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Characterization of stochastic processes which stabilize linear companion form systems ☆

John Kao[a,*], Volker Wihstutz[b]

[a] *University of San Fransisco, San Fransisco, CA 94117-1080, USA*
[b] *University of North Carolina at Charlotte, Charlotte, NC 28223, USA*

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Abstract

The class of stochastic processes is characterized which, as multiplicative noise with large intensity, stabilizes a linear system with companion form $d \times d$-matrix. This includes the characterization of parametric noise which stabilizes the damped inverse pendulum. The proof yields also an expansion of the top Lyapunov exponent in terms of the noise intensity as well as a criterion for a stationary diffusion process permitting a stationary integral and it shows that stabilizing noise averages the Lyapunov spectrum. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Goal of the paper

It is known that the damped inverse linearized pendulum cannot be stabilized via parameter excitation by means of white noise $\mathrm{d}W$, W a Wiener process, although improved stability behavior, i.e. lower instability, can be observed in presence of white noise with small intensity (Pardoux and Wihstutz, 1988). On the other hand, perturbed white noise such as $\mathrm{d}F = -\gamma_0 F\,\mathrm{d}t + \gamma_1\,\mathrm{d}W$, F an Ornstein–Uhlenbeck process ($\gamma_0, \gamma_1 > 0$), which has the same "amount of randomness" as white noise, does stabilize the inverse pendulum: if its intensity is high enough (see Kao and Wihstutz, 1994). This phenomenon also has been made visible by simulations as well as by means of physical experiments (recently Popp, 1995). Surprisingly, the same stabilizing effect can be achieved with the help of degenerate noise $\mathrm{d}F = f_0(\xi)\,\mathrm{d}t$, where ξ is a "very thin" stationary background noise (Kao and Wihstutz, 1994).

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* Corresponding author. Tel.: +1-415-422-6760.
E-mail address: kao@usfca.edu (J. Kao).

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To embellish the picture, we recall that the stable equilibrium point of the harmonic oscillator is destabilized by white noise as well as by noise $\mathrm{d}F_t = f(\xi_t)\,\mathrm{d}t$ from a wide class of stationary processes ξ_t whatever its intensity (see Molchanov, 1978; Kotani, 1984).

So naturally, the question arises: which kind of noise is stabilizing and which is destabilizing?

1.2. Noise perturbed system

We are interested in systems which can be derived from differential equations of order d, $y^{(d)} - a_d y^{(d-1)} - \cdots - a_2 y' - a_1 y = 0$, that is, putting $x = [y, y', \ldots, y^{(d-1)}]^{\mathrm{T}}$, in systems of the form

$$\mathrm{d}x = Ax\,\mathrm{d}t, \tag{1.1}$$

where A is a $d \times d$ companion form matrix,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ & & 1 \\ a_1 \ \cdots & a_{d-1} & a_d \end{bmatrix}. \tag{1.2}$$

Our goal is to introduce noise into the parameters a_i in such a way that the trivial solution $x(t) \equiv 0$ becomes globally exponentially stable, if it was unstable (and preserves and enhances its stability properties, if it was stable). So we consider the stochastic differential equation

$$\mathrm{d}x^\varepsilon = Ax^\varepsilon\,\mathrm{d}t + Ux^\varepsilon \circ \mathrm{d}F_t^\varepsilon \tag{1.3}$$

(Stratonovich form), $\varepsilon > 0$, where

$$U = \begin{bmatrix} 0 \ \cdots & 0 & 0 \\ & & \\ u_1 \ \cdots & u_{d-1} & u_d \end{bmatrix}. \tag{1.4}$$

As to the noise $\mathrm{d}F_t^\varepsilon$, in order to disallow systematic change we restrict ourselves to *mean zero noise,* that is, $\mathbb{E}F_t^\varepsilon = \mathrm{const} = 0$. (Here we assume without loss of generality that the constant is zero.) On the other hand, the permitted class of noise processes should be large enough to contain both real and white noise. Finally, we want to stay in the Markovian framework. This in mind, we begin with a background process ξ_t for which, for the sake of simplicity, we first assume properties which are more restricted than necessary. They will be relaxed later in order to include such common processes as, e.g. the Ornstein–Uhlenbeck process (see Section 6.2).

Background noise. More precisely, let ξ_t be a stationary and ergodic process on a connected compact Riemannian C^∞-manifold M satisfying the stochastic differential equation

$$\mathrm{d}\xi_t = X_0(\xi_t)\,\mathrm{d}t + \sum_{k=1}^{r} X_k(\xi_t) \circ \mathrm{d}W_t^k, \tag{1.5}$$

where $(W_t^1,\dots,W_t^r)$ are independent standard Wiener processes over the probability space $(\Omega,\mathscr{F},P)$ with filtration $(\mathscr{F}_t)$ and $X_0,X_1,\dots,X_r$ are C^∞-vector fields on M. We assume further that the generator

$$\mathscr{G}=(X_0\cdot\nabla_\xi)+\frac{1}{2}\sum_{k=1}^{r}(X_k\cdot\nabla_\xi)^2 \tag{1.6}$$

is uniformly elliptic, and denote the unique invariant probability measure with v.

We remark that under these conditions, $f(\xi)$ is p-integrable for any measurable function f on M and $p\geqslant 1$, and the Poisson equation $\mathscr{G}u=f-\mathbb{E}f$ has a solution on all of M which is unique up to a constant. (See e.g. Orey, 1971, p. 29 or Papanicolaou, 1978, p. 127.)

Random vibration. With help of ξ_t we define the semi-martingale

$$\mathrm{d}F_t=f_0(\xi_t)\,\mathrm{d}t+\sum_{k=1}^{r}f_k(\xi_t)\circ\mathrm{d}W_t^k, \tag{1.7}$$

where $f_0,f_1,\dots,f_r$ are real-valued C^∞-functions on M. If

$$\varphi_0(\xi)=f_0(\xi)+\frac{1}{2}\sum_{k=1}^{r}X_k(\xi)\cdot\nabla_\xi f_k(\xi) \tag{1.8}$$

denotes the Itô-drift, then due to stationarity of ξ_t

$$\mathbb{E}F_t-\mathbb{E}F_0=\int_0^t\mathbb{E}\varphi_0(\xi_\tau)\,\mathrm{d}\tau+\sum_{k=1}^{r}\mathbb{E}\int_0^t f_k(\xi_\tau)\,\mathrm{d}W_\tau^k=K_0t,$$

where $K_0:=\mathbb{E}\varphi_0(\xi_0)$, whence

$$\mathbb{E}\mathbb{F}_t=\text{const}\quad\text{iff}\quad K_0=0. \tag{1.9}$$

We have assumed $\mathbb{E}\mathbb{F}_t=\text{const}$ or $K_0=0$. That is, we are dealing with a stochastic analogue of what Meerkov (1980), Bellman et al. (1986) and others call vibrational control. Our noise can be considered as *random vibration*, meaning that the random vectorfield $x\mapsto[A+\varphi_0U+\sum_k f_kU\dot{W}]x$ averages out to the vectorfield $x\mapsto Ax$ of the unperturbed system $\mathrm{d}x=Ax\,\mathrm{d}t$.

Speeding up. Expecting the need of high energy for stabilization we introduce $\xi_t^\varepsilon:=\xi_{t/\varepsilon}$ and $F_t^\varepsilon:=F_{t/\varepsilon}$ $(\varepsilon\to 0)$.

1.3. Lyapunov exponents

Due to smoothness, for any $\mathscr{F}_0$-measurable initial condition x_0 there is a unique (strong) solution $x^\varepsilon(t,x_0,\omega)=x^\varepsilon(t,x_0,\omega;U,\mathrm{d}F^\varepsilon)$ of (1.3) for all $t\geqslant 0$ (see, e.g. Ikeda and Watanabe, 1987, p. 235) and we may define the pathwise exponential growth rates or Lyapunov exponents of (1.1) as

$$\lambda^\varepsilon(x_0,\omega):=\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|x^\varepsilon(t,x_0,\omega)\|. \tag{1.10}$$

1.4. Stabilizing noise

The question whether or not the trivial solution $x^\varepsilon(t) \equiv 0$ of (1.3) is globally exponentially stable is governed by the largest Lyapunov exponent of (1.3), given $\mathrm{d}F^\varepsilon$. We aim to make this exponent as small as possible. We introduce the following terminology.

Definition 1.1. $\mathrm{d}F^\varepsilon$ in Eq. (1.3) is called stabilizing noise, if for suitable entries $u_1,\dots,u_d$ (of U) the Lyapunov exponents (1.10) satisfy the following condition: for any $\delta > 0$ there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$ and all $x_0 \in \mathbb{R}^d$ P-a.s.

$$\lambda^\varepsilon(x_0,\omega) \leqslant \frac{1}{d}\mathrm{trace}(A) + \delta. \tag{1.11}$$

This is the best we can hope for, since in typical cases where Oseledec's theorem holds, the Lyapunov exponents sum up to $\sum \lambda_i^\varepsilon = \mathrm{trace}(A)$, whence $\lambda_{\max}^\varepsilon \geqslant \mathrm{trace}(A)/d$, for all $\varepsilon > 0$.

Stablizing mean zero noise will render the trivial solution globally exponentially stable if and only if $\mathrm{trace}(A) < 0$. In view of applications, we remark that in all mechanical or electrodynamical systems the trace of A is negative due to friction, damping, or resistance.

1.5. The main result

Theorem 1.2. *Let ξ_t be the stationary and ergodic background noise from* (1.5) *and F_t the semi-martingale* (1.7). *Then the following conditions are equivalent*:

(i) $\mathrm{d}F_t^\varepsilon$ *is stabilizing mean zero noise for* (1.3).

(ii) $\lim_{t\to\infty} \mathbb{E}F_t^2/t = 0$.

(iii) *There exists a measurable function ψ on M and an initial condition F_0 such that $F_t = \psi(\xi_t)$.*

(iv) *There exists an initial condition such that F_t is stationary and ergodic.*

1.6. Organization of the paper

The Lyapunov exponent λ^ε from (1.10) will be the starting point for our investigation. After having given λ^ε a representation which better suits our purposes, we will expand λ^ε in terms of ε by means of a homogenization procedure (Sections 2 and 3). In Section 4, we will draw from this expansion a necessary condition for $\mathrm{d}F^\varepsilon$ being stabilizing noise and give it a probabilistic interpretation. Moreover, this condition will enforce that the semi-martingale F_t is a function of the background noise, $F_t = \psi(\xi_t)$, therefore stationary and ergodic. In Section 5, we show that this property is sufficient for the noise to be stabilizing. This closes the circle. Examples are given in Section 6, where the assumptions on the background noise are relaxed. Section 7 concludes with observations for the whole Lyapunov spectrum and averaging properties of stabilizing noise.

2. Preliminaries for the proof

2.1. Trace-zero assumption

It suffices to consider trace$(A)=$ trace$(U)=0$, since if not, we proceed as follows. In order not to lose the companion form, we first transform $\mathbb{R}^d$ by a suitable trace and dimension-dependent linear transformation $T=T(\alpha)$, $\alpha=a_d/d$, and then subtract αI from $T^{-1}AT$. The first operation does not change either the Lyapunov exponents or the trace, while the second one changes both. The trace becomes zero and the Lyapunov exponents are shifted by $-\alpha$. For details, see Kao and Wihstutz (1994). The transformation does not change the form of U and we may choose from the beginning $u_d=0$. For the remainder of the paper we assume $a_d=u_d=0$.

2.2. Polar coordinates

For representing the Lyapunov exponent in a workable form it is convenient to rewrite our system

$$d\begin{bmatrix}\xi_t^\varepsilon\\ x_t^\varepsilon\end{bmatrix}=\left\{\begin{bmatrix}0\\ Ax^\varepsilon\end{bmatrix}+\frac{1}{\varepsilon}\begin{bmatrix}X_0(\xi^\varepsilon)\\ f_0(\xi^\varepsilon)Ux^\varepsilon\end{bmatrix}\right\}\mathrm{d}t+\frac{1}{\sqrt{\varepsilon}}\sum_{k=1}^{r}\begin{bmatrix}X_k(\xi^\varepsilon)\\ f_k(\xi^\varepsilon)Ux^\varepsilon\end{bmatrix}\circ\mathrm{d}W_t^k, \tag{2.1}$$

in polar coordinates. So let $s=x/\|x\|$ (s identified with $-s$ on the projective space $\mathbb{P}=\mathbb{P}^{d-1}$) and $\rho=\log\|x\|$, $x\neq 0$. Then we have

$$d\begin{bmatrix}\xi^\varepsilon\\ s^\varepsilon\\ \rho^\varepsilon\end{bmatrix}=\left\{\begin{bmatrix}0\\ h_A(s^\varepsilon)\\ q_A(s^\varepsilon)\end{bmatrix}+\frac{1}{\varepsilon}\begin{bmatrix}X_0(\xi^\varepsilon)\\ f_0(\xi^\varepsilon)h_U(s^\varepsilon)\\ f_0(\xi^\varepsilon)q_U(s^\varepsilon)\end{bmatrix}\right\}\mathrm{d}t$$

$$+\frac{1}{\sqrt{\varepsilon}}\sum_{k=1}^{r}\begin{bmatrix}X_k(\xi^\varepsilon)\\ f_k(\xi^\varepsilon)h_U(s^\varepsilon)\\ f_k(\xi^\varepsilon)q_U(s^\varepsilon)\end{bmatrix}\circ\mathrm{d}W_t^k \tag{2.2}$$

(Stratonovich form) where for any $d\times d$ matrix C

$$h_C(s):=Cs-q_C(s)s,\quad q_C(s):=s^{\mathrm{T}}Cs. \tag{2.3}$$

By integration, the component ρ_t^ε becomes

$$\rho_T^\varepsilon=\int_0^T Q^\varepsilon(\xi_t^\varepsilon,s_t^\varepsilon)\,\mathrm{d}t+\text{martingale}, \tag{2.4}$$

where

$$Q^\varepsilon(\xi,s)=q_A(s)+f_0(\xi)q_U(s)$$

$$+\frac{1}{2\varepsilon}\sum_{k=1}^{r}[(X_k\cdot\nabla_\xi f_k)(\xi)q_U(s)+f_k^2(\xi)(h_U\cdot\nabla_s q_U)(s)]. \tag{2.5}$$

Due to compactness there is at least one stationary and ergodic solution $(\xi_t^\varepsilon,s_t^\varepsilon)$ with invariant measure μ^ε on $M\times\mathbb{P}$ with marginal measure v on R. Therefore dividing

(2.4) by T and passing to the limit $T \to \infty$, noting that the martingale term goes to zero, we obtain the Furstenberg–Khasminsky-type representation

$$\lambda^{\varepsilon} = \int_{M\times\mathbb{P}} Q^{\varepsilon}(\xi,s)\mu^{\varepsilon}(\mathrm{d}\xi,\mathrm{d}s) =: \langle Q^{\varepsilon},\mu^{\varepsilon}\rangle, \tag{2.6}$$

which, possibly, is not the maximal exponential growth rate (see Has'minskii 1980, p. 225; Arnold et al., 1986; Kao and Wihstutz, 1994).

3. Expansion of λ^{ε}

In this section we will derive an expansion for λ^{ε} of the form

$$\lambda^{\varepsilon} = \varepsilon^{-1/3}\hat{\lambda}(K_1) + \varepsilon^{-1/6}\lambda_1(\varepsilon) + \lambda_2(\varepsilon) + \mathrm{O}(\varepsilon^{1/6}), \tag{3.1}$$

where $\hat{\lambda}(K_1)$, depending on a certain constant K_1, is positive, if K_1 is so, and $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ are bounded functions of ε. This form makes apparent that if $K_1 > 0$, then $\lambda^{\varepsilon} \to \infty$ as $\varepsilon \to 0$, from which we will obtain necessary conditions for stabilization.

3.1. Modifying the Furstenberg–Khasminsky representation of λ^{ε}

We consider the generators $\mathscr{A}^{\varepsilon}$ and $\mathscr{L}^{\varepsilon}$ of $(\xi_t^{\varepsilon}, s_t^{\varepsilon})$ and $(\xi_t^{\varepsilon}, s_t^{\varepsilon}, \rho_t^{\varepsilon})$, respectively, which are of the form

$$\mathscr{A}^{\varepsilon} = h_A\cdot\nabla_s + \frac{1}{\varepsilon}\left\{\mathscr{G} + \varphi_0(h_U\cdot\nabla_s) + \sum_{k=1}^{r} f_k(X_k\cdot\nabla_{\xi})(h_U\cdot\nabla_s) + \frac{1}{2}\sum_{k=1}^{r} f_k^2(h_U\cdot\nabla_s)^2\right\} \tag{3.2}$$

$$\mathscr{L}^{\varepsilon} = (h_A\cdot\nabla_s + q_A\cdot\nabla_{\rho}) + \frac{1}{\varepsilon}\left\{\mathscr{G} + \varphi_0(h_U\cdot\nabla_s + q_U\cdot\nabla_{\rho}) + \sum_{k=1}^{r} f_k(X_k\cdot\nabla_{\xi}) \times (h_U\cdot\nabla_s + q_U\cdot\nabla_{\rho}) + \frac{1}{2}\sum_{k=1}^{r} f_k^2(h_U\cdot\nabla_s + q_U\cdot\nabla_{\rho})^2\right\} \tag{3.3}$$

with $\mathscr{G}$ from (1.6) and Itô-drift φ_0 from (1.8). By virtue of the Fokker–Planck equation, $(\mathscr{A}^{\varepsilon})^{*}\mu^{\varepsilon} = 0$, we obtain from (2.6) for $g \in \operatorname{dom}\mathscr{A}^{\varepsilon}$,

$$\lambda^{\varepsilon} = \langle Q^{\varepsilon},\mu^{\varepsilon}\rangle + \langle g,(\mathscr{A}^{\varepsilon})^{*}\mu^{\varepsilon}\rangle = \langle \mathscr{A}^{\varepsilon}g + Q^{\varepsilon},\mu^{\varepsilon}\rangle. \tag{3.4}$$

A straightforward calculation shows for any smooth function $g(\xi,s)$, $\mathscr{A}^{\varepsilon}g + Q^{\varepsilon} = \mathscr{L}^{\varepsilon}(g+\rho)$, whence

$$\lambda^{\varepsilon} = \langle \mathscr{L}^{\varepsilon}(g+\rho),\mu^{\varepsilon}\rangle. \tag{3.5}$$

The task is now to find a suitable function $g(\xi,s)$ from which expansion (3.1) will become evident.

We first treat the *two-dimensional* case using a homogenization procedure. We then show that the *higher-dimensional* case can be reduced to dimension 2. This in mind, in the next three subsections we consider the matrices $A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix}$, $u \neq 0$.

3.2. Linear transformation

It will turn out that the following linear transformation of $\mathbb{R}^2$ yields the correct scaling for the asymptotics of λ^ε:

$$T = \begin{bmatrix} \varepsilon^{-1/6} & 0 \\ 0 & \varepsilon^{1/6} \end{bmatrix}. \tag{3.6}$$

We obtain

$$TAT^{-1} = \begin{bmatrix} 0 & \varepsilon^{-1/3} \\ \varepsilon^{1/3} a & 0 \end{bmatrix} = \varepsilon^{-1/3} N + \varepsilon^{1/3} \Gamma, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \tag{3.7}$$

$$TUT^{-1} = \varepsilon^{1/3} U. \tag{3.8}$$

Since the mappings $A \to h_A$ and $A \to q_A$ are both linear in A, in the new coordinates the generator $\mathscr{L}^\varepsilon$ reads

$$\widetilde{\mathscr{L}^\varepsilon} = \varepsilon^{1/3}(h_\Gamma \cdot \nabla_s + q_\Gamma \cdot \nabla_\rho) + \varepsilon^{-1/3}(h_N \cdot \nabla_s + q_N \cdot \nabla_\rho)$$
$$+ \frac{1}{\varepsilon}\{\mathscr{G} + \varepsilon^{1/3}\mathscr{G}_1 + \varepsilon^{2/3}\mathscr{G}_2\}, \tag{3.9}$$

where $\mathscr{G}$ is the generator of ξ from (1.6) and

$$\mathscr{G}_1 = \varphi_0(h_U \cdot \nabla_s + q_U \cdot \nabla_\rho) + \sum_{k=1}^{r} f_k(X_k \cdot \nabla_\xi)(h_U \cdot \nabla_s + q_U \cdot \nabla_\rho),$$

$$\mathscr{G}_2 = \frac{1}{2}\left(\sum_{k=1}^{r} f_k^2\right)(h_U \cdot \nabla_s + q_U \cdot \nabla_\rho)^2.$$

Since the Lyapunov exponents are invariant under linear transformation of the coordinates, we have

$$\lambda^\varepsilon = \langle \tilde{\mathscr{L}}^\varepsilon(g + \rho), \tilde{\mu}^\varepsilon \rangle \tag{3.10}$$

($\tilde{\mu}^\varepsilon$ the invariant measure in the new coordinates).

3.3. Homogenization

In order to average out the leading ξ-terms in $\tilde{\mathscr{L}}^\varepsilon$, as usual, we choose g of the form $g(\xi,s) = \gamma_0(s) + \varepsilon^{1/3} g_1(\xi,s) + \varepsilon^{2/3} g_2(\xi,s)$, put $g_0(s,\rho) = \gamma_0(s) + \rho$ and compute

$$(\mathscr{G} + \varepsilon^{1/3}\mathscr{G}_1 + \varepsilon^{2/3}\mathscr{G}_2)(g_0 + \varepsilon^{1/3} g_1 + \varepsilon^{2/3} g_2)$$
$$= \mathscr{G} g_0 + \varepsilon^{1/3}[\mathscr{G} g_1 + \mathscr{G}_1 g_0] + \varepsilon^{2/3}[\mathscr{G} g_2 + \mathscr{G}_1 g_1 + \mathscr{G}_2 g_0]$$
$$+ \varepsilon^{3/3}[\mathscr{G}_1 g_2 + \mathscr{G}_2 g_1] + \varepsilon^{4/3}[\mathscr{G}_2 g_2]. \tag{3.11}$$

Since g_0 depends only on s and ρ, $\mathscr{G}g_0=0$, and the choice of

$$g_1=-\mathscr{G}^{-1}(\mathscr{G}_1 g_0)=-\mathscr{G}^{-1}(\varphi_0)(h_U\cdot\nabla_s+q_U\cdot\nabla_\rho)g_0$$

yields $\mathscr{G}g_1+\mathscr{G}_1 g_0=0$. We may choose further

$$g_2=-\mathscr{G}^{-1}(\varphi_1-K_1)(h_U\cdot\nabla_s+q_U\cdot\nabla_\rho)^2 g_0,$$

where

$$\varphi_1=-\varphi_0\mathscr{G}^{-1}(\varphi_0)-\sum_{k=1}^{r}f_k(X_k\cdot\nabla_\xi)(\mathscr{G}^{-1}(\varphi_0))+\frac{1}{2}\sum_{k=1}^{r}f_k^2,$$

$$K_1=\mathbb{E}\varphi_1. \tag{3.12}$$

Then (3.11) is of order $\varepsilon^{2/3}$ and the coefficient of the $\varepsilon^{2/3}$-term on the right-hand side is given by $\mathscr{G}g_2+\mathscr{G}_1 g_1+\mathscr{G}_2 g_0=K_1(h_U\cdot\nabla_s+q_U\cdot\nabla_\rho)^2 g_0$. Note that for any matrix C,

$$(h_C\cdot\nabla_s+q_C\cdot\nabla_\rho)^2 g_0=(h_C\cdot\nabla_s)^2\gamma_0+h_C\cdot\nabla_s q_C,$$

taking into account that $g_0(s,\rho)=\gamma_0(s)+\rho$. With regards to (3.9) the homogenization results in

$$\tilde{\mathscr{L}}^\varepsilon(g+\rho)=\varepsilon^{-1/3}\{[K_1(h_U\cdot\nabla_s)^2+h_N\cdot\nabla_s]\gamma_0+[K_1(h_U\cdot\nabla_s)q_U+q_N]\}+\mathrm{O}(1). \tag{3.13}$$

3.4. Expansion, case $d=2$

We will see (Lemma 4.1) that for mean zero noise always $K_1\geqslant 0$. If $K_1>0$, then the operator

$$\hat{\mathscr{L}}=\frac{1}{2}(\sqrt{2K_1}h_U\cdot\nabla_s)^2+h_N\cdot\nabla_s,$$

can be regarded as generator of the projection onto the unit sphere, $\hat{s}=\hat{x}/\|\hat{x}\|$, of the diffusion process

$$\mathrm{d}\hat{x}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\hat{x}\,\mathrm{d}t+\sqrt{2K_1}\begin{bmatrix}0&0\\u&0\end{bmatrix}\hat{x}\circ\mathrm{d}W_t. \tag{3.14}$$

For this generator it is known (Pinsky and Wihstutz, 1991, p. 99) that one can solve the Poisson equation

$$\hat{\mathscr{L}}\gamma_0=-\hat{q}(s)+\hat{\lambda}, \tag{3.15}$$

where

$$\hat{\lambda}=\hat{\lambda}(K_1)=\langle\hat{q},\hat{\mu}\rangle>0,$$

$$\hat{q}=K_1(h_U\cdot\nabla_s)q_U+q_N \tag{3.16}$$

(and $\hat{\mu}$ the unique invariant measure on $\mathbb{P}^1$ for which $\hat{\mathscr{L}}^*\hat{\mu}=0$). Here $\hat{\lambda}$ is both the “Fredholm alternative” of (3.15) and the top Lyapunov exponent of the white-noise-driven system (3.14). With this choice of γ_0 (thus of $g=\gamma_0+\varepsilon^{1/3}g_1+\varepsilon^{2/3}g_2+\rho$) we obtain from (3.13) the expansion

$$\hat{\lambda}=\langle\tilde{\mathscr{L}}^\varepsilon(g+\rho),\tilde{\mu}^\varepsilon\rangle=\varepsilon^{-1/3}\hat{\lambda}(K_1)+\langle\mathrm{O}(1),\tilde{\mu}^\varepsilon\rangle \tag{3.17}$$

with $\hat{\lambda}(K_1)>0$ for $K_1>0$.

3.5. Expansion of λ^ε for general dimension d

The case of *general dimension*, $d \geqslant 2$, with A and U from (1.2) and (1.4), can be reduced to the two-dimensional case by choosing the linear transformation $T = \mathrm{diag}(\varepsilon^{-(d-1)/6}, \dots, \varepsilon^{-1/6}, \varepsilon^{1/6})$ — rather than (3.6) — and a suitable function $g(\xi, s) + \rho$ such that, after transformation, the leading term of $\tilde{\mathscr{L}}^\varepsilon(g+\rho)$ depends only on the last two components s_{d-1} and s_d and is of the same form as in (3.13).

To see that, we decompose $A = N^{(d)} + \Gamma^{(d)}$ with

$$N^{(d)} = \begin{bmatrix} 01 & 0 \\ \quad\diagdown & \\ & 1 \\ 0 & 0 \end{bmatrix}, \qquad \Gamma^{(d)} = \begin{bmatrix} 0 & \cdots & & 0 \\ a_1 & \cdots\ a_{d-2} & a_{d-1} & 0 \end{bmatrix} \tag{3.18}$$

and in order to split off the south-east 2×2 blocks we decompose further. We put $N^{(d)} = N_1 + N_2$, where

$$N_1 = \left[\begin{array}{c|c} 01 \quad & 0 \\ \diagdown & \\ & 10 \\ \hline 0 & 0 \end{array}\right], \qquad N_2 = \left[\begin{array}{c|c} 0 & 0 \\ \hline & 01 \\ 0 & 00 \end{array}\right]. \tag{3.19}$$

We introduce the matrices $\Gamma_1(\varepsilon)$ and Γ_2, whose first $(d-1)$ rows are zero and whose last rows are, respectively,

$$[\varepsilon^{(d-3)/6} a_1, \dots, \varepsilon^{1/6} a_{d-3}, a_{d-2}, 0, 0], \quad [0, \dots, 0, a_{d-1}, 0]$$

and we let $U_1(\varepsilon)$ and U_2 be defined analogously with last rows

$$[\varepsilon^{(d-3)/6} u_1, \dots, \varepsilon^{1/6} u_{d-3}, u_{d-2}, 0, 0] \quad \text{and} \quad [0, \dots, 0, u_{d-1}, 0].$$

Then

$$TAT^{-1} = \varepsilon^{-1/3}[N_2 + \varepsilon^{1/6} N_1] + \varepsilon^{1/3}[\Gamma_2 + \varepsilon^{1/6}\Gamma_1(\varepsilon)],$$

$$TUT^{-1} = \varepsilon^{1/3}[U_2 + \varepsilon^{1/6} U_1(\varepsilon)]. \tag{3.20}$$

Compare (3.20) with (3.7) and (3.8), note that N_1, $\Gamma_1(\varepsilon)$ and $U_1(\varepsilon)$ are of order ε^0 and that for $d=2$ these matrices vanish, while in that case $N_2 = N$, $\Gamma_2 = \Gamma$ and $U_2 = U$ from (3.7). Substituting $N_2 + \varepsilon^{1/6} N_1$, $\Gamma_2 + \varepsilon^{1/6}\Gamma_1(\varepsilon)$ and $U_2 + \varepsilon^{1/6} U_1(\varepsilon)$ for the matrices N, Γ and U in Section 3.2, we obtain

$$\begin{aligned} \tilde{\mathscr{L}}^\varepsilon(g^\varepsilon + \rho) &= \tilde{\mathscr{L}}^\varepsilon(\gamma_0(s) + \rho + \varepsilon^{1/3} g_1^\varepsilon + \varepsilon^{2/3} g_2^\varepsilon) \\ &= \varepsilon^{-1/3}\{[K_1(h_{U_2} \cdot \nabla_s)^2 + h_{N_2} \cdot \nabla_s]\gamma_0 + [K_1(h_{U_2} \cdot \nabla_s) q_{U_2} + q_{N_2}] \\ &\quad + \varepsilon^{1/6} R_1(\xi, s; \varepsilon) + R_2(\xi, s; \varepsilon)\}, \end{aligned} \tag{3.21}$$

where g_1^ε and g_2^ε are small perturbations of g_1 and g_2 from Section 3.3 and

$$\sup\{|R_i(\xi,s;\varepsilon)|;\, \xi\in M, s\in\mathbb{P}, \varepsilon>0\}\leqslant C<\infty, \quad i=1,2.$$

Now choose $\gamma_0(s)$ only depending on s_{d-1} and s_d with $s_{d-1}^2+s_d^2=1$. Then, as for $d=2$, if $K_1>0$, there is a function $\gamma_0(s_{d-1},s_d)$ such that the right-hand side in (3.21) becomes

$$\varepsilon^{-1/3}\{\hat{\lambda}(K_1)+\varepsilon^{1/6}R_1(\xi,s;\varepsilon)\}+R_2(\xi,s;\varepsilon)+\mathrm{O}(\varepsilon^{1/6}), \quad \hat{\lambda}(K_1)>0. \tag{3.22}$$

By integration with respect to $\tilde{\mu}^\varepsilon$ we obtain:

Lemma 3.1. *Given the stochastic differential equation* (1.3) *with stationary and ergodic background noise* (1.5), *the mean zero noise* $\mathrm{d}F^\varepsilon$ *from* (1.7) *and the function* φ_1 *from* (3.15), *then if* $K_1=\mathbb{E}\varphi_1>0$, *there is at least one Lyapunov exponent* λ^ε *of the form* (2.6) *which can be expanded as*

$$\lambda^\varepsilon=\varepsilon^{-1/3}\hat{\lambda}(K_1)+\varepsilon^{-1/6}\lambda_1(\varepsilon)+\lambda_2(\varepsilon)+\mathrm{O}(\varepsilon^{1/6}), \tag{3.23}$$

where $\hat{\lambda}(K_1)>0$, *and for* $i=1,2$

$$\sup_\varepsilon|\lambda_i(\varepsilon)|=\sup_{\varepsilon>0}|\langle R_i(\xi,s;\varepsilon),\mu^\varepsilon\rangle|\leqslant C<\infty.$$

4. Necessary conditions

4.1. Probabilistic interpretation of K_1

If $\mathrm{d}F^\varepsilon$ is stabilizing mean zero noise, then λ^ε is bounded from above, $\sup_{\varepsilon>0}\lambda^\varepsilon<\infty$, and therefore (by Lemma 3.1) K_1 cannot be positive. If we can show that on the other hand $K_1\geqslant 0$, then vanishing of the constant K_1 is necessary for $\mathrm{d}F^\varepsilon$ to be stabilizing.

What is the meaning of this crucial constant K_1?

The answer is easy, if $f_k=0$ $(k=1,\dots,r)$, thus $\varphi_0=f_0$ and $\varphi_1=-\varphi_0\mathscr{G}^{-1}(\varphi_0)$. Namely, this case, by the functional central limit theorem,

$$F_t/\sqrt{t}=(1/\sqrt{t})\left[F_0+\int_0^t\varphi_0(\xi_\tau)\,\mathrm{d}\tau\right]\to-2\langle\varphi_0\mathscr{G}^{-1}(\varphi_0),v\rangle W_1=2K_1W_1$$

in distribution, as $t\to\infty$. So, $2K_1$ is the limiting variance of $F_t/\sqrt{t}$,

$$\mathbb{E}F_t^2/t\to 2K_1 \quad (t\to\infty). \tag{4.1}$$

We now show that (4.1) holds generally in our framework. We allow thereby for the remainder of this subsection that $K_0=\mathbb{E}\varphi_0(\xi)$ is arbitrary (unless stated otherwise), to the effect that $\mathscr{G}^{-1}(\varphi_0-K_0)$ is substituted for $\mathscr{G}^{-1}(\varphi_0)$ in Defintion 3.12 for φ_1. First we observe that by Minkowsky's inequality, Jensen's inequality (with respect to the Lebesgue measure over $[0,t]$) and the stationarity of ξ_t for all $t\geqslant 0$

$$\begin{aligned}\|F_t\|_2 &:= \{\mathbb{E}|F_t|^2\}^{1/2}\\ &\leqslant \{\mathbb{E}|F_0|^2\}^{1/2}+\left\{\mathbb{E}\left|\int_0^t\varphi_0(\xi_\tau)\,\mathrm{d}\tau\right|^2\right\}^{1/2}+\sum_{k=1}^r\left\{\mathbb{E}\left|\int_0^t f_k(\xi_\tau)\,\mathrm{d}W_\tau^k\right|^2\right\}^{1/2}\end{aligned}$$

$$\leqslant \{\mathbb{E}|F_0|^2\}^{1/2} + \left\{\int_0^t \mathbb{E}|\varphi_0(\xi_\tau)|^2\,\mathrm{d}\tau\right\}^{1/2} + \sum_{k=1}^r \left\{\int_0^t \mathbb{E}|f_k(\xi_\tau)|^2\,\mathrm{d}\tau\right\}^{1/2}$$

$$\leqslant C_0 + C_1\sqrt{t} \tag{4.2}$$

with $0 \leqslant C_0, C_1 < \infty$, whence immediately

$$\sqrt{\mathbb{E}|F_t|^2/t} \to 0 \quad \text{as } t \to \infty. \tag{4.3}$$

To calculate $\mathbb{E}F_t^2$, where we assume without loss of generality that $\mathbb{E}F_0^2 = 0$, we consider the pair (ξ_t, F_t) with the generator

$$\mathscr{H} := \mathscr{G} + \varphi_0 \frac{\partial}{\partial F} + \sum_k f_k (X_k \cdot \nabla_\xi) \frac{\partial}{\partial F} + \frac{1}{2}\sum_k f_k^2 \frac{\partial^2}{\partial F^2}.$$

Then, putting $p_0(\xi, F) := F^2$, by Itô's formula, we have

$$\mathbb{E}F_t^2 = \mathbb{E}p_0(\xi_t, F_t) - \mathbb{E}p_0(\xi_0, F_0) = \int_0^t \mathbb{E}\mathscr{H}p_0(\xi_\tau, F_\tau)\,\mathrm{d}\tau$$

$$= \int_0^t \mathbb{E}\left[2\varphi_0(\xi_\tau)F_\tau + \sum_k f_k^2(\xi_\tau)\right]\mathrm{d}\tau. \tag{4.4}$$

In order to make $2K_1$ appear, we add and subtract this mean, obtaining

$$\mathbb{E}F_t^2 = 2K_1 t + \int_0^t \left\{\mathbb{E}\left[2\varphi_0(\xi_\tau)F_\tau + \sum_k f_k(\xi_\tau)^2\right] - 2K_1\right\}\mathrm{d}\tau, \tag{4.5}$$

and verify that the integrand on the right-hand side equals

$$-\mathbb{E}\mathscr{H}(p_1 + p_2)(\xi_\tau, F_\tau) + 2K_0\mathbb{E}F_\tau, \tag{4.6}$$

where

$$p_1(\xi, F) := -2\mathscr{G}^{-1}(\varphi_0 - K_0)\cdot F,$$

$$p_2(\xi, F) := 2\mathscr{G}^{-1}(\varphi_0\mathscr{G}^{-1}(\varphi_0 - K_0) - \mathbb{E}[\varphi_0\mathscr{G}^{-1}(\varphi_0 - K_0)]).$$

(Find p_1 and p_2 in a similar way as the "correctors" g_1 and g_2 in the homogenization procedure of Section 3 and compute $\mathscr{H}(p_1 + p_2) = \mathscr{H}(p_0 + p_1 + p_2) - \mathscr{H}(p_0)$.)

Then, again by Itô's formula together with $\mathbb{E}F_t = \mathbb{E}F_0 + K_0 t$ and the stationarity of $p_2(\xi, F) = p_2(\xi)$,

$$\mathbb{E}F_t^2 = 2K_1 t + 2K_0 \int_0^t [\mathbb{E}F_0 + K_0\tau]\,\mathrm{d}\tau - \mathbb{E}(p_1 + p_2)(\xi_t, F_t) + \mathbb{E}(p_1 + p_2)(\xi_0, F_0)$$

$$= 2K_1 t + 2K_0\left[\mathbb{E}F_0 t + \frac{1}{2}K_0 t^2\right]$$

$$+ \mathbb{E}\{\mathscr{G}^{-1}(\varphi_0 - K_0)(\xi_t)\cdot F_t\} - \mathbb{E}\{\mathscr{G}^{-1}(\varphi_0 - K)(\xi_0)\cdot F_0\}. \tag{4.7}$$

If, in addition, $K_0 = 0$, then by the Cauchy–Schwarz's inequality and (4.3), we obtain

$$\mathbb{E}F_t^2/t = 2K_1 + r(t), \quad \lim_{t\to\infty} r(t) = 0, \tag{4.8}$$

that is (4.1).

Lemma 4.1. *Let* $F_t = F_0 + \int_0^t \varphi_0(\xi_\tau)\,d\tau + \sum_k \int_0^t f_k(\xi_\tau)\,dW_\tau^k$ *be given by* (1.7) *with stationary ergodic noise* ξ_t *from* (1.5). *Then the second moment*, $\mathbb{E}F_t^2$, *is given by* (4.7) *and*

$$\lim_{t\to\infty} \frac{\mathbb{E}F_t^2}{t} = 2K_1 \quad \textit{iff} \quad K_0 = \mathbb{E}\varphi_0(\xi) = 0. \tag{4.9}$$

In that case $K_1 = \mathbb{E}\varphi_1 \geqslant 0$, φ_1 *given by* (3.12).

There is still another very useful representation of $K_1 = \mathbb{E}\varphi_1(\xi)$. [Again, we calculate here for arbitrary $K_0 \in \mathbb{R}$.] If we put

$$\psi := \mathcal{G}^{-1}(\varphi_0 - K_0) \tag{4.10}$$

and

$$\boldsymbol{f} := [f_1,\ldots,f_r]^{\mathrm{T}}, \qquad \boldsymbol{X}(\psi) := [X_1\cdot\nabla_\xi\psi,\ldots,X_r\cdot\nabla_\xi\psi]^{\mathrm{T}},$$

then

$$2\varphi_1 = \langle \boldsymbol{f},\boldsymbol{f}\rangle - 2\langle \boldsymbol{f},\boldsymbol{X}(\psi)\rangle - 2\varphi_0\psi,$$

thus $\|\boldsymbol{f} - \boldsymbol{X}(\psi)\|^2 = 2\varphi_1 + 2\varphi_0\psi + \|\boldsymbol{X}(\psi)\|^2$. Now using the identity

$$\begin{aligned}
\mathcal{G}(\psi^2) &= X_0\cdot\nabla_\xi\psi^2 + \frac{1}{2}\sum_{k=1}^{r}(X_k\cdot\nabla_\xi)^2(\psi^2)\\
&= (X_0\cdot\nabla_\xi\psi)(2\psi) + \frac{1}{2}\sum_{k=1}^{r} X_k\cdot\nabla_\xi((X_k\cdot\nabla_\xi\psi)(2\psi))\\
&= \mathcal{G}(\psi)(2\psi) + \sum_{k=1}^{r}(X_k\cdot\nabla_\xi\psi)^2\\
&= 2(\varphi_0 - K_0)\psi + \|\boldsymbol{X}(\psi)\|^2
\end{aligned}$$

and averaging with respect to v, while taking into account that $\mathcal{G}^* v=0$, thus $\mathbb{E}\mathcal{G}(\psi^2)=0$, yields

$$2K_1 = \mathbb{E}\|\boldsymbol{f} - \boldsymbol{X}(\psi)\|^2 + 2K_0\mathbb{E}\psi. \tag{4.11}$$

So, for *mean zero noise* ($K_0 = 0$),

$$2K_1 = \mathbb{E}\|\boldsymbol{f} - \boldsymbol{X}(\psi)\|^2. \tag{4.12}$$

That is to say, K_1 measures the difference between $X_k\cdot\nabla_\xi\psi$ and f_k.

4.2. Necessary conditions: vanishing of the limit variance of $F_t/\sqrt{t}$ and stationarity of F_t

From (4.9) and (4.12) we can easily draw necessary conditions for stabilizing mean-zero noise.

Theorem 4.2. *Let* $F_t = F_t(F_0,\xi_0) = F_0 + \int_0^t \varphi_0(\xi_\tau)\,d\tau + \sum_{k=1}^r \int_0^t f_k(\xi_\tau)\,dW_\tau^k$ *be given by* (1.7) *and* (1.8), *let* ξ_t *be the stationary and ergodic process from* (1.5) *with generator*

$\mathscr{G}$. Then for the mean zero noise $\mathrm{d}F_t^\varepsilon$ *to be stabilizing with respect to the companion form system* (1.3) *it is necessary that*

(i)

$$\frac{EF_t^2}{t} \to 0 \qquad (t \to \infty) \tag{4.13}$$

or, equivalently, that

(ii)

$$F_t = \psi(\xi_t) \quad \text{if} \quad F_0 = \psi(\xi_0) \tag{4.14}$$

with $\psi = \mathscr{G}^{-1}(\varphi_0)$. *That is to say, for suitable initial condition,* F_t *is a stationary and ergodic stochastic process* (*with finite variance*).

Proof. The necessity of (i) follows immediately from Lemmas 4.1 and 3.1.

If (i) holds, then $K_1=K_0=0$ and by (4.12), $f_k(\xi)=X_k(\xi)\cdot\nabla_\xi\psi(\xi)$ P-a.s. ($k=1,\dots,r$). Hence by Itô's formula

$$\begin{aligned} \mathrm{d}F_t &= \varphi_0(\xi_t)\,\mathrm{d}t + \sum_k f_k(\xi_t)\,\mathrm{d}W_t^k \\ &= \mathscr{G}(\psi)(\xi_t)\,\mathrm{d}t + \sum_k [X_k(\xi_t)\cdot\nabla_\xi\psi(\xi_t)]\,\mathrm{d}W_t^k \\ &= \mathrm{d}\psi(\xi_t) \end{aligned} \tag{4.15}$$

which implies (ii). (ii) → (i) is obvious, since $\mathbb{E}F_t^2 = \text{const}$. □

5. Sufficient conditions

The circle is closed, if stationarity and ergodicity of F_t implies the stabilizing property of $\mathrm{d}F^\varepsilon$. This is indeed the case.

Theorem 5.1. *Given* F_t *from* (1.7) *with* ξ_t *from* (1.5). *If* F_t *is a stationary and ergodic process in* $L^{2+\delta}$ *for some* $\delta > 0$, *then* $\mathbb{E}\varphi_0 = 0$ *and the mean zero noise* $\mathrm{d}F^\varepsilon$ *is a stabilizing noise for* (1.3).

For convenience, we outline here the main ideas of the *proof*. For details see Kao and Wihstutz (1994).

We consider first the mapping

$$(F,x) \mapsto (F,z), \quad z = T(-F)x, \quad T(F) = FU + I. \tag{5.1}$$

This mapping is linear in x, invertible with $T(F)^{-1} = T(-F)$, and it is tailored such that the diffusion term of

$$z^\varepsilon = z^\varepsilon(t,\omega) = T(-F^\varepsilon(t,\omega))x^\varepsilon(t,\omega), \tag{5.2}$$

vanishes. This is because by Itō's formula we obtain the family of ODEs

$$\mathrm{d}z^\varepsilon = B(F_t^\varepsilon)z^\varepsilon\,\mathrm{d}t, \tag{5.3}$$

where $B(F_t^\varepsilon) = B_0 + B_1^\varepsilon(t)$ (of course, still trace $B(F_t^\varepsilon) = 0$). Here $B_0 = \underline{B_0} + N$ is of companion form with

$$\underline{B_0} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ a_1 - u_1 u_{d-1}\mathbb{E}(F_t^\varepsilon)^2, & \ldots, & a_{d-1} - u_{d-1}^2\mathbb{E}(F_t^\varepsilon)^2, & 0 \end{bmatrix} \tag{5.4}$$

and U can be chosen such that all eigenvalues of B_0 are purely imaginary. Therefore we may assume without loss of generality that B_0 is skew-symmetric with $q_{B_0}(s) = 0$ (if necessary, after a deterministic linear transformation which does not change the Lyapunov exponents).

The entries of the matrix $B_t^\varepsilon(t)$ are 0, $\mathrm{const} \cdot F_t^\varepsilon$ or const $\cdot [-(F_t^\varepsilon)^2 + \mathbb{E}(F_t^\varepsilon)^2]$, that is to say $B_1^\varepsilon(t)$ is a fast matrix with mean zero, if $\mathbb{E}F_0 = 0$ (which can always be chosen without changing $\mathrm{d}F^\varepsilon$).

The important point of this mapping is that it preserves the Lyapunov exponents (since $\varphi_0(\xi_t)$ is mean zero stationary and ergodic and $\lim_{t\to\infty} M_t/t = 0$ for the martingale part M_t of F_t). This permits us to study the simpler system $(F_t^\varepsilon, z_t^\varepsilon)$ rather than $(F_t^\varepsilon, x_t^\varepsilon)$.

Second, since F_t^ε, thus $B(F_t^\varepsilon)$, is stationary and ergodic and in L^1, Osceledec's multiplicative ergodic theorem (MET, Oseledec, 1968) holds for z_t^ε from (4.3) (see, e.g. Arnold and Wihstutz 1986, p. 9). This entails that there are at most d distinct Lyapunov exponents (all being finite), which, with their multiplicities, sum up to trace $\mathbb{E}B(F_t^\varepsilon) = 0$. Moreover, there is an invariant measure $\tilde{\mu}^\varepsilon(\mathrm{d}F, \mathrm{d}s)$ on $\mathbb{R}^1 \times \mathbb{P}$ by means of which the top Lyapunov exponent (of z^ε, thus of x^ε) can be represented as

$$\lambda_{\max}^\varepsilon = \int_{\mathbb{R}\times\mathbb{P}} q(B(F), s)\tilde{\mu}^\varepsilon(\mathrm{d}F, \mathrm{d}s). \tag{5.5}$$

Intuitively, since $B_1^\varepsilon(t)$ oscillates very fast about $\mathbb{E}B_1^\varepsilon(t) = 0$, for $\varepsilon \to 0$ the dynamics of z_t^ε should be governed by $\mathrm{d}z^\varepsilon = B_0 z^\varepsilon\,\mathrm{d}t$ with $q_{B_0}(s) = 0$. Indeed, for F_t mean zero stationary and ergodic one can prove an averaging principle over the infinite time horizon $[0, \infty)$ from which one obtains that

$$\lambda_{\max}^\varepsilon = \left| \lambda_{\max}^\varepsilon - \int_{\mathbb{P}} q_{B_0}(s)\bar{\mu}^\varepsilon(\mathrm{d}s) \right| \to 0 \quad (\varepsilon \to 0), \tag{5.6}$$

where $\bar{\mu}^\varepsilon(\cdot) = \int_{\mathbb{R}} \tilde{\mu}^\varepsilon(\mathrm{d}F, \cdot)$ is the marginal measure associated with $\tilde{\mu}^\varepsilon$.

This proves Theorem 5.1 and concludes the proof of the main result, Theorem 2.1. □

6. Examples

6.1. Non-degenerate background noise with compact state space

(i) *White noise*, $\mathrm{d}F_t = \mathrm{d}W_t$. To treat white noise $\mathrm{d}F = \mathrm{d}W$ in our framework, we consider any non-degenerate stationary and ergodic diffusion process ξ_t on a compact manifold M, such as the Brownian motion on the unit circle, and put $f_0(\xi) = \mathrm{const} = 0$, $r = 1$ and $f_1(\xi) = \mathrm{const} = 1$. Since $\mathbb{E}F_t^2/t = \mathbb{E}W_t^2/t = 1$ does not converge to 0 as $t \to \infty$, white noise is *not stabilizing*. Or, to give an equivalent reason, since there is no initial condition $W_0(\omega)$ which would render W_t stationary and ergodic.

(ii) $F_t = f(\xi_t)$. If on the other hand for the semi-martingale F_t there is a smooth real-valued function f on M with $F_t = f(\xi_t)$, then $\lim_{t\to\infty} \mathbb{E}F_t^2/t = \lim(\text{const}/t) = 0$ and $\mathrm{d}F = (\mathscr{G}f)\mathrm{d}t + \sum_k (X_k \cdot \nabla_\xi f)\,\mathrm{d}W_t^k$ is *stabilizing* mean–zero noise. For instance, if ξ_t is the Brownian motion on the unit circle $M \subset \mathbb{R}^2$ and $f(\xi) = f(x_1,x_2) = x_1$, the projection on the first axis, then we have in local coordinates, for example, on the chart $U = \{(\sqrt{1-x_2^2}, x_2); |x_2| < 1\}$, $\theta_U(\xi) = x_2$, the stabilizing noise

$$\mathrm{d}F = \frac{\mathrm{d}}{\mathrm{d}x}(f \circ \theta_U^{-1})(x)\,\mathrm{d}t = (-2x_2)/\sqrt{1-x_2^2}\,\mathrm{d}t.$$

6.2. Relaxing the assumptions. Admissible noise

In order to include other types of noise, we broaden our class of noise processes. It is of interest, (for instance, for the physical experiments (Popp, 1995) to compare the unbounded standard Wiener process W_t with other unbounded non-degenerate processes, like the Ornstein–Uhlenbeck process, with respect to their stabilizing properties. So we relax our assumptions on the manifold M and the background noise ξ_t, which were introduced only for the sake of simplicity. All reasoning in Sections 2–5 goes through, if we allow integration by parts, $\langle \mathscr{G}f, v\rangle = \langle f, \mathscr{G}^* v\rangle$ for the representation (3.5) of λ^ε, if the Poisson equations

$$\mathscr{G}\psi_l = \varphi_l - \mathbb{E}\varphi_l, \quad l = 0,1,2, \qquad \mathscr{G}p = \varphi_0\psi_0 - \mathbb{E}(\varphi_0\psi_0). \tag{6.1a}$$

are solvable (for $l = 0$, $\psi_0 = \psi$ from (4.10)) and if the following integrability properties hold ($k = 1,\dots,r$):

$$f_0, f_k, X_k \cdot \nabla_\xi f_k \in L^{2+\delta}(v) \quad \text{for some } \delta > 0 \text{ [thus } \varphi_0(\xi), F_t \in L^{2+\delta}(P) \quad \text{if } F_0 \text{ is]}$$

$$\psi_0, X_k \cdot \nabla_\xi \psi_0,\ \varphi_1, \psi_1 \in L^2(v),\ f_k^2\psi_0,\ f_k^2\psi_1,\ p \in L^1(v). \tag{6.1b}$$

Of course, these conditions are automatically satisfied under our original assumptions. They also hold, if M is compact, but the generator $\mathscr{G}$ is degenerate and only weakly elliptic in the sense of Ichihara and Kunita (1974, 1977) meaning that

$$\dim J(\xi) = \dim M \quad \text{for all } \xi \in M,$$

where J is the ideal in the Lie-algebra $LA\{X_0, X_1, \dots, X_r\}$, generated by the diffusion vectorfields $X_1,\dots,X_r$. In this case, ξ_t is Doeblin and the Poisson equations are solvable.

There are many other situations, in which these conditions are easily proven to hold true.

Definition 6.1. We call dF from (1.7) *admissible*, if $K_0 = E\varphi_0 = 0$ (mean zero noise) and (3.5) as well as (6.1a) and (6.1b) hold. With this terminology, we have:

Theorem 6.2. *For* admissible *noise* dF *in* (1.3) *the assertions of Theorem* 1.2 *hold true*.

6.3. Examples, continued

Here we discuss *first* real noise, that is

$$dF_t^\varepsilon = \frac{1}{\varepsilon} f_0(\xi_t^\varepsilon)\,dt, \quad f_1 = \cdots = f_r = 0.$$

(iii) *Degenerate diffusion*: $\xi_t = (\alpha_t, \beta_t) \in M = T^2$, the two-dimensional torus, with

$$d\xi_t = d\begin{bmatrix}\alpha_t\\ \beta_t\end{bmatrix} = \begin{bmatrix}\cos\beta_t\\ 0\end{bmatrix}dt + \begin{bmatrix}0\\ 1\end{bmatrix}dW_t(\text{mod } 2\pi).$$

The vector fields $Y_1 := X_1 = \begin{bmatrix}0\\1\end{bmatrix}$, $Y_2 := [X_1, X_2] = \begin{bmatrix}-\cos\beta\\0\end{bmatrix}$ and $Y_3 := [X_1,[X_1,X_0]] = \begin{bmatrix}-\sin\beta\\0\end{bmatrix}$ belong to the ideal J generated in $LA\{X_0, X_1\}$ by X_1, and for any $\xi = (\alpha, \beta)$, rank $\{Y_1, Y_2, Y_3\}(\alpha, \beta) = 2$. Therefore the generator in weakly elliptic (in the Ichihara and Kunita sense). For $f_0(\xi) = f_0(\alpha, \beta) = -\sin\alpha\cos\beta$, $F_t = F_0 + \cos\alpha_t$ leads to admissible *stabilizing* noise.

(iv) *The Ornstein–Uhlenbeck process* ξ_t on $M = \mathbb{R}^1$,

$$d\xi_t = -\gamma_0 \xi_t\,dt + \gamma_1\,dW_t, \quad \gamma_0, \gamma_1 > 0$$

with uniformly elliptic generator $\mathscr{G} = (-\gamma_0\xi)\partial/\partial\xi + \frac{1}{2}\gamma_1^2(\partial/\partial\xi)^2$, mean $\mathbb{E}\xi_t = 0$, variance $\mathbb{E}\xi_t^2 = \Sigma^2 = \gamma_1^2/2\gamma_0$, together with $f_0(\xi) = \xi$ yields

$$dF_t^\varepsilon = \frac{1}{\varepsilon}\xi_{t/\varepsilon}\,dt,$$

which is admissible, but *not stabilizing*. This is because the functions $\varphi_0(\xi) = f_0(\xi) = \xi$, $\psi_0 = \mathscr{G}^{-1}(\varphi_0) = (-1/\gamma_0)\xi$, $\varphi_1 = -\varphi_0\psi_0 = (1/\gamma_0)\xi^2$ and $\psi_1 = \mathscr{G}^{-1}(\varphi_1 - K_1) = (-1)\mathscr{G}^{-1}(\varphi_0\psi_0 - \mathbb{E}(\varphi_0\psi_0)) = (-1/2\gamma_0^2)\xi^2$ are in $L^p(v)$, $p \geqslant 1$, but $K_1 = \mathbb{E}\varphi_1 = (1/\gamma_0)\Sigma^2 = \frac{1}{2}(\gamma_1^2/\gamma_0^2) \neq 0$.

Spectral behavior. We note that for admissible noise $dF = f(\xi_t)\,dt$, we have $\varphi_1 = -\varphi_0\psi_0 = -f\mathscr{G}^{-1}(f)$ and $2K_1 = -2\langle f\mathscr{G}^{-1}(f), v\rangle = 2\pi S_{ff}(0)$, where S_{ff} is the spectral density of the stationary process $f(\xi_t)$. So, we have shown the following corollary of Theorem 6.2, which relates our discussion on stabilization to the discussion in Orey (1981) and Arnold and Wihstutz (1983) (Theorem 2.3, Example 2.1) on the stationarity of integrals of stationary processes as non-resonance phenomenon.

Corollary 6.3. *Let* $dF = f(\xi)\,dt$ *be admissible noise*, S_{ff} *the spectral density of* f. *Then* $f(\xi_t)$ *has a stationary integral* $F_t = F_0 + \int_0^t f(\xi_\tau)\,d\tau$ *iff* dF *stabilizes* (1.3) *iff* $S_{ff}(0) = 0$.

For the Ornstein–Uhlenbeck process $f(\xi_t) = \xi_t$ there is no such stationary integral, since $K_1 = \frac{1}{2}(\gamma_1^2/\gamma_0^2) = S_{\xi\xi}(0) \neq 0$, where $S_{\xi\xi}(\Lambda) = \Sigma^2 2\gamma_0/(\gamma_0^2 + \Lambda^2)$.

If we consider, *second*, combinations of real and white noise, we encounter a different situation.

(v) $dF_t = d\xi_t = d$ (*Ornstein–Uhlenbeck process*) behaves very different from $dF = \xi_t\,dt$, since now $F_t = f(\xi_t) = \xi_t$ is stationary and ergodic. Therefore, by Theorem 6.2,

$$dF_t = d\xi_t = (-\gamma_0\xi)\,dt + \gamma_1\,dW_t \quad (\gamma_0, \gamma_1 > 0)$$

is *stabilizing* admissible noise. This stabilizing property is also equivalent to the other properties in Theorem 1.2, since for $F_t = f(\xi_t)$, f smooth on $\mathbb{R}$, we have $f_0 = (-\gamma_0\xi)f'$, $f_1 = \gamma_1 f'$, $\varphi_0 = \mathscr{G}f$, $\psi_0 = f$, $\varphi_1 = \mathscr{G}(-\frac{1}{2}f^2)$, $\psi_1 = -\frac{1}{2}f^2$ and $\mathscr{G}^{-1}(\varphi_0\psi_0 - \overline{\varphi_0\psi_0}) = \mathscr{G}^{-1}(f\cdot\mathscr{G}f - \overline{f\cdot\mathscr{G}f})$ where the bar denotes the expected value; and these functions are in $L^p(v)$, $p\geqslant 1$, if e.g. $f(\xi)$ is a polynomial in ξ or f bounded with bounded derivatives f' and f''.

The Ornstein–Uhlenbeck process ξ_t and the Wiener process W_t are both unbounded and "rich" stochastic processes. Their different stability behavior stems from the fact that the Wiener process spreads out too fast to allow an invariant probability measure. (Roughly speaking, this is a kind of resonance phenomenon. The power spectrum of white noise, $S_{\dot{W}\dot{W}}(\Lambda)$ is constant $\neq 0$ and thus does not vanish at $\Lambda = 0$, while for Ornstein–Uhlenbeck process ξ_t, $S_{\dot{\xi}\dot{\xi}}(0) = 0$.)

(vi) $\mathrm{d}F_t = \mathrm{arc}tg\xi$. Since it is of importance for applications we note explicitly that we may well insert the Ornstein–Uhlenbeck process ξ_t, say, into a bounded function f with bounded derivatives f' and f'', such as the arc tangent function:

$$\mathrm{d}F_t = d\,\mathrm{arc}tg\,(\xi_t) = (-1)[\gamma_0/(1+\xi^2) + \gamma_1^2/(1+\xi^2)^2]\xi\,\mathrm{d}t + \gamma_1/(1+\xi^2)\,\mathrm{d}W_t$$

is *stabilizing* admissible noise for (1.3) (while $\mathrm{d}F_t = \mathrm{arc}tg(\xi_t)\,\mathrm{d}t$ is admissible, but not stabilizing).

6.4. Sufficiently fast convergence of $\mu^\varepsilon_{\mathrm{d}F}$

We finish this section with the remark that $\mathrm{d}F$ is stabilizing iff the corresponding invariant measure $\mu^\varepsilon_{\mathrm{d}F}$ from (2.6) converges sufficiently fast to a suitable invariant μ^0, at least in some weak sense.

If $\mathrm{d}F$ is stabilizing, then $\lambda^\varepsilon_{\max} = \langle Q^\varepsilon, \mu^\varepsilon\rangle$, where by (2.5) $Q^\varepsilon = Q_0^\varepsilon + 1/\varepsilon Q_1$ with $Q_0^\varepsilon = q_A(s) + f_0(\xi)q_U(s) + 1/2\varepsilon\sum(X_k\cdot\nabla_\xi f_k)(\xi)q_U(s)$ and

$$Q_1 = \frac{1}{2}\sum f_k(\xi)(h_U\cdot\nabla_s q_U)(s).$$

Here the family $\{\mu^\varepsilon, \varepsilon\to 0\}$ of invariant measures on $M\times\mathbb{P}$ (with common marginal v on M and $\mathbb{P}$ being compact) is tight. Therefore, if $\varepsilon\to 0$ (suitably), $\mu^\varepsilon \Rightarrow \mu^0$ weakly for some invariant measure μ^0 on $M\times\mathbb{P}$. (Ethier and Kurtz, 1986). Necessarily, $\lim_{\varepsilon\to 0}\langle Q_0^\varepsilon, \mu^\varepsilon\rangle = \lim_{\varepsilon\to 0}\langle Q_1, \mu^\varepsilon\rangle = 0$. However, this is not sufficient. For a necessary and sufficient criterion we need, in addition, that $\mu^\varepsilon \Rightarrow \mu^0$ fast enough, at least in the weak sense that for Q_1 $\langle Q_1, \mu^\varepsilon\rangle \to \langle Q_1, \mu^0\rangle = 0$ faster than $\varepsilon\to 0$.

Inverted pendulum. For instance, in case of the *inverted pendulum*, $\ddot{y} + 2\beta\dot{y} - ay = 0$ ($a > 0$, $y=$ angle from the vertical), with trace zero form

$$\mathrm{d}x = \begin{bmatrix} 0 & 1 \\ \alpha+\beta^2 & 0 \end{bmatrix} x\,\mathrm{d}t + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x \circ \mathrm{d}F_t^\varepsilon,$$

we have for both white noise $\mathrm{d}F_t = \mathrm{d}W_t$ and $\mathrm{d}F_t = \mathrm{d}\xi_t = -\gamma_0\xi_t\,\mathrm{d}t + \gamma_1\,\mathrm{d}W_t$ ($\gamma_0, \gamma_1 > 0$) that $\lambda^\varepsilon_{\max} = \langle Q^\varepsilon, \mu^\varepsilon\rangle$, $\langle Q_0^\varepsilon, \mu^\varepsilon_{\mathrm{d}F}\rangle \to 0$, $\langle Q_1, \mu^\varepsilon_{\mathrm{d}F}\rangle \to 0$ ($\varepsilon\to 0$), and $Q_1 = Q_1(\xi,\phi) = \cos^2\phi$ (where the angle ϕ is determined by $s = [\cos\phi, \sin\phi]^{\mathrm{T}}$). But for non-stabilizing white noise: $\langle Q_1, \mu^\varepsilon_{\mathrm{d}W}\rangle \sim \varepsilon^{2/3}$ only ($\lambda^\varepsilon_{\max}(\mathrm{d}W) \sim \varepsilon^{-1/3}$; see Pinsky and Wihstutz (1991), while for the stabilizing noise $\mathrm{d}\xi = d$(Ornsten–Uhlenbeck): $1/\varepsilon\langle Q_1, \mu^\varepsilon_{\mathrm{d}\xi}\rangle \to 0$ ($\varepsilon\to 0$).

7. Impact on the Lyapunov spectrum

In this concluding section we discuss some simple implications of the main Theorem 1.2 in the context of the theory of random dynamical systems (in the sense of Arnold, 1998) and consider the full Lyapunov spectrum. For that purpose let $(C_0, \mathscr{B}(C_0), P_0)$ be the Wiener space (associated with the extended time set $\mathbb{R}$), let θ_t denote the measure preserving shift on C_0,

$$\theta_t w(\cdot) = w(\cdot + t) - w(t), \tag{7.1}$$

$w(\cdot) \in C_0$, $t \in \mathbb{R}$, and let $\xi_t^\varepsilon(\xi, w)$ be the stationary ergodic process solving (1.5) with initial condition $\xi_0^\varepsilon(\xi, w) = \xi \in M$. Then

$$\Theta_t^\varepsilon(\omega) = \Theta_t^\varepsilon(\xi, w) = (\xi_t^\varepsilon(\xi, w), \theta_t w) \tag{7.2}$$

is a measure preserving flow on $(\Omega, \mathscr{F}, P) := (M \times C_0,\ \mathscr{B}(M \times C_0),\ v \times P_0)$. If we put for any function f on M, $f(\omega) := f(\xi, w) := f(\xi)$ $(\xi \in M, w \in C_0)$, then we may consider the semi-martingale $F_t^\varepsilon - F_0$ from (1.7) as an additive co-cycle or helix (see Arnold, 1998, p. 73) over the flow Θ_t^ε, that is

$$(F_{s+t}^\varepsilon - F_0)(\omega) = (F_s^\varepsilon - F_0)(\omega) + (F_t^\varepsilon - F_0)(\Theta_s^\varepsilon \omega).$$

Cohomological equivalence of $F_t - F_0$ and zero. Theorem 1.2 can now be read as a necessary and sufficient criterion for the helix $F_t^\varepsilon - F_0$ to be cohomological equivalent to zero, meaning that there exists a measurable function γ on Ω such that

$$(F_t^\varepsilon - F_0)(\omega) = \gamma(\Theta_t^\varepsilon(\omega)) + 0 - \gamma(\omega). \tag{7.3}$$

We rewrite Theorem 2.1 as follows.

Proposition 7.1. *The semi-martingale*

$$F_t^\varepsilon - F_0 = \int_0^t \varphi_0(\xi_\tau)\,\mathrm{d}\tau + \sum_{k=1}^r f_k(\xi_\tau)\,\mathrm{d}W_\tau^k$$

from (1.7) *is cohomologically equivalent to zero iff* $\lim_{t\to\infty} \mathbb{E} F_t^2/t = 0$ *iff* $\mathrm{d}F$ *is stabilizing system* (1.3).

Validity of Oseledec's MET. It is of interest to know whether Oseledec's Multiplicative Ergodic Theorem (Oseledec, 1968; Arnold, 1998) can be applied to the linear stochastic system (1.3), that is to say to the associated fundamental matrix $\Psi^\varepsilon(t, \omega)$ with $\Psi^\varepsilon(0, \omega) = I$. Although we do not know the answer in general, that is for *arbitrary admissible noise* $\mathrm{d}F$, our reasoning (which is independent of the MET) provides a partial answer. In case of *stabilizing* noise it is easy to see that $\Psi^\varepsilon(t, \omega)$ meets the conditions of the MET. In this case namely the fundamental matrix $\Phi^\varepsilon(t, \omega)$ of the transformed linear system (5.3), $\dot{z}_t^\varepsilon \neq B(F_t^\varepsilon) z_t^\varepsilon$, is a (multiplicative) co-cycle over Θ_t^ε which satisfies the integrability conditions required by the MET (see, e.g. Arnold and Wihstutz, 1986. That also Ψ^ε is a co-cycle over Θ_t^ε satisfying the required integrability condition follows from the cohomological equivalence between Φ_t^ε and Ψ_t^ε, because we have

$$\Psi_t^\varepsilon x_0 = x_t^\varepsilon = T_t^\varepsilon z_t^\varepsilon = T_t^\varepsilon \Phi^\varepsilon(t,\omega) z_0$$
$$= T_t^\varepsilon \Phi^\varepsilon(t,\omega)(T_t^\varepsilon)^{-1} x_0, \tag{7.4}$$

where $T_t^\varepsilon := F_t^\varepsilon U + I$.

Proposition 7.2. *If the admissible noise from* (1.7) *stabilizes the stochastic linear system* (1.3), *then the fundamental matrix* $\Psi^\varepsilon(t,\omega)$ *with* $x^\varepsilon(t,\omega) = \Psi^\varepsilon(t,\omega)x_0(\omega)$ *is a* (*multiplicative*) *co-cycle which satisfies the conditions of Oseledec's multiplicative ergodic theorem.*

Dichotomy for $\lambda_{\max}^\varepsilon$ *and averaging property of stabilizing noise.* The maximum Lyapunov exponent of (1.3) exhibits a dichotomy corresponding to stabilizing/non-stabilizing admissible noise; and stabilizing noise averages the Lyapunov spectrum of (1.3) to its "center of gravity", as we can see from the following proposition.

Proposition 7.3. *Given the trace-zero companion form system* (1.3) *with admissible noise* dF *from* (1.7), *then*
- (i) *either* $\lim_{\varepsilon\to 0}\lambda_{\max}^\varepsilon = 0$ *or* $\lim_{\varepsilon\to 0}\lambda_{\max}^\varepsilon = \infty$, *according to whether or not the noise is stabilizing.*
- (ii) *If the noise is stabilizing, then for all Lyapunov exponents of the spectrum*: $\lambda_i^\varepsilon \to 0$ $(\varepsilon \to 0)$, $i = 1,\dots,p$.

Proof. (i) If dF is not stabilizing, then $K_1 = E\varphi_1 \neq 0$, and according to Lemma 3.1 there is a Lyapunov exponent $\lambda^\varepsilon = \langle Q^\varepsilon, \mu^\varepsilon\rangle$ with suitable invariant probability measure μ^ε, which grows like $\varepsilon^{-1/3} \to \infty$, if $\varepsilon \to 0$. Therefore any possibly larger Lyapunov exponent goes to ∞, as $\varepsilon \to 0$.

(ii) If dF is stabilizable, by Oseledec's theorem $\lambda_{\max}^\varepsilon \geqslant 0$ and together with Definition 1.1 for stabilizing noise, $\lambda_{\max}^\varepsilon \to 0$ $(\varepsilon \to 0)$. But then also $\lambda_i^\varepsilon = 0$ for all $i = 1, 2, \dots, p$, since all exponents sum up to 0. □

We see what the noise is really doing when stabilizing is averaging the Lyapunov spectrum. This means that stabilizing noise has destabilizing impact as well, namely on the solutions with negative exponential growth rates. But that is to say, on solutions which anticipate the future and therefore cannot be observed in praxis. So, from the practical point of view of applications the term "stabilizing" is justified.

8. For further reading

The following reference is also of interest to the reader: Katok and Hasselblatt, 1995.

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