DISCRETE MATHEMATICS

An n to $2n$ embedding of incomplete idempotent latin squares for small values of n

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Abstract

In 1983, necessary and sufficient conditions were obtained for an incomplete idempotent latin square of order n to be embedded in an idempotent latin square of order $2n$, providing $n > 16$. In this paper we consider the case where $n \leqslant 16$.

1. Introduction

A *partial latin square* of *order* n on the symbols $1,\dots,t$ is an $n \times n$ array L in which each cell contains at most one symbol, and each symbol appears at most once in each row and at most once in each column of L. An *incomplete latin square* is a partial latin square in which each cell contains exactly one symbol; so an incomplete latin square must have $t \geqslant n$. An incomplete latin square is said to be a *latin square* if $t = n$. Let $L(i,j)$ denote the symbol in cell (i,j) of L if one exists. A partial latin square of order n is said to be *idempotent* if $L(i,i) = i$ for $1 \leqslant i \leqslant n$. Let $N_L(i)$ be the number of cells in L that contain symbol i.

A partial latin square L is said to be *embedded* in the latin square M if $L(i,j)=M(i,j)$ for all filled cells (i,j) of L. The embedding of partial latin squares with various additional properties has a long history. For example, finding necessary and sufficient conditions for the embedding of a partial latin square of order n on the symbols $1,\dots,n$ in a latin square of order t was solved in 1960 by Evans [3], and such conditions for the embedding of incomplete latin squares were found by Ryser [8] in 1951 following the classic embedding theorem by Hall [5] in 1945.

In 1971, Lindner proved that there exists a finite embedding of a partial idempotent latin square of order n on the symbols $1,\dots,n$. After two further improvements, this

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Table 1
Empty cells indicate the only unsolved embeddings; the problem is solved for $n \geqslant 10$

$\mu \setminus n$	3	4	5	6	7	8	9
2				Theorem 3.1	Theorem 3.1	Theorem 3.1	Theorem 3.1
3					Theorem 3.1	Theorem 3.1	Theorem 3.1
4						Theorem 3.1	Theorem 3.1
5	Lemma 3.1						Theorem 3.1
6	Lemma 3.1	Lemma 3.2					

problem was finally completely settled for all $t \geqslant 2n+1$ by Andersen et al. [1]. The more general problem of embedding incomplete idempotent latin squares of order n on the symbols $1,\ldots,t$ in an idempotent latin square of order $t \geqslant 2n+1$ was settled by Rodger [7] in 1984 (where the even more general problem of a prescribed, but not necessarily idempotent diagonal for the containing latin square was considered).

When embedding an incomplete idempotent latin square L of order n in an idempotent latin square of order t with $t \geqslant 2n+1$, there are no necessary conditions. However, if $t \leqslant 2n$, it is necessary that $N_L(i) \geqslant 2n-t+f(i)$ for $1 \leqslant i \leqslant t$, where $f(i)=0$ or 1 if $i \leqslant n$ or $i > n$ respectively. In 1983, Rodger [6] proved that this condition is sufficient when $t=2n$ and $n>16$. However, if $t<2n$ then the problem becomes incredibly difficult, since numerical necessary conditions are no longer sufficient: the arrangement of the symbols within L can determine whether or not there exists an idempotent embedding of L [2]. Thus the case $t=2n$ is unique.

In this paper, we address the limitation in [6] of requiring $n>16$ when $t=2n$. We prove that this bound on n can be lowered to $n>9$ (see Corollary 3.1), and address many of the cases when $n \leqslant 9$ (see Theorem 3.1 and Table 1). It should be pointed out that, although we are only considering 'small' values of n, the number of partial idempotent latin squares of orders at most 16 is astronomical. Indeed, it is not even known how many complete latin squares of order 11 exist because there are too many for current computers to count. As is often the case, dealing with 'small' values raises many exceedingly difficult situations that are avoided once the problem becomes large enough; it is these problems we consider here.

2. Preliminary results

Throughout this paper, it will help to refer to the symbols in $\{1,\ldots,n\}$ as *small* symbols, the symbols in $\{n+1,\ldots,2n\}$ as BIG symbols, and the symbols in $\{n+1,\ldots,2n-2\}$ as *big* symbols. Furthermore, without loss of generality it will be assumed that $N_R(j-1) \geqslant N_R(j)$ for $n+2 \leqslant j \leqslant 2n$ for any $n \times n$ incomplete idempotent latin square R that is to be embedded in an idempotent latin square of order $2n$ (for if R is embedded in a latin square of order $2n$ with symbols $n+1, \ldots, 2n$ each occurring in a diagonal cell, then rows and columns $n+1, \ldots, 2n$ can be permuted to form the required idempotent embedding of R).

The following result is essentially the same as those used in [1,6,7] so is presented without proof here (see [4] for a proof).

Proposition 2.1. *Let R be an incomplete idempotent latin square of order n. Suppose a row 0 and two columns -1 and 0 can be added to R to form an $(n+1) \times (n+2)$ incomplete latin rectangle R^+ that contains two BIG symbols s and t with the properties that*:

(i) $N_{R^+}(j) \geqslant 3$ *for all small symbols j and BIG symbols $j \in \{s, t\}$*;

(ii) $N_{R^+}(j) \geqslant 4$ *for all BIG symbols j except $j \in \{s, t\}$*;

and R^+ satisfies either of the following

(iii) *cells $(0,-1), (a,0), (0,0)$, and $(a,-1)$ are filled with the two BIG symbols s and t for some row a, or*

(iv) *cells $(0,-1), (a,0), (0,0)$, and $(b,-1)$ are filled with the two BIG symbols s and t, and cells $(a,-1)$ and $(b,0)$ are filled with symbol v, for some rows a and b, and some small symbol v.*

Then R can be embedded in an idempotent latin square of order $2n$.

So it remains to embed R into an $(n+1) \times (n+2)$ incomplete latin rectangle R^+ that satisfies the conditions of Proposition 2.1. This embedding breaks into several cases, depending upon some properties of R. The next two lemmas deal with some special cases, and the third lemma deals with the most general case.

Lemma 2.1. *For $n \geqslant 5$, let R be an incomplete idempotent latin square of order n on the symbols $1, \ldots, 2n$ such that each symbol occurs at least once in R. Suppose $N_R(j-1) \geqslant N_R(j)$ for all $j \in \{n+2, \ldots, 2n\}$. If, in R, there does not exist a row or column missing two BIG symbols, then there exists an $(n+1) \times (n+2)$ incomplete latin rectangle R^+ with the properties that*

(a) $R(i,j) = R^+(i,j)$ *for all i and $j \in \{1, \ldots, n\}$,*

(b) *cells $(0,-1), (a,0), (0,0)$ and $(b,-1)$ are filled with BIG symbols $2n-1$ and $2n$, and cells $(a,-1)$ and $(b,0)$ are filled with symbol s, for some rows a and b, and some small symbol s,*

(c) $N_{R^+}(j) \geqslant 3$ *for all small symbols j and $j \in \{2n-1, 2n\}$, and*

(d) $N_{R^+}(j) \geqslant 4$ *for all big symbols j.*

Proof. Let R be an incomplete idempotent latin square satisfying the conditions of the lemma. Since R is idempotent and no row of R is missing two BIG symbols, each row of R contains exactly $n-1$ BIG symbols and one small symbol. Thus, each small symbol j occurs only once in R, namely in cell (j,j).

Now, since R is idempotent, it is impossible for all the n BIG symbols to occur in every row of R. Furthermore, since each of the $2n$ symbols must occur at least once in R, it is impossible for $n-1$ BIG symbols to occur in every row of R. Therefore, at most $n-2$ BIG symbols can occur n times in R, so at least two BIG symbols can each occur at most $n-1$ times in R. Since, $N_R(j-1) \geqslant N_R(j)$ for $j \in \{n+2,\dots,2n\}$, symbols $2n-1$ and $2n$ occur at most $n-1$ times in R. Hence, symbols $2n-1$ and $2n$ are each missing from at least one row. Since R does not contain a row missing two BIG symbols, symbols $2n-1$ and $2n$ are missing from different rows; without loss of generality, we can assume that they are missing from rows 2 and 1 respectively (that is; $a=1$ and $b=2$), and we can assume that $s=3$.

Note that since each row of R contains $n-1$ BIG symbols, n^2-n cells contain BIG symbols. Thus, if three BIG symbols each occur less than four times in R, the number of cells containing BIG symbols would be at most $(n-3)n+(3)3=n^2-3n+9$. But $n \geqslant 5$, so $n^2-n > n^2-3n+9$. Therefore, at most two BIG symbols occur less than four times in R, namely $2n-1$ and $2n$. Hence, $n-2$ BIG symbols are *finished*; that is, these symbols satisfy the conditions imposed on them by (a)–(d). Therefore, the only possible unfinished BIG symbols are $2n-1$ and $2n$.

Begin forming R^+ by adding row 0, column 0, and column -1. Place symbol $2n-1$ (symbol $2n$) in cells $(0,0)$ and $(2,-1)$ (in cells $(0,-1)$ and $(1,0)$). Both symbols $2n-1$ and $2n$ now occur at least three times in R^+. Hence, all BIG symbols are finished.

For $1 \leqslant j \leqslant n-1$, fill the empty cells in row 0 by placing symbol j in cell $(0,j+1)$. Then place symbol n in cell $(0,1)$. Each of the n small symbols have been added to row 0. So, currently, each small symbol occurs twice in R^+.

For $3 \leqslant j \leqslant n$, fill the empty cells in column 0 by placing symbol j in cell $(j-1,0)$. Then place symbol 1 in cell $(n,0)$. Now, all the small symbols except symbol 2 has been placed in column 0. Hence, each of these small symbols occurs three times in R^+ and are finished. The only unfinished symbol is symbol 2.

For $5 \leqslant j \leqslant n$, fill the empty cells in column -1 by placing symbol j in cell $(j-2,-1)$. Now, since we are assuming $n \geqslant 5$, we can place symbol 2 (symbol 4) in cell $(n-1,-1)$ (in cell $(n,-1)$). Lastly, completely fill column -1 by placing symbol 3 in cell $(1,-1)$. Thus, all the small symbols except symbol 1 have been placed in column -1. Hence, symbol 2 now occurs three times in R^+ and is finished. Therefore, all symbols are finished and the $(n+1)\times(n+2)$ incomplete latin rectangle R^+ has been formed. □

Lemma 2.2. *For $n \geqslant 5$, let R be an incomplete idempotent latin square of order n on the symbols $1,\dots,2n$ such that each symbol occurs at least once in R. Suppose $N_R(j-1) \geqslant N_R(j)$ for all $j \in \{n+2,\dots,2n\}$. If R contains an $n\times(n-1)$ latin rectangle defined on $n-1$ small symbols and one BIG symbol, then there exists a row, say row a, missing two BIG symbols, and there exists an $(n+1)\times(n+2)$ incomplete latin rectangle R^+ with the properties that:*

(a) *$R(i,j)=R^+(i,j)$ for all i and $j \in \{1,\dots,n\}$,*

(b) *cells* $(0,-1),(a,0),(0,0)$, *and* $(a,-1)$ *are filled with two BIG symbols* s *and* t,
(c) $N_{R^+}(j)\geqslant 3$ *for all small symbols* j *and BIG symbols* $j\in\{s,t\}$, *and*
(d) $N_{R^+}(j)\geqslant 4$ *for all BIG symbols* j, *except BIG symbols* $j\in\{s,t\}$.

Proof. Let R be an incomplete idempotent latin square satisfying the conditions of the lemma. Since $N_R(j-1)\geqslant N_R(j)$ for all $j\in\{n+2,\dots,2n\}$, each of the BIG symbols except symbol $n+1$ occurs only once in R and all must occur in the same column, say column n, while symbol $n+1$ occurs $n-1$ times in R in all columns except column n. Thus, since $n\geqslant 5$, symbol $n+1$ is *finished*; that is, this symbol satisfies the conditions imposed on it by (a)–(d). Furthermore, each of the small symbols except symbol n occurs $n-1$ times in R, occurring in all columns except column n, while symbol n occurs only once in R in column n. Again, since $n\geqslant 5$, all small symbols except symbol n are finished.

Now, since $n+1$ occurs exactly $n-1$ times in R, symbol $n+1$ is missing from one row, which we can assume is the first; so we can let $a=1$ and $t=n+1$. Let symbol s be the BIG symbol in cell $(n-1,n)$. Since s occurs only once in R, symbol s is also missing from row 1. Begin forming R^+ by adding row 0, column 0, and column -1. Place symbol s (symbol $t=n+1$) in cells $(0,0)$ and $(1,-1)$ (in cells $(0,-1)$ and $(1,0)$). Now, symbol s occurs three times in R^+ and is finished. Therefore, the set of unfinished symbols is precisely the set $S=\{n,\dots,2n\}\backslash\{s,t\}$. Note, the small symbol n must be added twice in any of row 0, column 0, or column -1, while BIG symbols in S must be added in each of row 0, column 0, and column -1.

To fill the empty cells in row 0, start by placing symbol 1 in cell $(0,n)$. The remaining $n-1$ empty cells in row 0 can be greedily filled with the $n-1$ unfinished symbols from column n. Therefore, since these symbols occurred once in R they now occur twice in R^+.

To fill columns -1 and 0, define:

$$R(2,0)=R(3,-1)=n,$$
$$R(2,-1)=R(n-2,n),$$
$$R(j,0)=R(j-2,n)\quad \text{for } 3\leqslant j\leqslant n,$$
$$R(j,-1)=R(j-3,n)\quad \text{for } 4\leqslant j\leqslant n.$$

Then clearly all symbols are finished and the $(n+1)\times(n+2)$ incomplete latin rectangle R^+ has been formed. □

We now present the most general situation. In the following lemma, symbols are placed in row 0, column 0, and column -1 in such a way to satisfy the conditions of Proposition 2.1 except that not all cells are completely filled (that is, a partial latin rectangle is produced). Lemma 2.4 then fills all these empty cells to produce the desired incomplete latin rectangle R^+. Lemma 2.3 also introduces the extremely useful parameter $\mu=N_R(2n-1)+N_R(2n)$. This parameter allows much to be said of the cases where $n\leqslant 9$ (see Table 1).

Lemma 2.3. *Let R be an incomplete idempotent latin square of order n on the symbols $1,\dots,2n$ such that each symbol occurs at least once in R. Suppose $N_R(j-1)\geqslant N_R(j)$ for all $j\in\{n+2,\dots,2n\}$, and suppose some row, say row a, of R is missing two BIG symbols s and t. Let $\mu=N_R(2n-1)+N_R(2n)$. If*
(i) *$\mu\leqslant 6$ and $n\geqslant\mu+4$, or*
(ii) *$\mu>6$,*

then there exists $R^*=R\cup\rho_0\cup c_0\cup c_{-1}$, an $(n+1)\times(n+2)$ partial latin rectangle on the symbols $1,\dots,2n$, with the following properties:
(a) *the only empty cells occur in row 0, column 0, or column -1,*
(b) *each symbol occurs at least once in $R^-\cup c_0^-$,*
(c) *cells $(0,-1),(a,0),(0,0)$ and $(a,-1)$ are filled with two BIG symbols s and t,*
(d) *$N_{R^*}(j)\geqslant 3$ for all small symbols j and BIG symbols $j\in\{s,t\}$, and*
(e) *$N_{R^*}(j)\geqslant 4$ for all BIG symbols j except BIG symbols $j\in\{s,t\}$.*

Proof. Let R be an incomplete latin square of order n on the symbols $1,\dots,2n$ satisfying the conditions of the lemma.

Case (i): First, suppose $\mu\leqslant 6$ and $n\geqslant\mu+4$. Then symbols $2n-1$ and $2n$ are both missing from some row, which for notational convenience we assume is the first (that is, $a=1$). So, we can let $s=2n-1$ and $t=2n$. Begin forming R^* by adding row 0, column 0, and column -1. Place symbol $2n-1$ (symbol $2n$) in cells $(0,0)$ and $(1,-1)$ (cells $(0,-1)$ and $(1,0)$). Both symbols $2n-1$ and $2n$ now occur at least three times in R^*. Hence, these symbols are *finished*; that is, these symbols satisfy the conditions imposed on them by (a)–(e) in the statement of this lemma.

To prove case (i), it will be necessary to investigate subcases:
(ia) at most $n-3$ of the $n-2$ big symbols occur exactly three times in R, and
(ib) all big symbols occur exactly three times in R.

Subcase (ia): Suppose at most $n-3$ of the $n-2$ big symbols occurs exactly three times in R. To fill some empty cells in row 0, start by creating a bipartite graph $G=(C,S)$ with vertex sets $C=\{c_1,c_2,\dots,c_n,c^*\}$ and $S=\{1,2,\dots,2n-2\}$. Let $E(G)=\{\{c_i,j\}$: column i in R is missing the symbol $j\}$. Then, in the graph G, for each $i\in\{1,\dots,n\}$ and each $j\in\{1,\dots,2n-2\}$, we have the following:

$$d_G(c_i)=n-2,\quad \text{if column } i \text{ contains neither } 2n-1 \text{ nor } 2n,$$
$$d_G(c_i)=n-1,\quad \text{if column } i \text{ contains either } 2n-1 \text{ or } 2n,$$
$$d_G(c_i)=n,\quad \text{if column } i \text{ contains both } 2n-1 \text{ and } 2n,$$
$$d_G(c^*)=0,$$
$$d_G(j)=n-N_R(j)\leqslant n-1.$$

Let N_i represent the number of big symbols occurring i times in R. Then, our immediate goal is to form a new graph G^* from G with maximum degree $n-2$ in such a way that all vertices in S with corresponding symbols occurring only once or twice in R have maximum degree, and $\min\{n-(\mu+3),N_3\}$ vertices with corresponding big symbols occurring three times in R also have maximum degree.

Form G^* from G as follows. If $d_G(j)=n-1$, remove any one edge incident with j. Furthermore, if $d_G(j)=n-3$ and $j\in\{n+1,\dots,2n-2\}$, then for an arbitrarily chosen set of $\min\{n-(\mu+3),N_3\}$ such vertices, add an edge $\{c^*,j\}$. It follows that all the vertices j mentioned above now have degree $n-2$. Lastly, for any c_i in the set C, if $d_G(c_i)=n-1$ or n, remove one or two edges, respectively, that are incident with c_i, and for each such edge $\{c_i,j\}$ add the edge $\{c^*,j\}$ (so G^* may be multigraph). This last step leaves the degree of vertex j unchanged, and results in the degree of each vertex c_i being at most $n-2$. It should also be noted that since $d_G(c_i)>n-2$ only for those vertices corresponding to columns containing $2n-1$ and/or $2n$, and since $\mu=N_R(2n-1)+N_R(2n)$, at most μ additional edges were added to vertex c^* in this last step. So, G^* has maximum degree $n-2$, and $d_{G^*}(c^*)\leqslant\mu+[n-(\mu+3)]=n-3<n-2$.

Properly edge-color the graph G^* with $n-2$ colors (König showed in 1916 that every bipartite graph has a Δ-edge-coloring; see [9]). Let k be a color not occurring on an edge incident with vertex c^*. Fill the cells in row 0 of R^* by placing symbol j in cell $(0,i)$ if and only if edge $\{c_i,j\}$ is colored k. Thus, all symbols with corresponding vertices of maximum degree have been placed in row 0 of R^*.

Note, small symbols occurring twice in R were placed in row 0 and are finished. Small symbols occurring only once in R were also placed in row 0, so must be placed at least once more in either column 0 or column -1. Big symbols occurring twice or once in R were also placed in row 0, so must be placed once or twice more respectively in either column 0 or column -1. Lastly, since $N_3\leqslant n-3$ in this subcase, and since $\min\{n-(\mu+3),N_3\}$ big symbols occurring three times in R were placed in row 0 (and are finished), at most $\max\{N_3-(n-(\mu+3)),0\}\leqslant\mu$ big symbols occurring three times in R have to occur once more in either column 0 or column -1. Therefore, since both symbols $2n-1$ and $2n$ are finished, the set of unfinished symbols consists of small symbols occurring once in R, big symbols occurring once or twice in R and at most μ big symbols occurring three times in R. Note, in this case $n\geqslant\mu+4$, so $n-(\mu+3)\geqslant 1$. Hence, if there exists at least one big symbol occurring three times in R then $\min\{n-(\mu+3),N_3\}\geqslant 1$ (that is, one such big symbol would be finished).

Next, recall that R^- represents the $(n-1)\times n$ latin rectangle formed from R by deleting its first row. To fill some empty cells in column 0, start by creating a bipartite graph $H=(P,S)$ with vertex sets $P=\{\rho_2,\dots,\rho_n,\rho^*\}$ and $S=\{1,2,\dots,2n-2\}$. Let $E(H)=\{\{\rho,j\}$: row i in R^- is missing the symbol $j\}$. Then, in the graph H, for each $i\in\{2,\dots,n\}$ and each $j\in\{1,2,\dots,2n-2\}$, we have the following:

$d_H(\rho_i)=n-2,$ if row i contains neither $2n-1$ nor $2n$,

$d_H(\rho_i)=n-1,$ if row i contains either $2n-1$ or $2n$,

$d_H(\rho_i)=n,$ if row i contains both $2n-1$ and $2n$,

$d_H(\rho^*)=0,$

$d_H(j)=(n-1)-N_{R^-}(j)\leqslant n-1.$

Now, the immediate goal is to form a new graph H^* from H with maximum degree $n-2$ in such a way that all vertices in S with corresponding symbols occurring at most once in R^- have maximum degree.

Form H^* from H as follows. If $d_H(j)=n-1$, remove any one edge incident with j. All these vertices now have degree $n-2$. Next, if for any ρ_i in the set P, $d_H(\rho_i)=n-1$ or n, then remove one or two edges, respectively, that are incident with ρ_i and for each such edge $\{\rho_i,j\}$ add the edge $\{\rho^*,j\}$ (so H^* may be a multigraph). This last step leaves the degree of vertex j unchanged and results in the degree of each vertex ρ_i being at most $n-2$. Now, since $d_H(\rho_i)>n-2$ only for those vertices corresponding to rows containing $2n-1$ and/or $2n$, and since $\mu=N_R(2n-1)+N_R(2n)$, at most μ edges were added to vertex ρ^* in this last step. So, H^* has maximum degree $n-2$, and $d_{H^*}(\rho^*)\leqslant\mu\leqslant n-4<n-2$.

Properly edge-color the graph H^* with $n-2$ colors. Let k be a color not occurring on an edge incident with vertex ρ^*. Fill the cells in column 0 of R^* by placing symbol j in cell $(i,0)$ if and only if edge $\{\rho_i,j\}$ is colored k. Thus, all symbols with corresponding vertices of maximum degree have been placed in column 0 of R^*.

Note, if symbol j occurred once in R, then it was added to both row 0 and column 0 and now occurs three times in R^*. Furthermore, if symbol j occurred twice in R then it was added to row 0 (and possibly added to column 0) and now occurs at least three times in R^*. Therefore, all symbols currently occur at least three times in R^*. Hence all small symbols and the BIG symbols $2n-1$ and $2n$ are finished. Thus, the set of unfinished symbols contains only big symbols from the set $\{n+1,\dots,2n-2\}$.

Notice further that if a symbol j did not occur in R^- then the corresponding vertex had maximum degree, so such a symbol was placed in column 0, and so all symbols in $\{1,2,\dots,2n-2\}$ occur at least once in $R^-\cup c_0^-$. Also symbols $2n-1$ and $2n$ occur in R^-, so all symbols occur at least once in $R^-\cup c_0^-$; that is, all symbols satisfy condition (b).

Therefore, after placing symbols in column 0, we have an $(n+1)\times(n+2)$ partial latin rectangle with the only empty cells occurring in any of row 0, column 0, and column -1, each symbol occurs at least once in $R^-\cup c_0^-$, and cells $(0,1),(1,0),(0,0)$, and $(1,-1)$ are filled with the BIG symbols $2n-1$ and $2n$. Thus, $R^-\cup c_0^-$ satisfies the conditions of Lemma 2.4. Using Lemma 2.4, completely fill all empty cells in row 0, column 0, and column -1. Next, remove the symbols in cells $(i,-1)$ for all $i\in\{2,\dots,n\}$. This last step leaves the cells of column -1 empty except for the cells $(0,-1)$ and $(1,-1)$ which contains symbols $2n-1$ and $2n$, and leaves row 0 and column 0 with no empty cells.

To finish all the symbols (and fill some cells of column -1), it will be necessary to consider two separate situations. Suppose first that at least one big symbol is finished. Then at most $n-3$ symbols must be placed in column -1.

Recall, $R^-\cup c_0^-$ represents the $(n-1)\times(n+1)$ latin rectangle formed by rows $2,\dots,n$ of $R\cup c_0$. To fill some empty cells in column -1, start by creating a bipartite graph $F=(P,S)$ with vertex sets $P=\{\rho_2,\dots,\rho_n,\rho^*\}$ and $S=\{n+1,\dots,2n-2\}$. Let $E(F)=\{\{\rho_i,j\}$: row i in $R^-\cup c_0^-$ is missing symbol $j\}$. Then, in the graph F, for each $i \in \{2,\dots,n\}$ and each $j \in \{n+1,\dots,2n-2\}$, we have the following:

$$d_F(\rho_i) \leqslant n-2 = |S|,$$

$$d_F(\rho^*) = 0,$$

$$d_F(j) = (n-1) - N_{R^- \cup c_0^-}(j) \leqslant n-2.$$

Now, the immediate goal is to form a new graph F^* from F of maximum degree $n-3$ in such a way that all vertices corresponding to big symbols occurring less than four times in R^* have maximum degree.

Form F^* from F as follows. If $d_F(j) = n-2$, remove an edge incident with j. Furthermore, if $d_F(j) = n-4$ and if the corresponding symbol j occurs only three times in R^*, add an edge $\{\rho^*, j\}$. Since at most μ big symbols occurring three times in R are unfinished, we have at most μ vertices j satisfying the previous condition. It follows that all the vertices j mentioned above now have degree $n-3$ and the degree of ρ^* is at most μ. Next, if $d_F(\rho_i) = n-2$ for any ρ_i in the set P, then there exists an edge $\{\rho_i, j\}$ for all ρ_i in the set P and j in the set S. Let f be a vertex from the set S corresponding to a finished big symbol (one such symbol exists by assumption). Remove the vertex f and all edges incident with f. This last step leaves the degree of each ρ_i at most $n-3$. So, F^* has maximum degree $n-3$, and $d_{F^*}(\rho^*) = \mu \leqslant n-4 < n-3$.

Properly edge-color the graph F^* with $n-3$ colors. Let k be a color not occurring on an edge incident with vertex ρ^*. Fill the cells in column -1 of R^* by placing symbol j in cell $(i,-1)$ if and only if edge $\{\rho_i, j\}$ is colored k. Thus, all the symbols with corresponding vertices of maximum degree have been placed in column -1 of R^*.

Note, all big symbols occurring three times in R^* were added to column -1. So, all symbols are finished. Therefore, in the case when $\mu \leqslant 6$ and $n \geqslant \mu + 4$, if there exists one finished big symbol after filling column 0, then a partial idempotent latin rectangle R^* has been formed satisfying conditions (a)–(e) of the Lemma.

Otherwise, it must be that no big symbol from the set $\{n+1,\dots,2n-2\}$ is finished. Hence, each of these $n-2$ big symbols must be placed in one of the $n-1$ empty cells of column -1.

Recall, if there exists at least one big symbol occurring three times in R then at least one such big symbol has been placed in row 0 and, hence, is finished. Thus, since in this case not one of the symbols $n+1,\dots,2n-2$ is finished, all these big symbols must occur once or twice in R. A big symbol occurring once in R was placed in both row 0 and column 0, so occurs at most twice in $R^- \cup c_0^-$. A big symbol occurring twice in R was placed in row 0 and possibly column 0. If such a symbol was placed in column 0, then it would occur four times in R^* and would be finished. Thus no big symbols occurring twice in R were placed in column 0, so these symbols also occur at most twice in $R^- \cup c_0^-$. Therefore, each of the big symbols $n+1,\dots,2n-2$ occurs at most twice in $R^- \cup c_0^-$.

To fill some empty cells in column -1, start by creating a bipartite graph $F = (P, S)$ with vertex sets $P = \{\rho_2,\dots,\rho_n,\rho^*\}$ and $S = \{1,\dots,2n-2\}$. Let $E(F) = \{\{\rho_i, j\}$: row

i in $R^- \cup c_0^-$ is missing symbol $j\}$. Now, since the $n+1$ cells in each row of $R^- \cup c_0^-$ are filled, each of these rows is missing exactly $n-1$ symbols. Therefore, in the graph F, for each $i \in \{2,\ldots,n\}$ and each $j \in \{1,\ldots,2n-2\}$, we have the following:

$$\begin{aligned}
&d_F(\rho_i) = n-3 && \text{if row } i \text{ contains neither } 2n-1 \text{ nor } 2n,\\
&d_F(\rho_i) = n-2 && \text{if row } i \text{ contains either } 2n-1 \text{ or } 2n,\\
&d_F(\rho_i) = n-1 && \text{if row } i \text{ contains both } 2n-1 \text{ and } 2n,\\
&d_F(\rho^*) = 0, &&\\
&d_F(j) = (n-1) - N_{R^- \cup c_0^-}(j) \leqslant n-2. &&
\end{aligned}$$

Now, the immediate goal is to form a new graph F^* from F of maximum degree $n-3$ in such a way that all vertices corresponding to big symbols have maximum degree.

Form F^* from F as follows. If $d_F(j)=n-2$, remove an edge incident with j. Thus, all these vertices now have degree $n-3$. Next, if for any ρ_i in the set P, $d_F(\rho_i)=n-2$ or $n-1$, then remove one or two edges, respectively, that are incident with ρ_i and for each such edge $\{\rho_i, j\}$ add the edge $\{\rho^*, j\}$ (so F may be a multigraph). This last step leaves the degree of vertex j unchanged and results in the degree of each ρ_i being at most $n-3$. Now, since $d_F(\rho_i) > n-3$ only for those vertices corresponding to rows containing $2n-1$ and/or $2n$, and since $\mu = N_R(2n-1) + N_R(2n)$, at most μ edges were added to vertex ρ^* in this last step. So, F^* has maximum degree $n-3$, and $d_{F^*}(\rho^*) \leqslant \mu \leqslant n-4 < n-3$.

Properly edge-color the graph F^* with $n-3$ colors. Let k be a color not occurring on an edge incident with vertex ρ^*. Fill the cells in column -1 of R^* by placing symbol j in cell $(i,-1)$ if and only if edge $\{\rho_i, j\}$ is colored k. Thus, all the symbols with corresponding vertices of maximum degree have been placed in column -1 of R^*.

Note, each of the big symbols $n+1,\ldots,2n-2$ was added to column -1. So, all symbols are finished. Therefore, in the case when $\mu \leqslant 6$ and $n \geqslant \mu+4$ if each of the $n-2$ big symbols $n+1,\ldots,2n-2$ must be placed in column -1, then a partial idempotent latin rectangle R^* has been formed satisfying conditions (a)–(e) of the Lemma.

Subcase (ib): Suppose all of the $n-2$ big symbols occur exactly three times in R. Then exactly $3(n-2)+\mu = 3n-6+\mu$ cells in R are filled with BIG symbols. Since we are under the assumption that $n \geqslant \mu+4$, $\mu \leqslant n-4$ and $3n-6+\mu \leqslant 3n-6+(n-4) = 4n-10$. That is, the number of cells in R filled with BIG symbols is at most $4n-10$. If only two small symbols are finished, then at most $2(n-2)+2n = 4n-4$ cells in R are filled with small symbols. So at most $(4n-10)+(4n-4) = 8n-14$ cells in R are filled. Now since $n \geqslant 6$, $n^2 > 8n-14$, so there must be at least three small symbols that are finished. Therefore, in this case, the set of unfinished symbols consists of the $n-2$ big symbols and at most $n-3$ small symbols.

Each of the $n-2$ big symbols must be added once to any of row 0, column 0, or column -1. We will place $n-3$ of these symbols in row 0. To fill some cells in row 0, start by creating a bipartite graph $G=(C,S)$ with vertex sets $C=\{c_1,\ldots,c_n\}$ and $S=\{n+2,\ldots,2n-2\}$. Let $E(G)=\{\{c_i, j\}$: column i in R is missing symbol $j\}$.

Then in the graph G, for each $i \in \{1,\dots,n\}$ and $j \in \{n+2,\dots,2n-2\}$, we have the following:

$$d_G(c_i) \leqslant n-3=|S|,$$
$$d_G(j)=n-N_R(j)=n-3.$$

Properly edge-color the graph G with $n-3$ colors. Let k be any color. Fill the cells in row 0 of R^* by placing symbol j in cell $(0,i)$ if and only if edge $\{c_i,j\}$ is colored k. Thus, all of the symbols with corresponding vertices of maximum degree have been placed in row 0 of R^*. That is, big symbols $n+2,\dots,2n-2$ have been placed in row 0 and are finished. Hence, the set of unfinished symbols consists of only one big symbol, symbol $n+1$, and at most $n-3$ small symbols.

To fill some empty cells in columns 0 and -1, start by creating a bipartite graph $H=(P,S)$ with vertex sets $P=\{\rho_2,\dots,\rho_n\}$ and $S=\{1,\dots,n\}\backslash\{f_1,f_2,f_3\}$; where f_1,f_2,f_3 represent three finished small symbols. Let $E(H)=\{\{\rho_i,j\}$: row i in R^- is missing the symbol $j\}$. Then, in the graph H, for each $i \in \{2,\dots,n\}$ and each $j \in \{1,\dots,n\} \setminus \{f_1,f_2,f_3\}$, we have the following:

$$d_H(\rho_i)= \leqslant n-3=|S|,$$
$$d_H(j)=(n-1)-N_{R^-}(j) \leqslant n-1.$$

Now, our immediate goal is to form a new graph H^* from H with maximum degree $n-3$ in such a way that all vertices in S with corresponding symbols occurring at most two times in R^- have maximum degree.

Form H^* from H as follows. If $d_H(j)=n-2$ or $n-1$, remove an edge or two, respectively, so that the resulting degree is $n-3$. Properly color the graph with $n-3$ colors. Let k_1 and k_2 be two colors. Fill the cells in column 0 (column -1) of R^* by placing symbol j in cell $(i,0)$ (in cell $(i,-1)$) if and only if edge $\{c_i,j\}$ is colored k_1 (colored k_2). Thus, all the symbols with corresponding vertices of maximum degree have been placed in both column 0 and column -1 of R^*.

Note, if a small symbol occurred less than three times in R^- then the corresponding vertex had maximum degree. Such a symbol was placed in both columns 0 and -1 and is finished. Therefore, the only unfinished symbol is the big symbol $n+1$. This symbol must be placed once in any of row 0, column 0, or column -1.

Notice further that if a small symbol j did not occur in R^- then it was added to column 0, so all small symbols occur at least once in $R^- \cup c_0^-$. Each of the $n-2$ big symbols occur three times in R so must occur at least once in R^-. Also, symbols $2n-1$ and $2n$ occur in R^-, so all symbols occur at least once in $R^- \cup c_0^-$; that is, all symbols satisfy condition (b).

To finish all the symbols, notice that no big symbols occur in either column 0 or column -1, so we can place symbol $n+1$ in either column. If there exists an empty cell contained in a row missing $n+1$, place the symbol $n+1$ in such a cell. Thus, we can assume that all the rows with empty cells contain the symbol $n+1$. Since symbol $n+1$ occurs three times in R, at least $(n-1)-3=n-4$ cells in column -1 are filled. Since $n \geqslant 6$ and at least $n-4 \geqslant 2$ cells of column -1 are filled, say cells $(n-1,1)$ and $(n,-1)$. Now, since exactly $n-3$ big symbols have been placed in row 0, row 0 has three empty cells and contains no small symbols. If $R^*(n-1,-1)$ does not occur in one of the columns, say column x, with an empty cell in row 0, move symbol $R^*(n-1,1)$ to cell $(0,x)$. Otherwise, if $R^*(n-1,-1)$ occurs in all three columns with an empty cell, $R^*(n-1,-1)$ is finished, so remove this symbol from column -1. In either case, cell $(n-1,-1)$ is now empty, so we can place symbol $n+1$ in cell $(n-1,-1)$. So, all symbols are finished.

Therefore, in the case when $\mu \geqslant 6$ and $n \geqslant \mu + 4$, if all $n-2$ big symbols occur exactly three times in R, then a partial idempotent latin rectangle R^* has been formed satisfying the conditions (a)–(e) of the Lemma.

Case (ii): Secondly, suppose $\mu > 6$. Among all rows missing two BIG symbols, if possible, choose row a to be one that is missing symbol $2n$.

Since $N_R(j-1) \geqslant N_R(j)$ for all $j \in \{n+2,\dots,2n\}$ and since $\mu/2 > 3$, at least $3n$ cells of R are filled with BIG symbols. Furthermore, since row a is missing two BIG symbols, at most n^2-n-1 cells can be filled with BIG symbols. Hence, we must have $3n \leqslant n^2-n-1$. Therefore, if $\mu > 6$, it must be that $n \geqslant 5$. Also, since $N_R(j-1) \geqslant N_R(j)$ for all $j \in \{n+2,\dots,2n\}$ and since $N_R(2n-1) \geqslant \mu/2 > 3$, all BIG symbols with the possible exception of $2n$ occur at least four times in R. That is, at least $n-1$ BIG symbols are finished; leaving at most one BIG symbol, $2n$, unfinished. Again we consider two cases.

Subcase (iia): Suppose first that $N_R(2n) \geqslant 4$, or that $N_R(2n) \leqslant 3$ and $t = 2n$ (that is, one of the two BIG symbols missing from row a is symbol $t = 2n$). Again, for notational convenience (and without loss of generality) we assume $a=1$. Begin forming R^* by adding row 0, column 0, and column -1. Place symbol s (symbol t) in cells $(0,0)$ and $(1,-1)$ (cells $(0,-1)$ and $(1,0)$).

Therefore, in either case, all BIG symbols are finished and only small symbols occurring less than three times in R need to be placed in any of row 0, column 0, and column -1.

To fill some empty cells in row 0, start by creating a bipartite graph $G=(C,S)$ with vertex sets $C=\{c_1,c_2,\dots,c_n,c^*\}$ and $S=\{1,2,\dots,n\}$. Let $E(G)=\{\{c_i,j\}$: column i in R is missing the symbol $j\}$. Then, in the graph G, for each $i \in \{1,\dots,n\}$ and each $j \in \{1,\dots,n\}$, we have the following:

$$d_G(c_i) \leqslant n-1 = |S \setminus \{i\}|,$$

$$d_G(c^*) = 0,$$

$$d_G(j) = n - N_R(j) \leqslant n-1.$$

Let n_i represent the number of small symbols occurring exactly i times in R. Then, our immediate goal is to form a new graph G^* from G with maximum degree $n-1$ in such a way that all vertices in S with corresponding symbols occurring only once in R have maximum degree and all but at most two vertices in S with corresponding symbols occurring twice in R also have maximum degree.

Form G^* from G as follows. If $d_G(j) = n-2$, add the edge $\{c^*, j\}$ for exactly $\min\{n-2, n_2\}$ such vertices. If $\min\{n-2, n_2\} = n-2$, arbitrarily choose these $n-2$ such vertices except in the case when $d_G(1) = n-2$. In this special case, include the edge $\{c^*, 1\}$. It follows that all the vertices j mentioned above now have degree $n-1$. So, G^* has maximum degree $n-1$, and $d_{G^*}(c^*) = \min\{n-2, n_2\} < n-1$.

Properly edge-color the graph G^* with $n-1$ colors. Let k be a color not occurring on an edge incident with vertex c^*. Fill the cells in row 0 of R^* by placing symbol j in cell $(0,i)$ if and only if edge $\{c_i, j\}$ is colored k. Thus, all symbols with corresponding vertices of maximum degree have been placed in row 0 of R^*.

Note, $\min\{n-2, n_2\}$ small symbols (including symbol 1, if $N_R(1) = 2$) occurring twice in R were placed in row 0 and are finished, so at most 2 such symbols need to be placed once more in either column 0 or column -1. Small symbols occurring only once in R were also placed in row 0, so these symbols must also be placed once more in either column 0 or column -1. Therefore, since all BIG symbols are finished, the set of unfinished symbols consists of small symbols occurring once in R and at most 2 small symbols occurring twice in R.

Since all BIG symbols with the possible exception of $2n$ occur at least four times in R and since if $N_R(2n) \leqslant 3$, symbol $2n$ is missing from row 1, all BIG symbols occur at least once in R^-. Furthermore, since R is idempotent, symbols $2, \dots, n$ also occur at least once in R^-. Therefore, symbol 1 is the only symbol that may not occur in R^-.

To fill some empty cells in column 0, start by creating a bipartite graph $H = (P, S)$ with vertex sets $P = \{\rho_2, \dots, \rho_n, \rho^*\}$ and $S = \{2, \dots, n\}$. Let $E(H) = \{\{\rho_i, j\}$: row i in R^- is missing the symbol $j\}$. Then, in the graph H, for each $i \in \{2, \dots, n\}$ and each $j \in \{2, \dots, n\}$, we have the following:

$$d_H(\rho_i) \leqslant n-2 = |S \setminus \{i\}|,$$

$$d_H(\rho^*) = 0,$$

$$d_H(j) = (n-1) - N_R(j) \leqslant n-2.$$

Now, the immediate goal is to form a new graph H^* from H with maximum degree $n-2$ in such a way that all vertices with corresponding symbols occurring once or twice in R^- have maximum degree.

Form H^* from H as follows. If $d_H(j) = n-3$ and j does not occur in row 0, add the edge $\{\rho^*, j\}$. Since there are at most two such vertices, at most two edges were added at this step. It follows that the vertices j mentioned above now have degree $n-2$ and the degree of ρ^* is at most 2. So, H^* has maximum degree $n-2$, and $d_{H^*}(\rho^*) \leqslant 2 < n-2$.

Properly edge-color the graph H^* with $n-2$ colors. Let k be a color not occurring on an edge incident with vertex ρ^*. Fill the cells in column 0 of R^* by placing symbol j in cell $(i,0)$ if and only if edge $\{\rho_i, j\}$ is colored k. Thus, all the symbols with corresponding vertices of maximum degree were placed in column 0 of R^*.

Note, small symbols, except symbol 1, occurring once in R were placed in both row 0 and column 0, and are finished. Small symbols occurring twice in R were placed in either row 0 or column 0, and also are finished. So, all small symbols except symbol 1 are finished. Therefore, since BIG symbols are finished, symbol 1 is the only possible unfinished symbol, and the only symbol that may not occur in $R^- \cup c_0^-$.

Recall, if symbol 1 occurs twice in R it occurs once in R^- and was added to row 0; so it would be finished. Hence, in this case, all symbols would occur in $R^- \cup c_0^-$ and would be finished.

Otherwise, we can suppose symbol 1 occurs only once in R (in row 1). Then it does not occur in R^-, but was added to row 0. Thus, symbol 1 occurs only twice in R^* and is unfinished. Now, to satisfy the conditions imposed by (a)–(e), symbol 1 must be placed in column 0.

If column 0 has an empty cell, say cell $(x, 0)$. Certainly, x is neither 1 nor 0. Thus, since symbol 1 does not occur in R^-, it does not occur in row x. Place symbol 1 in cell $(x, 0)$.

So, we can suppose column 0 has no empty cells. Then, by construction, the $n-1$ cells of column 0 in $R^- \cup c_0^-$ contain exactly the symbols $2, \ldots, n$. Now, since row 1 is missing two BIG symbols it must contain at least two small symbols; symbol 1 and some other small symbol, say symbol v. Since R is idempotent, symbol v also occurs in cell (v, v). Thus, v occurs at least once in R^- and at least twice in $R^- \cup c_0^-$. Let cell $(y, 0)$ be the cell containing v in column 0. Move symbol v to cell $(y, -1)$ and then place symbol 1 in cell $(y, 0)$. After this last step, symbol v and symbol 1 both occur once in $R^- \cup c_0^-$ and are both finished. Therefore, all symbols occur at least once in $R^- \cup c_0^-$ and all symbols are finished.

Thus, the partial the partial idempotent latin rectangle R^* has been formed satisfying the conditions (a)–(e) of this lemma in the case where $\mu > 6$ and either there exists a row missing the symbol $2n$ and some other BIG symbol or all BIG symbols occur at least four times in R.

Subcase (iib): Finally, we can assume that $N_R(2n) \leqslant 3$, and that every row missing symbol $2n$ contains every other BIG symbol. Thus, in this special case, row a cannot be chosen to be missing symbol $2n$ (that is, row a contains symbol $2n$). Again, for notational convenience (and without loss of generality) we assume $a = 1$. Begin forming R^* by adding row 0 and column 0, and column -1. Place symbol s (symbol t) in cells $(0, 0)$ and $(1, -1)$ (cells $(0, -1)$ and $(1, 0)$).

Thus, the only unfinished BIG symbol, $2n$, must be placed exactly $4 - N_R(2n)$ times in any of row 0, column 0, and column -1. Furthermore, all small symbols occurring once or twice in R need to be placed twice or once respectively in any of row 0, column 0, and column -1.

Again, since row 1 is missing two BIG symbols, it must contain at least two small symbols; symbol 1 and some other small symbol, which we have called symbol v. This symbol v occurs at least twice in R (in rows 1 and v).

To fill some empty cells in row 0, start by creating a bipartite graph $G = (C, S)$ with vertex sets $C = \{c_1, c_2, \ldots, c_n, c^*\}$ and $S = \{1, 2, \ldots, n, 2n\} \setminus \{v\}$. Let $E(G) = \{\{c_i, j\}:$

column i in R is missing the symbol $j\}$. Then, in the graph G, for each i in the set $\{1,\ldots,n\}$ and each j in the set $\{1,\ldots,n,2n\}\backslash\{v\}$, we have the following:

$$d_G(c_i)\leqslant n-1=|S\setminus\{i\}| \quad \text{for all } i \text{ different from } v,$$
$$d_G(c_v)\leqslant n=|S|,$$
$$d_G(c^*)=0,$$
$$d_G(j)=n-N_R(j)\leqslant n-1.$$

Now, the immediate goal is to form a new graph G^* from G with maximum degree $n-1$ in such a way that vertex $2n$ has maximum degree, all vertices with corresponding symbols occurring only once in R have maximum degree, and $\min\{n-(N_R(2n)+2),n_2\}$ vertices with corresponding symbols occurring twice in R have maximum degree.

Form G^* from G as follows. Add exactly $N_R(2n)-1$ edges of the form $\{c^*,2n\}$ so that the resulting degree of $2n$ is $n-1$ (so G^* may be a multigraph). Note, since $N_R(2n)$ is at most three, at most two edges were added to c^*. Next, if $d_G(c_v)=n$, then vertex c_v is adjacent to all vertices in the set S; remove the edge $\{c_v,v-1\}$ and add an edge $\{c^*,v-1\}$. This last step leaves the degree of $v-1$ unchanged, and results in the degree of c_v being at most $n-1$ and the degree of c^* being at most $N_R(2n)$. Lastly, if the $d_G(j)=n-2$, add an edge $\{c^*,j\}$ for exactly $\min\{n-(N_R(2n)+2),n_2\}$ such vertices. It follows that the vertices j mentioned above now have degree $n-1$ and that at most an additional $n-(N_R(2n)+2)$ edges are added to c^*. Thus, $d_{G^*}(c^*)\leqslant(N_R(2n)-1)+1+(n-(N_R(2n)+2))=n-2<n-1$.

Properly edge-color the graph G^* with $n-1$ colors. Let k be a color not occurring on an edge incident with vertex c^*. Fill the cells in row 0 of R^* by placing symbol j in cell $(0,i)$ if and only if edge $\{c_i,j\}$ is colored k. Thus, all symbols with corresponding vertices of maximum degree have been placed in row 0 of R^*.

Note, small symbols occurring once in R were placed in row 0, so must be placed once more in either column 0 or column -1. Furthermore, $\min\{n-(N_R(2n)+2),n_2\}$ small symbols occurring twice in R were also placed in row 0 and are finished; so at most $N_R(2n)+2$ such small symbols, including v, have to occur once more in either column 0 or column -1. Therefore, all small symbols occur at least twice in R^*, so each must be added at most once in column 0 or column -1. Lastly, the BIG symbol $2n$ was placed in row 0, so it must be added exactly $3-N_R(2n)\leqslant 2$ times in any of column 0 and column -1. This is the only symbol that may need to be added to both column 0 and column -1. Therefore, since symbol $2n$ is the only unfinished BIG symbol, the set of unfinished symbols consists of small symbols occurring once in R the BIG symbol $2n$, and at most $N_R(2n)+2$ small symbols occurring twice in R.

Now, there are $2n-2$ empty cells in columns 0 and -1 and, at most $n+2$ not necessarily distinct symbols (n small symbols and two copies of symbol $2n$) still to be placed in these cells in order that all symbols satisfy conditions (a)–(e). Assuming $n\geqslant 5$, $2n-2\geqslant n+2$; so there are enough cells to accommodate all of these $n+2$ symbols.

Since, in this case, every row missing symbol $2n$ contains every other BIG symbol, symbols $n+1,\ldots,2n-1$ are all contained in R^-. Furthermore, since R is idempotent, symbols $2,\ldots,n$ also occur at least once in R^-. Therefore, symbols $2,\ldots,2n-1$ satisfy condition (b), while symbols 1 and $2n$ may not satisfy this condition.

Suppose $N_R(2n)=1$. Then row 1 is the only row in R containing symbols $2n$ and 1, so each of the rows i for $i\in\{2,\ldots,n\}$, contain the symbols i, $n+1,\ldots,2n-1$. Fill empty cells of column 0 and column -1 by first placing symbol $2n$ (symbol 1) in cell $(n,0)$ (cell $(2,0)$). Thus all symbols now occur at least once in $R^-\cup c_0^-$ (that is, all symbols satisfy condition (b)). Next, place symbol $2n$ (symbol 2) in cell $(2,-1)$ (cell $(3,-1)$). Lastly, we can place symbol j in cell $(j-1,0)$ for $j\in\{3,\ldots,n\}$. Therefore, all symbols are finished.

Secondly, suppose $N_R(2n)=2$. Then row 1 and some other row, say row b, in R contain symbol $2n$. Hence, symbol $2n$ occurs in R^- and satisfies condition (b). If symbol 1 does not occur in row b, then it is the only symbol not occurring in R^-, and we can place symbol 1 in cell $(b,0)$. Otherwise, if symbol 1 occurs in row b, then all symbols occur in R^-. Thus, in either case, all symbols satisfy condition (b). Next, to fill some empty cells of columns 0 and -1 start by placing symbol j in cell $(j-1,0)$ for $j\in\{3,\ldots,n\}\setminus\{b+1\}$. Since row b is different from row 1, we can place symbol $b+1$ in cell $(2,-1)$. Lastly, since $n\geqslant 5$, $n-2\geqslant 3$, and since the three symbols 1,2, and $2n$ do not occur in any of the rows i for $i\in\{3,\ldots,n\}\setminus\{b\}$, these symbols can be greedily placed in column -1. Therefore, all symbols are finished.

Lastly, suppose $N_R(2n)=3$. Then row 1 and two other rows, say rows b and c, in R contain symbol $2n$. Hence, symbol $2n$ occurs in R^- and satisfies condition (b). Furthermore, symbol $2n$ has been placed in row 0, so symbol $2n$ is finished. If symbol 1 does not occur in one of these two rows, say row b, then it is the only symbol not occurring in R^-, and we can place symbol 1 in cell $(b,0)$. Otherwise, if symbol 1 occurs in both rows b and c, then $N_R(1)=3$ and this symbol is finished. Thus, in either case, all symbols satisfy condition (b). Next, to fill some empty cells of columns 0 and -1 start by placing symbol j in cell $(j-1,0)$ for $j\in\{3,\ldots,n\}\setminus\{b+1,c+1\}$. Since row b is different from row 1, we can place symbol $b+1$ in cell $(2,-1)$. If symbol $c+1$ does not occur in one of the rows b and c, say row b, place symbol $c+1$ in cell $(b,-1)$. Otherwise, if symbol $c+1$ occurs in both rows b and c, then $N_R(c+1)=3$ and this symbol is finished. Lastly, since $n\geqslant 5$, $n-3\geqslant 2$, and since the symbol 2 does not occur in any of the rows i for $i\in\{3,\ldots,n\}\setminus\{b,c\}$, this symbol can be greedily placed in column -1. Therefore, all symbols are finished.

Therefore, when $N_R(2n)\leqslant 3$ and every row missing symbol $2n$ contains every other BIG symbol, the partial idempotent latin rectangle R^* has been formed satisfying the conditions (a)–(e) of the Lemma. □

The following lemma address the difficulty of filling the empty cells of R^+ formed in Lemma 2.3.

Lemma 2.4. *For* $n\geqslant 5$, *let* $R^*=R\cup\rho_0\cup c_0\cup c_{-1}$ be an $(n+1)\times(n+2)$ *partial latin rectangle on the symbols* $1,\ldots,2n$ *in which*:

(i) *the only empty cells occur in row* 0, *column* 0, *or column* -1,

(ii) *each symbol occurs at least once in* $R^- \cup c_0^-$ *and at least once in* R,

(iii) R *is an incomplete idempotent latin square of order* n *on the symbols* $1,\dots,2n$, *and*

(iv) *cells* $(0,-1),(a,0),(0,0)$, *and* $(a,-1)$ *are filled with two BIG symbols* s *and* t *for some row* a *in* R.

Then there exists an $(n+1)\times(n+2)$ *incomplete latin rectangle* R^+ *with the properties that*:

(a) $R(i,j)=R^+(i,j)$ *for all* $i, j\in\{1,\dots,n\}$ *and for* $(i,j)\in\{(0,-1),(0,0),(a,-1),(a,0)\}$, *and*

(b) $N_{R^+}(j)\geqslant N_{R^*}(j)$ *for all* $j\in\{1,\dots,2n\}$.

Proof. Let R^* be a partial latin rectangle satisfying the conditions of the lemma. The incomplete latin rectangle R^+ will be formed from R^* by first filling any empty cells in row 0, then any in column 0, and lastly any in column -1.

If row 0 has three or more empty cells then one of the empty cells $(0,j)$ can be greedily filled. That is, since column j contains n symbols and row 0 currently contains $n-1$ symbols, at least one symbol occurs in neither column j nor row 0.

Suppose row 0 has two empty cells, say cells $(0,n-1)$ and $(0,n)$. If row 0 and column $n-1$ of R have at least one common symbol, then cell $(0,n-1)$ can be greedily filled. So, suppose cells $(0,n-1)$ and $(0,n)$ are empty and row 0 and column $n-1$ have no common symbols. Now, since each symbol occurs at least once in R, symbol s must occur in some column, say column x of R. Hence, row 0 and column x have at least one symbol in common, namely symbol s. Furthermore, since row 0 and column $n-1$ have no common symbols, symbol $R^*(0,x)$ is not contained in column $n-1$. Thus, symbol $R^*(0,x)$ can be moved to cell $(0,n-1)$. The empty cell $(0,x)$ can then be greedily filled.

Lastly, suppose exactly one cell of row 0 is empty, say cell $(0,n)$. Then row 0 contains exactly $n+1$ symbols. Since each of these $n+1$ symbols must occur at least once in the n columns of R there must be at least one column in R with two symbols in common with row 0. Furthermore, since there are $2n+1$ filled cells within row 0 and each column of R and only $2n$ available symbols, row 0 has at least one symbol in common with each of the columns of R. In particular, row 0 and column n have at least one common symbol.

Now, if row 0 and column n have at least two common symbols, then cell $(0,n)$ can be greedily filled. Thus, suppose row 0 and column n have exactly one common symbol. If there are two columns of R with at least two symbols in common with row 0, say columns y and z, then at least one of the symbols $R^*(0,y)$ or $R^*(0,z)$ does not occur in column n, say $R^*(0,y)$. Thus, symbol $R^*(0,y)$ can be moved to cell $(0,n)$. The empty cell $(0,y)$ can then be greedily filled.

Furthermore, if there is a column y, with at least two symbols in common with row 0 and if the symbol $R^*(0,y)$ does not occur in column n, then symbol $R^*(0,y)$ can be moved to cell $(0,n)$ and the empty cell $(0,y)$ can then be greedily filled.

Thus, we can suppose column y is the only column in R with at least two symbols in common with row 0 and we can suppose symbol $R^*(0,y)$ is the only symbol common to both row 0 and column n. In this case, each column of R other than column y, has exactly one symbol in common with row 0.

If there exists a column q of R that does not contain symbol $R^*(0,y)$, then certainly symbol $R^*(0,q)$ does not occur in column n. Thus, symbol $R^*(0,q)$ can be moved to cell $(0,n)$ and then symbol $R^*(0,y)$ can be moved to cell $(0,q)$. The empty cell $(0,y)$ can then be greedily filled.

Otherwise, suppose each column of R other than column y contains the symbol $R^*(0,y)$. Then symbol $R^*(0,y)$ is the only symbol common to row 0 and each of the columns of R except column y. Thus no symbol in row 0 other than $R^*(0,y)$ occurs in any column of R other than column y. That is, the n symbols in row 0, excluding $R^*(0,y)$, occur only once in R and that occurrence is in column y. Hence, each of the other columns in R contain exactly the same set of n symbols. Now, since R is idempotent, each column i contains the symbol i. Thus, column y contains exactly one small symbol, namely symbol y, and each of the other columns contain all the other $n-1$ small symbols. Therefore, there is exactly one big symbol in the set of n symbols contained in each of the columns of R excluding column y. Hence R satisfies the conditions of Lemma 2.2 and can be embedded into the incomplete latin rectangle R^+.

Therefore, row 0 has been completely filled. Next, we show how to fill the empty cells of column 0.

If column 0 has two or more empty cells, then one of the empty cells can be greedily filled. Thus, suppose only one cell of column 0 is empty, say cell $(n,0)$. If row n and column 0 have at least one common symbol, then this cell can be greedily filled.

So, we can suppose column 0 and row n have no common symbols. Recall that symbol s or t, say s, occurs in cell $(1,0)$. Hence, symbol s does not occur in row 1 of R. Now, since every symbol occurs at least once in R it must be that symbol s occurs in some row x of R with x different from 1. So, row x and column 0 have at least one symbol in common, namely symbol s. Furthermore, since row n and column 0 have no common symbols, symbol $R^*(x,0)$ is not contained in row n. Thus, symbol $R^*(x,0)$ can be moved to cell $(n,0)$. The empty cell $(x,0)$ can then be greedily filled.

Therefore, column 0 has been completely filled. Lastly, we will fill the empty cells of column -1.

If column -1 has three or more empty cells, then one of the empty cells can be greedily filled. Thus, we first suppose exactly two cells of column -1 are empty, say cells $(n-1,-1)$ and $(n,-1)$. If row $n-1$ and column -1 have at least one common symbol, then cell $(n-1,-1)$ can be greedily filled.

So, suppose column -1 and row $n-1$ have no common symbols. Since symbol s occurs at least once in R^-, then there exists a row of R^- containing the symbol s, say row x. Hence column -1 and row x of $R^- \cup c_0^-$ have at least one common symbol, namely symbol s. Furthermore, since we are assuming column -1 and row $n-1$ have no common symbols, symbol $R^*(x,-1)$ does not occur in row $n-1$. Thus, symbol $R^*(x,-1)$ can be moved to cell $(n-1,-1)$. The empty cell $(x,-1)$ can then be greedily filled.

Lastly, suppose exactly one cell of column -1 is empty, say cell $(n,-1)$. Then column -1 contains exactly n symbols. Since each of these n symbols must occur at least once in the $n-1$ rows of $R^- \cup c_0^-$, there must be at least one row in $R^- \cup c_0^-$ with at least two symbols in common with column -1. Furthermore, since there are $2n+1$ filled cells within column -1 and each row of $R^- \cup c_0^-$, and since there are only $2n$ available symbols, each of the rows in $R^- \cup c_0^-$ has at least one symbol in common with column -1. In particular, column -1 and row n have at least one common symbol.

Now, if column -1 and row n have at least two common symbols, then cell $(n,-1)$ can be greedily filled. So, we can suppose column -1 and row n have exactly one common symbol.

If there are two rows of $R^- \cup c_0^-$ with at least two symbols in common with column -1, say row y and row z, then one of the symbols $R^*(y,-1)$ or $R^*(z,-1)$ does not occur in row n of $R^- \cup c_0^-$, say $R^*(y,-1)$. Thus, symbol $R^*(y,-1)$ can be moved to cell $(n,-1)$. The empty cell $(y,-1)$ can then be greedily filled.

Furthermore, if there is a row of $R^- \cup c_0^-$, say row y, with at least two symbols in common with column -1 and if the symbol $R^*(y,-1)$ does not occur in row n of $R^- \cup c_0^-$, then symbol $R^*(y,-1)$ can be moved to cell $(n,-1)$ and the empty cell $(y,-1)$ can then be greedily filled.

Thus, we can suppose: that row y is the only row of $R^- \cup c_0^-$ with at least two symbols in common with column -1; that symbol $R^*(y,-1)$ is the only symbol common to both column -1 and row n; and that each row of $R^- \cup c_0^-$, other than row y, has exactly one symbol in common with column -1.

If there exists a row q of $R^- \cup c_0^-$, that does not contain the symbol $R^*(y,-1)$, then certainly symbol $R^*(q,-1)$ does not occur in row n. Thus, symbol $R^*(q,-1)$ can be moved to cell $(n,-1)$ and then symbol $R^*(y,-1)$ can be moved to cell $(q,-1)$. The empty cell $(y,-1)$ can now be greedily filled.

Therefore, we can suppose each row of $R^- \cup c_0^-$ other than row y contains the symbol $R^*(y,-1)$. Then for each row j of $R^- \cup c_0^-$ other than y, the symbol in common to row j and column -1 is $R^*(y,-1)$. Thus, no symbol in column -1 of R^* other than $R^*(y,-1)$ occurs in any row of $R^- \cup c_0^-$ other than row y. That is, the $n-1$ symbols in column -1, other than $R^*(y,-1)$, occur only once in $R^- \cup c_0^-$, namely in row y. Hence, each of the other rows in $R^- \cup c_0^-$ contain exactly the same set S of $n+1$ symbols which includes $R^*(y,-1)$. Furthermore, since there are $n-2$ rows of $R^- \cup c_0^-$ containing the symbols in the set S, column 0 contains at least $n-2$ of the symbols in S. Symbols $R^*(0,0)$, $R^*(1,0)$, and $R^*(y,0)$ may not be in the set S.

Now, let $v \in \{2,\dots,n-1\}\backslash\{y\}$. Note, v exists since $n>4$. Choose w from the set $\{n,v\}$ in such a way that $R^*(w,0)$ is not the same symbol as $R^*(y,-1)$. Hence, $R^*(w,0)$ has been chosen from the set $S\backslash R^*(y,-1)$.

First, suppose $w=v$. Then, since no symbol in column -1 except symbol $R^*(y,-1)$ is an element of the set S, symbol $R^*(w,-1)$ is not in S. Hence, $R^*(w,-1)$ is not in row n of $R^- \cup c_0^-$. Thus, symbol $R^*(w,-1)$ can be moved to the empty cell $(n,-1)$.

Furthermore, since the symbol $R^*(w,0)$ from the set $S\backslash R^*(y,-1)$ is not contained in column -1, it can be moved to the empty cell $(w,-1)$. Otherwise, if $w=n$, then symbol $R^*(w,0)=R^*(n,0)$ is not contained in column -1, so $R^*(w,0)$ can be directly moved to cell $(n,-1)$. In either case, column 0 contains at least $n-3$ symbols from the set S. So, column 0 and row w have at least $n-3$ symbols in common. Thus, since $n\geqslant 5$, the remaining empty cell $(w,0)$ can then be greedily filled.

Therefore, column -1 has been completely filled, and the incomplete latin rectangle R^+ has been formed satisfying the conditions of the lemma. □

3. The main result

We are finally ready to present the main result of this paper.

Theorem 3.1. *For* $n\geqslant 5$, *let* R *be an incomplete idempotent latin square of order* n *on the symbols* $1,\ldots,2n$ *such that each symbol occurs at least once in* R. *Suppose* $N_R(j-1)\geqslant N_R(j)$ *for all* $j\in\{n+2,\ldots,2n\}$ *and let* $\mu=N_R(2n-1)+N_R(2n)$. *If any of the following is true*:

(i) *there does not exist a row or column in* R *missing two BIG symbols*, *or*

(ii) R *contains an* $n\times(n-1)$ *latin rectangle defined on* $n-1$ *small and* 1 *BIG symbols*, *or*

(iii) *there exists a row in* R *missing two BIG symbols when* $\mu\leqslant 6$ *and* $n\geqslant\mu+4$, *or*

(iv) *there exists a row in* R *missing two BIG symbols when* $\mu>6$,

then R *can be embedded in an idempotent latin square* T *of order* $2n$.

Proof. Let R be an incomplete idempotent latin square of order n on the symbols $1,\ldots,2n$ such that each symbol occurs at least once in R. Suppose $N_R(j-1)\geqslant N_R(j)$ for all $j\in\{n+2,\ldots,2n\}$ and let $\mu=N_R(2n-1)+N_R(2n)$.

In case (i), apply Lemma 2.1 to R to form an $(n+1)\times(n+2)$ incomplete latin rectangle R^+. In case (ii), apply Lemma 2.2 to R to form R^+. Lastly in cases (iii) and (iv), first apply Lemma 2.3 and then Lemma 2.4 to form R^+. Finally apply Proposition 2.1 to obtain the desired result.

An immediate consequence of Theorem 3.1 is the following corollary.

Corollary 3.1. *Let* $n>9$. *An incomplete idempotent latin square* R *of order* n *on* $2n$ *symbols can be embedded in an idempotent latin square* T *of order* $2n$ *if and only if* $N_R(j)\geqslant 1$ *for all* $j\in\{1,\ldots,2n\}$.

It is also worth noting that Theorem 3.1 addresses many of the cases where $n\leqslant 9$ (see Table 1). Some of the smallest values of n can be handled separately by the following lemmas.

Lemma 3.1. *Suppose R is an incomplete idempotent latin square of order* 3 *on the symbols* $1,\dots,6$ *such that each symbol occurs at least once in R and suppose* $N_R(j-1) \geqslant N_R(j)$ *for all* $j \in \{4,5,6\}$. *Let* $\mu = N_R(5) + N_R(6)$. *Then* $\mu \leqslant 4$.

Proof. Let R be an incomplete idempotent latin square satisfying the conditions of the lemma. Suppose $\mu > 4$. Then $N_R(6) \geqslant 2$ and $N_R(5) = 3$, so $N_R(4) \geqslant 3$. Therefore the number of cells containing BIG symbols is at least 8; but since R is idempotent, at most 6 cells in R can contain BIG symbols. Therefore, this is a contradiction to the original assumption. So, $\mu \leqslant 4$. □

Lemma 3.2. *Let R be an incomplete idempotent latin square of order n on the symbols* $1,\dots,2n$ *such that each symbol occurs at least once in R. Suppose* $N_R(j-1) \geqslant N_R(j)$ *for all* $j \in \{n+2,\dots,2n\}$. *Let* $\mu = N_R(2n-1) + N_R(2n)$. *If*

(i) $n = 2$, *or*

(ii) $n = 4$ *and* $\mu = 6$,

then R can be embedded into an idempotent latin square T of order $2n$.

Proof. Case (i): Let R be an incomplete idempotent latin square of order 2 satisfying the conditions of the lemma. The latin square R is of the following form (or is the transpose):

$R =$

1	3
4	2

R can be embedded in the following idempotent latin square T.

$T =$

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

Case (ii): Let R be an incomplete idempotent latin square of order 4 on the symbols $1,\dots,8$ satisfying the conditions of the lemma. We will first add row 0 and columns 0 and -1 to R to form and 5×6 latin rectangle R^+ satisfying the conditions of Proposition 2.1. Then using Proposition 2.1, R will be embedded into an idempotent latin square T of order 8.

Now, since R is idempotent, at most 12 cells in R can contain BIG symbols. Furthermore, since $N_R(5) \geqslant N_R(6) \geqslant N_R(7) \geqslant N_R(8)$ and since $\mu = 6$, $N_R(5) = N_R(6) = N_R(7) = N_R(8) = 3$ and $N_R(1) = N_R(2) = N_R(3) = N_R(4) = 1$. Thus, each row in R contains exactly three BIG symbols and one small symbol. So, since each BIG symbol occurs three times in R and since each row of R is missing exactly one BIG symbol, symbols $5, 6, 7,$

and 8 are missing from different rows. Without loss of generality, we can assume that symbol 7 (symbol 8) is missing from row 2 (row 1) and we can assume that $v=3$ (v is defined in Proposition 2.1 (iv)).

Begin forming R^+ by adding row 0, column 0, and column -1. Place symbol 7 (symbol 8) in cells $(0,0)$ and $(2,-1)$ (in cells $(0,-1)$ and $(1,0)$). Next, since symbol 3 occurs only in cell $(3,3)$, 3 is missing from both rows 1 and 2. Place symbol 3 in cells $(1,-1)$ and $(2,0)$. So, symbols 7 and 8 both occur four times in R^+ and symbol 3 occurs three times in R^+. Hence, each of these symbols is finished.

Now, since each BIG symbol is missing from exactly one column, let symbol 5 be missing from column x. Place symbol 5 in cell $(0,x)$. Then the small symbols $1,2$, and 4 can easily be placed in the remaining empty cells of row 0. Symbol 5 now occurs four times in R^+ and is finished. Small symbols $1,2$, and 4 each occur two times in R^+ so must be added once more in any of column 0 and column -1.

Lastly, the four remaining empty cells in columns 0 and -1 will be filled. Since each of the symbols $5,6,7$ and 8 is missing from a different row and since symbol 7 (symbol 8) is missing from row 2 (row 1), symbol 6 must be missing from either row 3 or row 4. First, place symbol 4 in cell $(3,0)$ and place symbol 6 in cell $(3,-1)$ or cell $(4,-1)$. Then, the symbols 1 and 2 can be greedily placed in empty cells. Symbol 6 now occurs four times in R^+ and is finished. Small symbols $1,2$, and 4 now occur three times in R^+ and is finished. Therefore, a 5×6 latin rectangle R^+ has been formed satisfying the conditions of Proposition 2.1. Apply Proposition 2.1 to obtain the desired embedding. □

The Table 1 indicates the cases that remain to be studied. The unsolved cases are represented by empty cells.

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