Periodic Solutions of Symmetric Perturbations of the Kepler Problem

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It is proved that for any symmetric perturbation of the Kepler problem, a circular solution of the unperturbed system with inclination different from 0 and \( \pi \) gives rise to a periodic solution of the reduced dynamics which is defined in the quotient space of the action by the subgroup that fixes the symmetry axis.

1. INTRODUCTION

Periodic solutions of the satellite problem of an oblate planet have been a subject of research for a long time. At the beginning of the century, Darwin and his collaborators [2] searched for periodic orbits in celestial mechanics, including the satellite problem. More recently, Kyner [5] applied the theorem of Moser on invariant curves to show that there exists a periodic solution of the satellite problem of an oblate planet for every inclination different from the so-called critical inclination [3]. In this paper, we consider the more general setting of an arbitrary perturbation of the Kepler problem, requiring only that it have symmetry with respect to the \( z \)-axis and the \( xy \)-plane, hence also with respect to the origin. This includes the important case of the satellite of an oblate planet.

We therefore consider a perturbation of the Kepler problem with the potential function

\[
U_{\mu}(r) = \frac{\kappa}{|r|} + \mu f(r, \mu),
\]

where \( f \) is a real analytic function invariant under rotations about the \( z \)-axis and under reflections with respect to the \( xy \)-plane, hence also symmetric with respect to the origin. In (1), \( \kappa \) is a positive constant and \( \mu \) is a small parameter.
The subgroup $G$ of $SO(3)$ that fixes the $z$-axis acts diagonally on the phase space $M = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ by the rule $g \cdot (r, \dot{r}) = (g \cdot r, g \cdot \dot{r})$. Due to the rotational symmetry of $U_\mu(r)$, the flow $\Phi(t, (r, \dot{r}), \mu)$ of the equation of motion

$$\ddot{r} = \nabla U_\mu(r)$$

(2)

is equivariant with respect to this action, which means that

$$\Phi(t, g \cdot (r, \dot{r}), \mu) = g \cdot \Phi(t, (r, \dot{r}), \mu),$$

and therefore we get an induced flow on the quotient space $M/G$ of this action,

$$\tilde{\Phi}(t, [r, \dot{r}], \mu) = [\Phi(t, (r, \dot{r}), \mu)],$$

where $[r, \dot{r}]$ stands for the equivalent class of the element $(r, \dot{r})$. This is the process of elimination of the node first used by Jacobi given by Smale in [7].

The goal of this paper is to prove the following theorem.

**Theorem.** Let $r^0(t)$ be a circular solution of the Kepler problem which corresponds to $\mu = 0$. Set $\varepsilon = \sqrt{\mu}$, and let $\tau$ be the period and $\iota$ the inclination of $r^0(t)$. If $\iota \neq 0$ and $\pi$, then there exists a $\tau$-periodic solution $\tilde{\Phi}(t, [r(\varepsilon), \dot{r}(\varepsilon), \varepsilon]$ of the reduced dynamics such that $\Phi(t, (r(0), \dot{r}(0)), 0) = (r^0(t), \dot{r}^0(t))$.

Notice that if $(r(t, \varepsilon), \dot{r}(t, \varepsilon)) = \Phi(t, (r(\varepsilon), \dot{r}(\varepsilon), \varepsilon)$, then $r(t, \varepsilon)$ does not remain in a neighborhood of $r^0(t)$, since $r(t, \varepsilon)$ differs from $r^0(0)$ by a (small) rotation $R_{\varepsilon(\varepsilon)}$ about the $z$-axis and the successive points $r(2\tau, \varepsilon), r(3\tau, \varepsilon), ...$ rotate about this axis at the same height with respect to the $xy$-plane as the point $r^0(0)$. Of course, this solution may or may not be periodic in physical space, depending on a commensurability relation between the motion along $r^0(t)$ and the rotation about the $z$-axis.

In the proof of this theorem we will use the Poincaré continuation method in the presence of first integrals, as is described in [6]. The difficulty lies in the high degeneracy of the Kepler problem, since for the usual variational equations the rank is one in a six-dimensional system. We take advantage of the fact that the generating orbit is circular to get a certain symmetry (see Section 3), which helps to raise the rank of the periodicity system to 3.

Then, using a parameter associated with the reduction of the node (the passage to the reduced dynamics), we can break the degeneracy even further, raising the rank of the system to 4. We then apply the implicit function theorem to the six-dimensional periodicity system (Eq. (12),...
Section 3) to solve it with respect to four of its equations. Finally, using the integrals of motion given by the energy and the polar component of the angular moment, we are able to prove that the two remaining equations of the periodicity system are also satisfied, thereby guaranteeing the existence of the periodic solution in the reduced dynamics. This program is carried out in the next sections, the proof of the theorem being completed in the last one, Section 5.

2. THE EQUATIONS OF MOTION

Let $r^0(t)$ be a circular solution of the Kepler problem which corresponds to $\mu = 0$. Let $i$ be the inclination of its orbital plane with respect to the $xy$-plane, $\omega$ its angular speed, and $a$ its radius.

For $\varepsilon = \sqrt{\mu} \neq 0$, we set

$$r(t, \varepsilon) = r^0(t) + \varepsilon s(t, \varepsilon) \quad (3)$$

and notice that $r(t, \varepsilon)$ is a solution of Eq. (2) if, and only if, $s(t, \varepsilon)$ is a solution of the equation

$$\ddot{s} = -\frac{\kappa}{|r|^3} s + \frac{3 \varepsilon (r^0 \cdot s)}{|r|^3} r^0 + O(\varepsilon), \quad (4)$$

where $r^0 \cdot s$ means the usual inner product of these vectors. Consider the orthonormal frame $e_1(t), e_2(t), e_3(t)$ defined by

$$e_1 = \frac{r^0}{|r^0|}, \quad e_2 = e_3 \times e_1, \quad e_3 = \frac{r^0 \times r^0}{|r^0 \times r^0|},$$

for which we have

$$\dot{e}_1 = \omega e_2, \quad \dot{e}_2 = -\omega e_1, \quad \dot{e}_3 = 0.$$

Setting

$$s = x_1 e_1 + x_2 e_2 + x_3 e_3, \quad \dot{s} = y_1 e_1 + y_2 e_2 + y_3 e_3,$$

the differential equation (4) is equivalent to the system

$$\dot{z} = Az + O(\varepsilon), \quad (5)$$
where \( z = (x_1, x_2, x_3, y_1, y_2, y_3)^T \), and

\[
A = \begin{pmatrix}
0 & \omega & 0 & 1 & 0 & 0 \\
-\omega & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
2\omega^2 & 0 & 0 & 0 & \omega & 0 \\
0 & -\omega^2 & 0 & -\omega & 0 & 0 \\
0 & 0 & -\omega^2 & 0 & 0 & 0
\end{pmatrix}.
\]

The eigenvalues of \( A \) are \( 0, i\omega \) and \( -i\omega \), each with multiplicity 2. Consider the basis \( B = \{v_1, ..., v_6\} \) of \( \mathbb{R}^6 \), where \( v_j \) is the \( j \)th column of the matrix

\[
F = \begin{pmatrix}
2 & 0 & 1 & 0 & 0 & 0 \\
0 & -3\omega & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 3\omega^2 & 0 & -\omega & 0 & 0 \\
-\omega & 0 & -\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega
\end{pmatrix}.
\]

Then, the matrix of \( A \) in this basis is \( \mathcal{A} = F^{-1}AF \), and it is the real Jordan form of \( A \), namely,

\[
\mathcal{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega & 0 & 0 \\
0 & 0 & -\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega \\
0 & 0 & 0 & 0 & -\omega & 0
\end{pmatrix}.
\]

Letting \( z = F\zeta \), the equation of motion (5) becomes

\[
\dot{\zeta} = \mathcal{A}\zeta + O(\epsilon),
\]

and its flow is given by

\[
\psi(t, \zeta, \epsilon) = e^{t\mathcal{A}}\zeta + O(\epsilon),
\]

(7)
where
\[
e^{\mathbf{J} t} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \omega t & \sin \omega t & 0 & 0 \\
0 & 0 & -\sin \omega t & \cos \omega t & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \omega t & \sin \omega t \\
0 & 0 & 0 & 0 & -\sin \omega t & \cos \omega t
\end{pmatrix}.
\]

3. THE PERIODICITY EQUATION

Because the potential \( U_g \) is symmetric with respect to the origin, we have the following property of the flow of Eq. (2):
\[
\Phi(t, -(r, \dot{r}), \varepsilon) = -\Phi(t, (r, \dot{r}), \varepsilon).
\]

Now, assume that for some initial condition \((r, \dot{r})\) and some \( g \in G \), the solution \( \Phi(t, (r, \dot{r}), \varepsilon) \) satisfies the equation
\[
\Phi \left( \frac{\tau}{2}, (r, \dot{r}), \varepsilon \right) = g(r, \dot{r}). \tag{8}
\]

Then, by uniqueness of solutions of ordinary differential equations, we have
\[
\Phi \left( t + \frac{\tau}{2}, (r, \dot{r}), \varepsilon \right) = \Phi(t, -(r, \dot{r}), \varepsilon),
\]
for all \( t \). By the equivariance of the flow \( \Phi(t, (r, \dot{r}), \varepsilon) \) and the above property, it follows that
\[
\Phi(t + \tau, (r, \dot{r}), \varepsilon) = g^2 \cdot \Phi(t, (r, \dot{r}), \varepsilon),
\]
which means that
\[
\Phi(t + \tau, [r, \dot{r}], \varepsilon) = \Phi(t, [r, \dot{r}], \varepsilon), \quad \text{for all } t;
\]
that is to say, \( \Phi(t, [r, \dot{r}], \varepsilon) \) is a periodic solution with period \( \tau \) for the reduced dynamics on \( M/G \).
We will call (8) the periodicity equation and next we will see how it is expressed in the \( z \) variables. If we let \((r(t, e), \dot{r}(t, e)) = \Phi(t, (r, \dot{r}, e)),\) Eq. (8) can be written as
\[
    r \left( \frac{\tau}{2}, e \right) = -g r \quad \text{and} \quad \dot{r} \left( \frac{\tau}{2}, e \right) = -g \dot{r} .
\]  
Considering the fixed frame \( e^{(1)} = e_1(0), \) \( e^{(2)} = k \times e_1(0), \) \( e^{(3)} = k, \) where \( k = (0, 0, 1), \) any \( g \in G \) is given by
\[
    g e^{(1)} = \cos \theta e^{(1)} + \sin \theta e^{(2)}, \quad g e^{(2)} = -\sin \theta e^{(1)} + \cos \theta e^{(2)}, \quad g e^{(3)} = e^{(3)},
\]
where \( \theta \) is the angle of rotation about the \( z \)-axis. For small \( e, \) the position and velocity of the particle at time \( \tau \) will be close to their original values at time zero. Therefore, for a periodic solution of the reduced system with period \( \tau, \) the rotation that sends the initial to the final positions and velocities is close to the identity. This is the motivation to take the angle of rotation of the form \( \theta = e + O(e^2), \) where \( v \) is a parameter. Since
\[
    \cos \theta = 1 - \frac{1}{2} e^2 v^2 + O(e^4), \quad \sin \theta = e v + O(e^3),
\]
the rotation \( g \) is given by
\[
    g = I + e N,
\]
where
\[
    N e^{(1)} = \frac{\cos \theta - 1}{e} e^{(1)} + \frac{\sin \theta}{e} e^{(2)},
\]
\[
    N e^{(2)} = -\frac{\sin \theta}{e} e^{(1)} + \frac{\cos \theta - 1}{e} e^{(2)},
\]
\[
    N e^{(3)} = 0 .
\]
Letting \( N = (n_{ij}) \) be the matrix of this linear operator in the basis \( \{e_i(0)\}, \) that is,
\[
    N e_j(0) = \sum_{i=1}^{3} n_{ij} e_i(0) \quad (j = 1, 2, 3),
\]
the periodicity equation (8), in view of (9), can be written as
\[
    z \left( \frac{\tau}{2}, e \right) = R(z + b + e N z),
\]
where
\[ b = (a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32})^T, \]
\[ N = \text{diag}[N, N], \]
and
\[ R = \text{diag}[R, R], \]
with
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

Therefore, if \( \phi(t, z, \varepsilon) \) denotes the flow of (5), the periodicity equation assumes the form
\[
\phi \left( \frac{\tau}{2}, z, \varepsilon \right) - R(z + b + \varepsilon N z) = 0,
\]
or, in the \( \zeta \) variables,
\[
\psi \left( \frac{\tau}{2}, \zeta, \varepsilon \right) - R(\zeta + \beta + \varepsilon \cdot \zeta) = 0, \quad (11)
\]
where \( \psi(t, \zeta, \varepsilon) \) is the flow of (6), \( \beta = T^{-1} b, \ N' = T^{-1} N T, \) and \( R = T^{-1} R T \) is the diagonal matrix
\[
R = \text{diag}(1, 1, 1, -1, -1).
\]

The linear operator \( N \) defined in (10) can be written as
\[
N = \nu N_0 + O(\varepsilon),
\]
where
\[
N_0 e^{(1)} = e^{(2)}, \quad N_0 e^{(2)} = -e^{(1)}, \quad \text{and} \quad N_0 e^{(3)} = 0.
\]
The matrix of \( N_0 \) in the basis \( \{e_i(0)\} \) is
\[
N_0 = \begin{pmatrix}
0 & -\cos \tau & \sin \tau \\
\cos \tau & 0 & 0 \\
-\sin \tau & 0 & 0
\end{pmatrix},
\]
where \( \tau \) is the inclination of the orbit \( r^0(t) \). Then, the vector \( b \) introduced above is given by
\[
b = \nu Y + O(\varepsilon),
\]
where
\[ Y = (0, a \cos t, -a \sin t, -a \omega \cos t, 0, 0)^T \]
and, therefore,
\[ \beta = vY + O(\varepsilon), \]
with
\[ Y = \left(0, -\frac{a}{3\omega} \cos t, 0, 0, -a \sin t, 0\right)^T. \]

Let us denote by \( P(\zeta, v, \varepsilon) \) the left-hand side of the periodicity equation (11); that is, let
\[ P(\zeta, v, \varepsilon) = \psi \left( \frac{\tau}{2}, \zeta, \varepsilon \right) - R, \quad (\zeta + \beta + v\varepsilon) = 0 \quad (i = 1, \ldots, 6). \]

Using (7) and the expression of \( e^{\varepsilon\beta} \) given in Section 2, we notice that the requirement
\[ P(\zeta^*, 0, 0) = (e^{\varepsilon(2)\beta} - R) \zeta^* = 0, \]
implies the restrictions \( \zeta^*_2 = \zeta^*_3 = \zeta^*_4 = 0 \). Therefore, we take
\[ \zeta^* = (0, \zeta^*_5, 0, 0, \zeta^*_6)^T, \]
with \( \zeta^*_5 \), \( \zeta^*_6 \), and \( \zeta^*_5 \) arbitrary, for the moment. Now, using the above expression for \( \beta \) and the formula (7) for the flow of (6), we see that the Jacobian matrix of \( P \) with respect to the variables \( v \) and \( \zeta \) evaluated at the point \( (\zeta^*, 0, 0) \) is given by
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a}{3\omega} \cos t & \frac{\tau}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Consider the system of four equations formed by those in (12) corresponding to the indices \( i = 2, 3, 4, 5 \); in this system we fix the variables \( \zeta_2 = \zeta_3 \), \( \zeta_5 = \zeta_6 \), and \( \zeta_6 = \zeta_5 \). Its Jacobian matrix has determinant equal to
\[ -2a\tau \sin t \]
and, therefore, if \( t \neq 0, \pi \), the Implicit Function Theorem guarantees the existence of analytic functions \( r = r(\varepsilon) \) and \( \zeta_i = \zeta_i(\varepsilon) \), \( i = 1, 3, 4 \), in a neighborhood of \( \varepsilon = 0 \), satisfying the equations

\[
\mathcal{P}(\zeta, r, \varepsilon) = 0 \quad (i = 2, 3, 4, 5),
\]

where

\[
\zeta(\varepsilon) = (\zeta_1(\varepsilon), \zeta_2^*, \zeta_3(\varepsilon), \zeta_4^*, \zeta_5^*),
\]

and such that

\[
\nu(0) = 0, \quad \zeta_i(0) = 0, \quad i = 1, 3, 4, \quad \text{and} \quad \zeta_i(0) = \zeta_i^*, \quad i = 2, 5, 6.
\]

We must show, in order to have periodicity, that the remaining two equations,

\[
\mathcal{P}_i(\zeta, \nu, \varepsilon) = 0, \quad (i = 1, 6),
\]

are also satisfied in a possibly smaller neighborhood of \( \varepsilon = 0 \). That will be done in the next two sections, making use of the integrals of motion.

### 4. INTEGRALS OF MOTION

If \( z = (x_1, x_2, x_3, y_1, y_2, y_3)^T \), let

\[
s(z, t) = x_1 e_1(t) + x_2 e_2(t) + x_3 e_3(t),
\]

\[
\dot{s}(z, t) = y_1 e_1(t) + y_2 e_2(t) + y_3 e_3(t).
\]

Notice that if we set \( s(t) = s(z(t), t), \dot{s}(t) = \dot{s}(z(t), t) \) and \( \mathbf{r}(t) = \mathbf{r}^0(t) + \varepsilon s(t), \dot{\mathbf{r}}(t) = \dot{\mathbf{r}}^0(t) + \varepsilon \dot{s}(t) \), then \( z(t) \) is a solution of the system (5) if, and only if, \( \mathbf{r}(t) \) is a solution of (2).

Since

\[
\mathbf{r}^0 \left( t + \frac{\tau}{2} \right) = -\mathbf{r}^0(t) \quad \text{and} \quad \dot{\mathbf{r}}^0 \left( t + \frac{\tau}{2} \right) = -\dot{\mathbf{r}}^0(t), \quad (13)
\]

we have

\[
s \left( z, t + \frac{\tau}{2} \right) = -x_1 e_1(t) - x_2 e_2(t) + x_3 e_3(t),
\]

\[
\dot{s} \left( z, t + \frac{\tau}{2} \right) = -y_1 e_1(t) - y_2 e_2(t) + y_3 e_3(t),
\]
for all \( z \) and for all \( t \), so it follows that

\[
\begin{align*}
    s \left( z, t + \frac{\tau}{2} \right) &= -s(0, z, t), \\
    \dot{s} \left( z, t + \frac{\tau}{2} \right) &= -\dot{s}(0, z, t)
\end{align*}
\]

(14)

for all \( z \) and for all \( t \).

If \( I(r, \dot{r}, \epsilon) \) is a first integral for the differential equations of motion (2), for instance the energy or the polar angular momentum, then defining

\[
I_s(z, t) = I(r, \dot{r}),
\]

where \( r = r^0(t) + \epsilon s \) and \( \dot{r} = \dot{r}^0(t) + \epsilon \dot{s} \), it is clear that \( I_s(z, t) \) is a time-dependent, \( r \)-periodic first integral for the system (5).

From the invariance of the integral \( I \) under the action of the group \( G \), the above integral \( I_s \) satisfies the relation

\[
I_s(\phi(t, z, \epsilon), t) = I_s(z + b + \epsilon N z, 0).
\]

(15)

Indeed, with \( g = I + \epsilon \)

\[
I_s(\phi(t, z, \epsilon), t) = I(r^0(0) + \epsilon s(0, z, 0), \dot{r}^0(0) + \epsilon \dot{s}(0, z, 0), \epsilon) = I(g(r^0(0) + \epsilon s(0, z, 0)), g(\dot{r}^0(0) + \epsilon \dot{s}(0, z, 0), \epsilon) = I(r^0(0) + \epsilon(s(0, z, 0) + \epsilon N^0(0) + \epsilon N s(0, z, 0), \epsilon) = I_s(z + b + \epsilon N z, 0),
\]

where \( \cdots \) stands for the analogous expressions involving the derivatives \( \dot{r}^0 \) and \( \dot{r} \).

As consequence of (13) and (14) we have the following result

\[
I_s \left( z, t + \frac{T}{2} \right) = I_s(0, s(0, z, t)).
\]

(16)

In fact, by (14), and the symmetry given by (13), we have

\[
\begin{align*}
    r^0 \left( t + \frac{T}{2} \right) + \epsilon s \left( z, t + \frac{T}{2} \right) &= -r^0(t) - \epsilon s(0, z, t), \\
    \dot{r}^0 \left( t + \frac{T}{2} \right) + \epsilon \dot{s} \left( z, t + \frac{T}{2} \right) &= -\dot{r}^0(t) - \epsilon \dot{s}(0, z, t).
\end{align*}
\]

The result now follows from the equality

\[
\bar{r} - r = \bar{r}(t, \dot{r}).
\]
a fact which is true for the integrals of energy and angular momentum. Passing to $\zeta$-coordinates, we have the integral
\[ A(\zeta, t) = I(\mathcal{F}_\zeta, t) \]
for the system (6) and the above Eq. (15) is written as
\[ A(\phi(t, \zeta, \epsilon), 0) = A(\zeta + \beta + eN'\zeta, 0). \tag{17} \]
Using (15) we obtain now, in the $\zeta$ variables,
\[ A\left( \mathcal{R}\phi\left( \frac{\tau}{2}, \zeta, \epsilon \right), 0 \right) - A(\zeta + \beta + eN'\zeta, 0) = 0, \tag{18} \]
and by the Mean Value Theorem and the fact that $\mathcal{R}^{-1} = \mathcal{R}^T = \mathcal{R}$, we have
\[ \mathcal{R} \nabla_{\zeta} A(\zeta, 0) \cdot \mathcal{P}(\zeta, \nu, \epsilon) = 0, \tag{19} \]
where $V_{\zeta} A$ is the gradient of $A$ with respect to $\zeta$, $\mathcal{P}$ is the function on the left-hand side in (12), and $\mathcal{Z}$ is a point on the line segment joining $\zeta + \beta + eN'\zeta$ to $\mathcal{R}\phi(z, \mathcal{Z}, \epsilon)$.

We now expand in powers of $\epsilon$ the functions
\[ w(\epsilon) = \zeta + \beta + eN'\zeta \quad \text{and} \quad \Psi(\epsilon) = \mathcal{R}\phi\left( \frac{\tau}{2}, \zeta, \epsilon \right), \]
where $\zeta = \zeta(\epsilon)$ and $\nu = \nu(\epsilon)$.

Clearly, $w(0) = \zeta^*$, while $\Psi(0) = \zeta^*$ by the choice of $\zeta^*$. It follows that
\[ w(\epsilon) = \zeta^* + O(\epsilon) \quad \text{and} \quad \Psi(\epsilon) = \zeta^* + O(\epsilon), \]
so that
\[ \bar{\zeta} = sw(\epsilon) + (1 - s) \Psi(\epsilon) = \zeta^* + O(\epsilon), \]
for some $s \in (0, 1)$. Using the definition of $I(z, 0)$ we expand it in powers of $\epsilon$ and find that
\[ I(z, 0) = I_0 + \epsilon I_1 \cdot z + O(\epsilon^2), \tag{20} \]
where the scalar $I_0$ and the vector $I_1$ depend only on the elements of the Kepler orbit. Passing to $\zeta$ coordinates, we have
\[ A(\zeta, 0) = A_0 + \epsilon A_1 \cdot \zeta + O(\epsilon^2), \]
where $A_0 = I_0$ and $A_1 = \mathcal{F}^T I_1$. 

Consequently, we have
\[
\frac{1}{\varepsilon} \# \nabla_{\zeta} \mathcal{E}(\zeta, 0) = \mathcal{R}_1 + O(\varepsilon). \tag{21}
\]

In the next section we will work out this expression in the case of the energy and the polar component of the angular momentum.

5. THE ENERGY AND THE POLAR ANGULAR MOMENTUM

The total energy of the system is given by
\[
E = \frac{1}{2} |\dot{r}|^2 - U_\mu(r),
\]
with \( U_\mu \) as in (1). Developing \( E \) in Taylor series about \( \varepsilon = 0 \), we obtain
\[
E(x, t) = E_0 + \varepsilon E_1 \cdot x + O(\varepsilon^2),
\]
where \( E_0 \) is the energy of the orbit \( r^0(t) \) of the Kepler problem and
\[
E_1 = (a \omega^2, 0, 0, a\omega, 0).
\]
The polar component of the angular momentum, \( C = (r \times \dot{r}) \cdot \mathbf{k} \), is given by
\[
C(x, t) = C_0 + \varepsilon C_1 \cdot x + O(\varepsilon^2),
\]
where \( C_0 \) is the angular momentum corresponding to \( r^0(t) \) and
\[
C_1 = (a \omega \cos t, 0, 0, a \cos t, -a \sin t).
\]

Passing to the new coordinate system via the transpose of \( \mathcal{F} \), we find the corresponding vectors. For the energy, we get
\[
\mathcal{E}_0(\zeta, t) = E_0 + \varepsilon \mathcal{E}_1 \cdot \zeta + O(\varepsilon), \quad \text{with} \quad \mathcal{E}_1 = (a \omega^2, 0, 0, 0, 0),
\]
and for the angular momentum we have
\[
\mathcal{C}_0(\zeta, t) = C_0 + \varepsilon \mathcal{C}_1 \cdot \zeta + O(\varepsilon), \quad \text{with} \quad \mathcal{C}_1 = (a \omega \cos t, 0, 0, 0, a \omega \sin t).
\]

Using (21), we compute at the point \( \zeta \) corresponding to each one of these two integrals,
\[
\frac{1}{\varepsilon} \# \nabla_{\zeta} \mathcal{E}_0(\zeta(\varepsilon)) = (a \omega^2, 0, 0, 0, 0) + O(\varepsilon)
\]
and
\[
\frac{1}{\varepsilon} \# \nabla_{\zeta} \mathcal{C}_0(\zeta(\varepsilon)) = (a \omega \cos t, 0, 0, 0, a \omega \sin t) + O(\varepsilon).
\]
Since, for small $\varepsilon$, we already have $P_i = 0$ ($i = 2, 3, 4, 5$), the equation (19) reduces to the system of two equations in the unknowns $P_1$, $P_6$

\[
[a \varepsilon^2 + O(\varepsilon)] P_1 + [0 + O(\varepsilon)] P_6 = 0
\]

\[
[a \varepsilon \cos \theta + O(\varepsilon)] P_1 + [a \varepsilon \sin \theta + O(\varepsilon)] P_6 = 0.
\]

For $\varepsilon = 0$ this system has determinant $a^2 \varepsilon^3 \sin \theta$. We have already made the restriction that $\sin \theta \neq 0$. Consequently, by continuity the determinant is different from zero for $\varepsilon$ sufficiently small. For such values of $\varepsilon$, therefore, the unique solution of this linear system is the trivial one, which means that the two remaining equations

\[
P_i(\zeta(\varepsilon), \theta(\varepsilon), \varepsilon) = 0, \quad (i = 1, 6),
\]

are also satisfied. Therefore, all the equations of the periodicity system (12) are satisfied when $\zeta = \zeta(\varepsilon)$ and $\theta = \theta(\varepsilon)$, as long as $\varepsilon$ is sufficiently small. This proves the theorem.

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