \in I_x.$$

From assumption (iv), we have

$$V(t, x, x') \leq r(t, t_0, x_0, x'_0)$$

where $r(t)$ is the maximal solution of (2). This shows that $x \in S$ and so S is nonempty. If $(x_\beta)_\beta$ is a chain $(S, \leq)$, then there is uniquely defined mapping y on $I_y = [t_0, \sup_\beta c_{x_\beta}]$ that coincides with x_β on I_{x_β}. Clearly $y \in S$ and hence y is an upper bounded of $(x_\beta)_\beta$ in $(S, \leq)$. The proof of the theorem is complete if we show that $c_z < \infty$. Suppose that it is not true, so that $c_z < \infty$. Since $r(t)$ is assumed to exist on $[t_0, \infty)$, $r(t)$ is bounded on I_z and there is an $M > 0$ such that

$$D(f(t, z(t), z'(t)), \hat{0}) \leq M, \quad t \in I_z.$$

Then, for all $t_1, t_2 \in I_z$, $t_1 \leq t_2$,

$$\begin{aligned}
&D(z(t_2, t_0, z_0, z'_0), z(t_1, t_0, z_0, z'_0)) \\
&\quad = D\Bigg(k_1 \Theta(-1)\bigg(k_2(t_2 - t_0)\Theta(-1)\int_{t_0}^{t_2}\bigg(\int_{t_0}^{w} f(s, z(s), z'(s))\mathrm{d}s\bigg)\mathrm{d}w\bigg), \\
&\qquad k_1 \Theta(-1)\bigg(k_2(t_1 - t_0)\Theta(-1)\int_{t_0}^{t_1}\bigg(\int_{t_0}^{w} f(s, z(s), z'(s))\mathrm{d}s\bigg)\mathrm{d}w\bigg)\Bigg) \\
&\quad \leq D(k_2(t_1 - t_0), k_2(t_2 - t_0)) + D\bigg(\int_{t_0}^{t_2}\bigg(\int_{t_0}^{w} f(s, x(s), x'(s))\mathrm{d}s\bigg)\mathrm{d}s, \int_{t_0}^{t_1}\bigg(\int_{t_0}^{w} f(s, x(s), x'(s))\mathrm{d}s\bigg)\mathrm{d}s\bigg) \\
&\quad \leq \int_{t_1}^{t_2}\int_{t_0}^{w} D(f(s, x(s), x'(s)), \hat{0})\mathrm{d}s\mathrm{d}s \\
&\quad \leq M|t_2 - t_1|\left|\frac{t_1 + t_2 - t_0}{2}\right|.
\end{aligned}$$

That is to say $\lim_{t \to T^-} x(t, t_0, x_0, x'_0)$ exists and is finite. Now let $x_1 = \lim_{t \to T^-} x(t, t_0, x_0, x'_0)$, $x'_1 = \lim_{t \to T^-} x'(t, t_0, x_0)$ and consider the initial problem

$$\begin{cases} x'' = f(t, x, x') \\ x(T) = x_1 \\ x'(T) = x'_1. \end{cases}$$

According to Theorem 2.2, its solution exists on $[T, T_1]$ and we define

$$x^*(t) = \begin{cases} x(t, t_0, x_0, x_0'), & t_0 \le t < T, \\ x_1(t) & T \le t < T_1. \end{cases}$$

Obviously, $x^*(t)$ is a solution of the initial value problem (1) on $[t_0, T_1]$. This contradicts with the definition of T. $\square$

4. Conclusions

In this paper, the global existence of solutions for second-order fuzzy differential equations with initial conditions under generalized H-differentiability is studied. Two theorems for global existence of solutions are given and proved on $[t_0, \infty]$.

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