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Rates of A-statistical convergence of positive linear operators

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Abstract

In this paper we study the rates of A-statistical convergence of sequences of positive linear operators mapping the weighted space C_{ρ_1} into the weighted space B_{ρ_2}.

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1. Introduction

In the classical summability setting rates of summation have been introduced in several ways (see, e.g., [1–3]). The concept of statistical rates of convergence, for nonvanishing two null sequences, is studied in [4]. Unfortunately no single definition seems to have become the "standard" for the comparison of rates of summability transforms. The situation becomes even more uncharted when one considers rates of A-statistical convergence. For this reason various ways of defining rates of convergence in the A-statistical sense are introduced in [5].

In the present paper, using the concepts of [5], we study rates of A-statistical convergence of sequences of positive linear operators defined on weighted spaces. We note that the classical Korovkin-type

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approximation theory may be found in [6–8] while its further extensions studied via A-statistical convergence may be viewed in [5,9,10].

Now we turn to introducing some notation and the basic definitions used in this paper. Let $A = (a_{jn})$ be an infinite summability matrix. For a given sequence $x := (x_n)$, the A-transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$, provided the series converges for each j. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$ [11]. Assume now that A is a nonnegative regular summability matrix and K is a subset of $\mathbb{N}$, the set of all natural numbers. The A-density of K is defined by $\delta_A(K) := \lim_j \sum_{n=1}^{\infty} a_{jn} \chi_K(n)$ provided the limit exists, where χ_K is the characteristic function of K. Then the sequence $x := (x_n)$ is said to be A-statistically convergent to the number L if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$; or equivalently $\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim x = L$ [12–15]. For the case in which $A = C_1$, the Cesáro matrix, A-statistical convergence reduces to statistical convergence [16–18]. Also, taking $A = I$, the identity matrix, A-statistical convergence coincides with the ordinary convergence. We also note that if $A = (a_{jn})$ is a nonnegative regular summability matrix for which $\lim_j \max_n\{a_{jn}\} = 0$, then A-statistical convergence is stronger than convergence [19]. A sequence $x = (x_n)$ is said to be A-statistically bounded provided that there exists a positive number M such that $\delta_A\{n \in \mathbb{N} : |x_n| \leq M\} = 1$. Recall that $x = (x_n)$ is A-statistically convergent to L if and only if there exists a subsequence $\{x_{n(k)}\}$ of x such that $\delta_A\{n(k) : k \in \mathbb{N}\} = 1$ and $\lim_k x_{n(k)} = L$ (see [15,19]). Note that the concept of A-statistical convergence is also given in normed spaces [20].

Now let $\mathbb{R}$ denote the set of real numbers. The function ρ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim_{|x|\to\infty} \rho(x) = \infty$ and $\rho(x) \geq 1$ (for all $x \in \mathbb{R}$). Then the space of real valued functions f defined on $\mathbb{R}$ and satisfying $|f(x)| \leq M_f.\rho(x)$ (for all $x \in \mathbb{R}$) is called weighted space and denoted by B_ρ, where M_f is a constant depending on the function f. The weighted subspace C_ρ of B_ρ is given by $C_\rho := \{f \in B_\rho : f$ is continuous over $\mathbb{R}\}$. It is known [21] that the spaces B_ρ and C_ρ are Banach spaces with the norm $\|f\|_\rho := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\rho(x)}$.

Assume that ρ_1 and ρ_2 are two weight functions and that they satisfy

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0. \tag{1.1}$$

If T is a positive linear operator such that $T : C_{\rho_1} \to B_{\rho_2}$, then the operator norm $\|T\|_{C_{\rho_1}\to B_{\rho_2}}$ is given by $\|T\|_{C_{\rho_1}\to B_{\rho_2}} := \sup_{\|f\|_{\rho_1}=1} \|T\ f\|_{\rho_2}$.

Using a functional analytic technique, Duman and Orhan [9] proved the following Korovkin-type approximation theorem via A-statistical convergence.

Theorem A. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix and let ρ_1 and ρ_2 be weight functions satisfying (1.1). Assume that $\{T_n\}$ is a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2}. Then, for all $f \in C_{\rho_1}$, $st_A - \lim_n \|T_n f - f\|_{\rho_2} = 0$ if and only if $st_A - \lim_n \|T_n F_v - F_v\|_{\rho_1} = 0$, $(v = 0, 1, 2)$, where $F_v(x) = \frac{x^v \rho_1(x)}{1+x^2}$, $(v = 0, 1, 2)$.*

Recall that the classical case of Theorem A may be found in [21] and [22]. We note that an example is also presented in [9] so that Theorem A holds but the classical Korovkin theorem fails.

2. Rates of A-statistical convergence

In this section, using the modulus of continuity, we study rates of A-statistical convergence in Theorem A.

The concepts of the rates of A-statistical convergence have been introduced in [5] as follows.

Let $A = (a_{jn})$ be a nonnegative regular summability matrix and let (a_n) be a positive nonincreasing sequence of real numbers. Then a sequence $x = (x_n)$ is A-statistically convergent to the number L with the rate of $o(a_n)$ if for every $\varepsilon > 0$, $\lim_j \frac{1}{a_j} \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$. In this case we write $x_n - L = st_A - o(a_n)$, (as $n \to \infty$). If for every $\varepsilon > 0$, $\sup_j \frac{1}{a_j} \sum_{n:|x_n| \geq \varepsilon} a_{jn} < \infty$, then x is A-statistically bounded with the rate of $O(a_n)$ and it is denoted by $x_n = st_A - O(a_n)$, (as $n \to \infty$). In the above two definitions the "rate" is controlled more by the entries of the summability method than by the terms of the sequence $x = (x_n)$. For instance, when one takes the identity matrix I, if $a_{nn} = o(a_n)$ then $x_n - L = st_A - o(a_n)$ for any convergent sequence $(x_n - L)$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation we may consider the concept of convergence in measure from measure theory to define the rate of convergence as follows: $x = (x_n)$ is A-statistically convergent to L with the rate of $o_\mu(a_n)$, denoted by $x_n - L = st_A - o_\mu(a_n)$, (as $n \to \infty$), if for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n - L| \geq \varepsilon a_n} a_{jn} = 0$. Finally, the sequence $x = (x_n)$ is A-statistically bounded with the rate of $O_\mu(a_n)$ provided that there is a positive number M such that $\lim_j \sum_{n:|x_n| \geq M a_n} a_{jn} = 0$. In this case we write $x_n = st_A - O_\mu(a_n)$, (as $n \to \infty$).

Throughout the paper the weight function ρ_1 will be defined by $\rho_1(x) = 1 + x^2$ on $\mathbb{R}$. Also, we consider the following weighted modulus of continuity: $w_{\rho_1}(f, \delta) = \sup_{|x-y| \leq \delta} \frac{|f(y) - f(x)|}{\rho_1(x)}$, where δ is a positive constant and $f \in C_{\rho_1}$ (see [23]). It is easy to see that, for any $c > 0$ and all $f \in C_{\rho_1}$,

$$w_{\rho_1}(f, c\delta) \leq (1 + [c]) w_{\rho_1}(f, \delta), \tag{2.1}$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

To obtain our main result we need the following two lemmas.

Lemma 2.1. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix. Assume that $\{T_n\}$ is a sequence of positive linear operators defined on C_{ρ_1} such that the sequence $\{\|T_n\|_{C_{\rho_1} \to B_{\rho_1}}\}$ is A-statistically bounded, i.e., $\delta_A(K) = 1$ with $K := \{n \in \mathbb{N} : \|T_n\|_{C_{\rho_1} \to B_{\rho_1}} \leq M\}$ for some $M > 0$. Let $T_n \varphi_x$ and $T_n F_0$ be in C_{ρ_1} for each n, where $\varphi_x(y) = (y - x)^2$ and $F_0(y) = 1$. Then, for any $s > 0$ and all $n \in K$, the inequality*

$$\sup_{\|f\|_{\rho_1}=1} \left(\sup_{|x| \leq s} |T_n(f; x) - f(x)| \right) \leq C \left\{ \sup_{\|f\|_{\rho_1}=1} (w_{\rho_1}(f, \alpha_n)) + \|T_n F_0 - F_0\|_{\rho_1} \right\} \tag{2.2}$$

holds, where $\alpha_n := \sqrt{\|T_n \varphi_x\|_{\rho_1}}$ and C is a positive constant depending on s.

Proof. Using linearity and positivity of T_n, for all $n \in \mathbb{N}$ and any $\delta > 0$, we get

$$\begin{aligned} |T_n(f; x) - f(x)| &\leq T_n(|f(y) - f(x)|; x) + |f(x)||T_n(F_0; x) - F_0(x)| \\ &\leq T_n \left(\rho_1(x) w_{\rho_1} \left(f, \delta \frac{|y - x|}{\delta} \right); x \right) + |f(x)||T_n(F_0; x) - F_0(x)|. \end{aligned}$$

From (2.1) it follows that

$$\begin{aligned} |T_n(f; x) - f(x)| &\leq \rho_1(x) w_{\rho_1}(f, \delta) T_n \left(1 + \left[\frac{|y - x|}{\delta} \right]; x \right) \\ &\quad + |f(x)||T_n(F_0; x) - F_0(x)| \\ &\leq \rho_1(x) w_{\rho_1}(f, \delta) T_n \left(1 + \frac{(y - x)^2}{\delta^2}; x \right) \end{aligned}$$

$$
\begin{aligned}
&+|f(x)||T_n(F_0;x)-F_0(x)| \\
&\le \rho_1(x) w_{\rho_1}(f,\delta)\left\{T_n(\rho_1;x)+\frac{1}{\delta^2}T_n(\varphi_x;\ x)\right\} \\
&+|f(x)||T_n(F_0;x)-F_0(x)|.
\end{aligned} \tag{2.3}
$$

Since $\varphi_x \in C_{\rho_1}$, for any $s > 0$ and all $n \in \mathbb{N}$, (2.3) yields that

$$
\sup_{|x|\le s} |T_n(f;x)-f(x)| \le C_1 w_{\rho_1}(f,\delta)\left(C_1\|T_n\rho_1\|_{\rho_1}+\frac{C_1}{\delta^2}\|T_n\varphi_x\|_{\rho_1}\right)+C_2\|T_nF_0-F_0\|_{\rho_1}, \tag{2.4}
$$

where $C_1 = \sup_{|x|\le s}\rho_1(x) = 1+s^2$ and $C_2 = \sup_{|x|\le s}(|f(x)|\rho_1(x))$. Since $\|T_n\rho_1\|_{\rho_1} = \|T_n\|_{C_{\rho_1}\to B_{\rho_1}}$, by the hypothesis, for all $n \in K$, we obtain

$$
\|T_n\rho_1\|_{\rho_1} \le M. \tag{2.5}
$$

Now putting $\delta := \alpha_n = \sqrt{\|T_n\varphi_x\|_{\rho_1}}$ and combining (2.4) and (2.5), we conclude, for all $n \in K$, that

$$
\sup_{|x|\le s}|T_n(f;x)-f(x)| \le (1+M)C_1^2 w_{\rho_1}(f,\alpha_n)+C_2\|T_nF_0-F_0\|_{\rho_1}.
$$

This implies that

$$
\sup_{\|f\|_{\rho_1}=1}\left(\sup_{|x|\le s}|T_n(f;x)-f(x)|\right) \le (1+M)C_1^2 \sup_{\|f\|_{\rho_1}=1}(w_{\rho_1}(f,\alpha_n))+C_2\|T_nF_0-F_0\|_{\rho_1}. \tag{2.6}
$$

Hence, taking $C := \max\{(1+M)C_1^2, C_2\}$, (2.2) follows from (2.6) immediately, which completes the proof. □

Lemma 2.2. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix, and let ρ_1 and ρ_2 satisfy (1.1). Assume that $\{T_n\}$ is a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that $\{\|T_n\|_{C_{\rho_1}\to B_{\rho_1}}\}$ is A-statistically bounded. Assume further that (c_n) is a positive nonincreasing sequence. If, for any $s \in \mathbb{R}$,*

$$
\sup_{\|f\|_{\rho_1}=1}\left(\sup_{|x|\le s}|T_n(f;x)-f(x)|\right) = st_A - o(c_n), \qquad \text{as } n\to\infty,
$$

then $\|T_nf-f\|_{\rho_2} = st_A - o(c_n)$, as $n\to\infty$. Furthermore, similar results hold when little "o" is replaced by big "O", little "o_μ" or big "O_μ", respectively.

Proof. Using the same technique as in the proof of Lemma 2 in [9], one can get the result immediately. □

Theorem 2.3. *Let $A = (a_{jn})$, ρ_1 and ρ_2 be the same as in Lemma 2.2, and let $\{T_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that $\{\|T_n\|_{C_{\rho_1}\to B_{\rho_1}}\}$ is A-statistically bounded. Let $T_n\varphi_x$ and T_nF_0 be in C_{ρ_1} for each n where $\varphi_x(y) = (y-x)^2$ and $F_0(y) = 1$. Assume that the operators T_n satisfy the conditions*

(i) $\|T_nF_0-F_0\|_{\rho_1} = st_A - o(a_n)$, *as $n\to\infty$*
(ii) $\sup_{\|f\|_{\rho_1}=1}\left(w_{\rho_1}(f,\alpha_n)\right) = st_A - o(b_n)$, *as $n\to\infty$ with $\alpha_n = \sqrt{\|T_n\varphi_x\|_{\rho_1}}$,*

where (a_n) and (b_n) are positive nonincreasing sequences. Then, for all $f \in C_{\rho_1}$, $\|T_nf-f\|_{\rho_2} = st_A - o(c_n)$, as $n\to\infty$, where $c_n := \max\{a_n, b_n\}$. Similar results hold when little "o" is replaced by big "O".

Proof. Let

$$u_n := \sup_{\|f\|_{\rho_1}=1}\left(\sup_{|x|\le s}|T_n(f;x)-f(x)|\right),\ v_n := \sup_{\|f\|_{\rho_1}=1}(w_{\rho_1}(f,\alpha_n)) \text{ and } z_n := \|T_nF_0-F_0\|_{\rho_1}.$$

Then, by (2.2), we have $u_n \le C(v_n+z_n)$ for some $C>0$ and all $n\in K$, where K is the same as in Lemma 2.1. Given $\varepsilon>0$, define the following sets: $D=\{n\in K : v_n+z_n\ge \frac{\varepsilon}{C}\}$, $D_1=\{n\in K : v_n\ge \frac{\varepsilon}{2C}\}$ and $D_2=\{n\in K : z_n\ge \frac{\varepsilon}{2C}\}$. Then clearly we have $D\subseteq D_1\cup D_2$. Hence, observe that the inequality

$$\frac{1}{c_j}\sum_{n\in K:u_n\ge\varepsilon} a_{jn} \le \frac{1}{c_j}\sum_{n\in D} a_{jn} \le \frac{1}{c_j}\sum_{n\in D_1} a_{jn} + \frac{1}{c_j}\sum_{n\in D_2} a_{jn} \tag{2.7}$$

holds for all $j\in\mathbb{N}$. Since $c_j=\max\{a_j,b_j\}$, we get from (2.7) that

$$\frac{1}{c_j}\sum_{n\in K:u_n\ge\varepsilon} a_{jn} \le \frac{1}{b_j}\sum_{n\in D_1} a_{jn} + \frac{1}{a_j}\sum_{n\in D_2} a_{jn}, \qquad \text{for all } j\in\mathbb{N}. \tag{2.8}$$

Letting $j\to\infty$ in (2.8), and using (i) and (ii), for any $s>0$, we conclude that

$$\sup_{\|f\|_{\rho_1}=1}\left(\sup_{|x|\le s}|T_n(f;x)-f(x)|\right) = st_A - o(c_n), \qquad \text{as } n\to\infty.$$

So, the result follows from Lemma 2.2. $\square$

Replacing "o" by "o_μ" one can get the following result immediately.

Theorem 2.4. *Let $A=(a_{jn})$, ρ_1, ρ_2 be the same as in Lemma 2.2 and let $\{T_n\}$ be the same as in Theorem 2.3. Assume that the operators T_n satisfy the conditions*

(i) $\|T_nF_0-F_0\|_{\rho_1} = st_A - o_\mu(a_n)$, *as* $n\to\infty$
(ii) $\sup_{\|f\|_{\rho_1}=1}\big(w_{\rho_1}(f,\alpha_n)\big) = st_A - o_\mu(b_n)$, *as* $n\to\infty$ *with* $\alpha_n=\sqrt{\|T_n\varphi_x\|_{\rho_1}}$,

where (a_n) and (b_n) are positive nonincreasing sequences. Then, for all $f\in C_{\rho_1}$, $\|T_nf-f\|_{\rho_2} = st_A - o_\mu(c_n)$, as $n\to\infty$, where $c_n := \max\{a_n,b_n,a_nb_n\}$. Similar conclusions hold when little "o_μ" is replaced by big "O_μ".

Now, specializing the sequences (a_n) and (b_n) in Theorem 2.3 or 2.4, we can easily get Theorem A. So, Theorems 2.3 and 2.4 give the rates of A-statistical convergence of the operators T_n from C_{ρ_1} into B_{ρ_2}. Of course, when $A=(a_{jn})$ is replaced by the identity matrix I, we get the following ordinary rate of convergence of these operators.

Corollary 2.5. *Let ρ_1 and ρ_2 be the same as in Lemma 2.2 and let $\{T_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that the sequence $\{\|T_n\|_{C_{\rho_1}\to B_{\rho_1}}\}$ is bounded. Let $T_n\varphi_x$ and T_nF_0 be in C_{ρ_1} for each n where $\varphi_x(y)=(y-x)^2$ and $F_0(y)=1$. Assume that the operators T_n satisfy the conditions*

(i) $\lim_n \|T_nF_0-F_0\|_{\rho_1}=0$ *with* $F_0(y)=1$,
(ii) $\lim_n(\sup_{\|f\|_{\rho_1}=1} w_{\rho_1}(f,\alpha_n))=0$ *with* $\alpha_n=\sqrt{\|T_n\varphi_x\|_{\rho_1}}$.

Then, for all $f\in C_{\rho_1}$, we have $\lim_n\|T_nf-f\|_{\rho_2}=0$.

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