Discrete Mathematics 68 (1988) 9–13
North-Holland

BOUNDS OF THE LONGEST DIRECTED CYCLE LENGTH FOR MINIMAL STRONG DIGRAPHS

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Received 13 November 1985
Revised 26 February 1987

In this paper we present the upper and lower bounds of the longest directed cycle length for minimal strong digraphs in terms of the numbers of vertices and arcs. These bounds are both sharp. In addition, we give analogous results for minimal 2-edge connected graphs.

A variety of results of the length of the longest directed cycle of directed graph have been found. In particular, Bermond et al. [2] obtained the lower bound for strong digraphs with a sufficient number of arcs. Bondy [4] gave the lower bound for any strong digraph by means of the chromatic number. Thomassen [9] discussed this problem for digraph with constraints on degrees. For review of the concerned results see [3]. We use the terminology of [1].

In this paper we examine this problem for minimal strong digraphs. Our result is established on the most fundamental parameters, i.e., vertex number and arc number. In addition, we also obtain analogous results for minimal 2-edge connected graphs.

We use standard terminology. For the sake of clarity we repeat some important definitions. Let D be a digraph. If D has only one vertex, it is called trivial; otherwise, it is called nontrivial. The cyclomatic number of D, denoted by $\nu(D)$, is the cyclomatic number of its underlying graph. D is a strong digraph if it contains a directed (u, v)-path for any ordered pair of vertices (u, v). Let D be a digraph. A maximal strong subdigraph of D is called a dicomponent of D. If D is a strong digraph but $D-e$ is not strong for any arc e in D, then D is a minimal strong digraph. Let $w=(x, v_1, \ldots, v_k, y)$ be a directed path ($k \geq 1$) in a strong digraph such that both indegree and outdegree of each internal vertex v_i equal one. If $D-\{v_1, \ldots, v_k\}$ is also strong, then w is called a reducible chain of D. $D-\{v_1, \ldots, v_k\}$ is denoted by $D-w$, and called a digraph obtained from D by eliminating the reducible chain w.

Lemma 1 ([5; 1, p. 30–32]). *A nontrivial minimal strong digraph has at least one reducible chain.*

Lemma 2. *A minimal strong digraph D has $\nu(D)$ directed cycles such that every arc of D is contained in at least one of them.*

Proof. By induction on the cyclomatic number $v(D)$.

It holds trivially for $v(D)=0, 1$.

Suppose $v(D)\geqslant 2$. By Lemma 1, D has a reducible chain w. Clearly, $D-w$ is a minimal strong digraph with cyclomatic number equal to $v(D)-1$. By the induction hypothesis, $D-w$ has $v(D)-1$ directed cycles $\{C_i\}$ such that each arc of $D-w$ is contained in at least one of $\{C_i\}$. On the other hand, since D is a strong digraph there must be a directed cycle containing w in D. Adding it to $\{C_i\}$, we obtain desired $v(D)$ directed cycles. □

Lemma 3. *Any minimal strong digraph D with $v(D)\geqslant 2$ has at least two reducible chains.*

Proof. If $v(D)=2$, then D must be one of the two kinds of digraphs indicated in Fig. 1.

It is clear that such digraph has two reducible chains w_1 and w_2.

Now assume that Lemma 3 is not true. Let D be the counterexample with fewest arcs. Obviously $v(D)\geqslant 3$. Let $w=(x, v_1, \ldots, v_k, y)$ be a reducible chain of D. Then $v(D-w)=v(D)-1\geqslant 2$. It follows that $D-w$ has at least two reducible chains, w_1 and w_2. Since D has only one reducible chain w, each of w_1 and w_2 must contain exactly one end vertex of w as its internal vertex. Therefore we may assume $w_1=(x', v_1', \ldots, x, \ldots, v_m', y')$. Consequently $(x, \ldots, v_m', y')$ is a reducible chain of D (otherwise, D would not be a minimal strong digraph) which is different from w. This contradicts the assumption that D is a counterexample. □

Lemma 4. *Let D be a minimal strong digraph with $v(D)\geqslant 2$ and C denote a longest directed cycle of D. Then D has a reducible chain w such that it is arc-disjoint with C.*

Proof. Since $v(D)\geqslant 2$. If D is minimal, then a directed cycle in D has no chord thus there must exist an arc (u, v) in D such that $u\in V(C)$ and $v\notin V(C)$. Note that $D-(u, v)$ is not a strong digraph. Let $\bar{u}$ denote the di-component containing u. Clearly, $C\subset \bar{u}$. Condensing the dicomponent $\bar{u}$ into a vertex, still denoted as $\bar{u}$, we obtain a new digraph D' which is also a minimal strong digraph. If $v(D')=1$, then D' is a directed cycle. This cycle, consider as a reducible chain from $\bar{u}$ to $\bar{u}$,

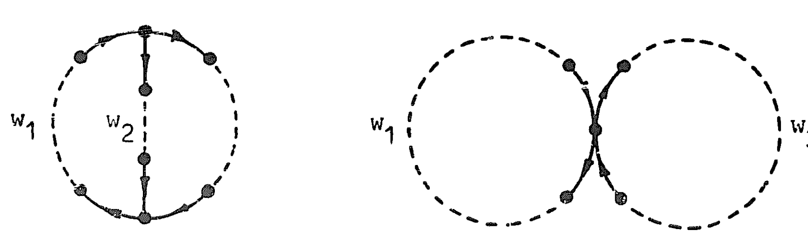

Fig. 1. Two kinds of minimal strong digraphs with $v(D)=2$.

is also a reducible chain in D. It is the desired reducible chain in D. It is the desired reducible chain because it is arc-disjoint with C. If $v(D') \geqslant 2$, by Lemma 3, D' has at least two reducible chains. Obviously, there must exist one chain w_1 such that $\tilde{u}$ is not an internal vertex of w_1. Then w_1 is also the desired reducible chain of D. Lemma 4 is thus proved. □

Now we give our main result.

Theorem 1. *Let D be a nontrivial minimal strong digraph with n vertices and m arcs. Let $l(D)$ denote the length of the longest directed cycle in D. Then*

$$\left\lceil \frac{m}{m-n+1} \right\rceil \leqslant l(D) \leqslant 2n - m.$$

Moreover, both these bounds are sharp.

Proof. By Lemma 2, the arc set of D can be covered by $v(D)$ directed cycles of D. Thus $m \leqslant l(D) \cdot v(D)$. Note that $v(D) = m - n + 1$. We have

$$\left\lceil \frac{m}{m-n+1} \right\rceil \leqslant l(D).$$

Now we use induction on $v(D)$ to prove $l(D) \leqslant 2n - m$. It holds trivially for $v(D) = 1$.

Suppose $v(D) \geqslant 2$. Let C be a longest directed cycle of D. By Lemma 4, there is a reducible chain w of D which is arc-disjoint with C. Let r be the length of w. It is not difficult to see the following:

(1) $D - w$ is a minimal strong digraph with $(m - r)$ arcs and $(n - r + 1)$ vertices;

(2) C is also a longest directed cycle of $D - w$;

(3) the cyclomatic number of $D - w$ is equal to $v(D) - 1$;

(4) $r \geqslant 2$.

By the induction hypothesis, we have

$$l(D) \leqslant 2(n - r + 1) - (m - r) = (2n - m) - (r - 2).$$

Therefore

$$l(D) \leqslant 2n - m.$$

Finally, we can see that these bounds are sharp from the following examples (Fig. 2).

The proof of Theorem 1 is thus completed. □

Remark. Given n and m, the two kinds of extremal digraphs are not uniquely determined.

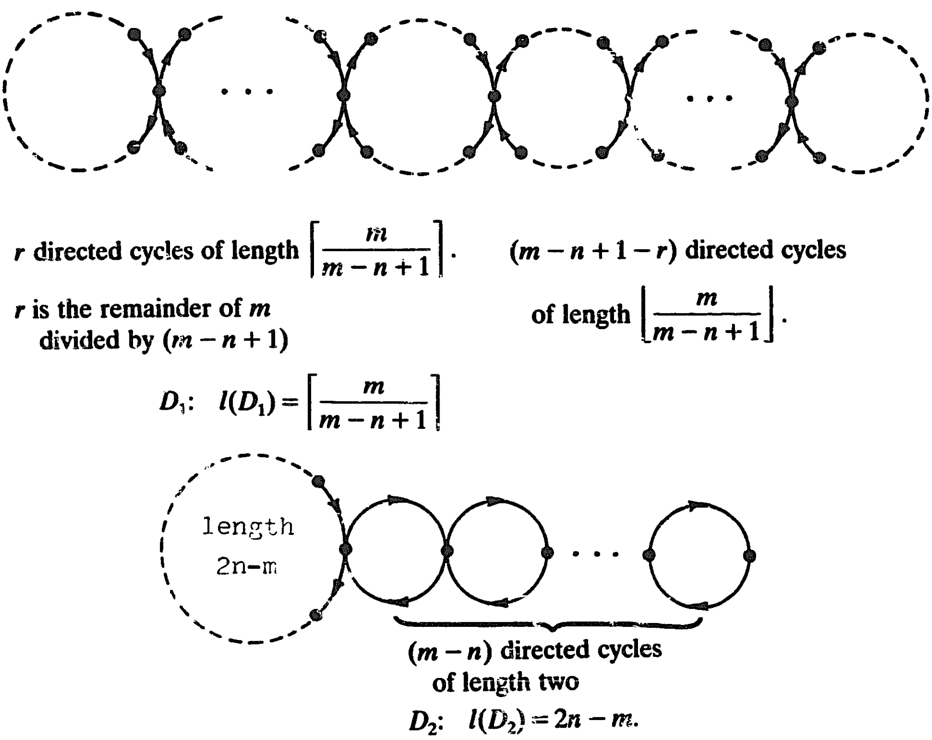

Fig. 2. Examples of two kinds of extremal digraphs.

The following proposition characterizes the extremal digraphs reaching the upper bound.

Proposition 1. *Under the condition of Theorem* 1, $l(D) = 2n - m$ *iff D can be converted into a directed cycle by successively eliminating the reducible chains of length* 2.

Proof. Suppose that $l(D) = 2n - m$. Because every non-trivial minimal strong digraph has a reducible chain, we can successively eliminate the shortest reducible chains until a directed cycle C is finally obtained. From Lemma 4, we can make C be a longest directed cycle in D. Hence, the number of the eliminated arcs is $m - l(D) = 2(m - n)$. On the other hand, the number of the eliminated chains is $v(D) - 1 = m - n$. Note that each chain at at least two arcs. Therefore, all eliminated reducible chains have length 2.

The converse is very clear. Our proof is thus completed. □

As for the case of reaching the lower bound, we can not present a simple characterization as above. We do not intend to discuss it here.

From Theorem 1 we can obtain an immediate corollary which is a known result due to Gupta [7].

Corollary 1. *If D is a nontrivial minimal strong digraph, then* $n \leqslant m \leqslant 2(n-1)$.

Proof. It is obvious that $n \leqslant m$. The rest can be easily induced by the inequality of Theorem 1. Since $m > n - 1$ and $n > 1$, $0 < m - n + 1 < m$. Then $m/(m - n + 1) > 1$. □

By using a theorem of Nash–Williams [8], the above results can be transferred to the minimal 2-edge connected graphs which had been considered by Zhu Biwen and Niu Yanyin [6]. Their deeper results are for minimal 2-edge connected graphs without cut vertices. We have

$$\left\lceil \frac{2m}{m-n+2} \right\rceil \leqslant l(G) \leqslant 2n - m.$$

We shall not discuss it here.

Remark. There is an example without cut vertices other than D_1 (the underlying graph is 2-connected): take q internally disjoint paths of length p from a to b, plus an arc ba.

Acknowledgment

The authors are grateful to the referee for his (her) comments which is described as the remark in this paper.

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