A note on finite-sheeted covering maps from 2-dimensional compact Abelian groups

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We consider finite-sheeted covering maps from 2-dimensional compact connected abelian groups to Klein bottle weak solenoïdal spaces, metric continua which are not groups. We show that whenever a group covers a Klein bottle weak solenoïdal space it covers groups as well, moreover it covers the product of two solenoïds. The converse is not true, we give an example of group which covers groups with any finite number of sheets, but does not cover any Klein bottle weak solenoïdal space.

1. Introduction and the main result

In his paper [10] M.C. McCord introduced a notion of (weak) solenoïdal space. A (weak) solenoïdal space is the limit space of an inverse sequence, where each term is a connected, locally pathwise connected, semilocally 1-connected space and each bonding map is a (covering map) regular covering map. Recall that each compact connected 2-dimensional abelian group $X$ is a solenoïdal space obtained as the limit of an inverse sequence, where each term is 2-torus $T^2$ and each bonding map is a covering homomorphism. That is why such groups are called toroidal groups for short. A Klein bottle weak solenoïdal space is the limit of an inverse sequence, where each term is Klein bottle $K$ and each bonding map is a covering map from $K$ to $K$. In his paper [11] C. Tezer introduced and studied Klein bottle weak solenoïdal spaces $\Sigma(p, q, r)$, where $p = (p_i)$, $q = (q_i)$, $r = (r_i)$ are sequences of integers, $p_i \neq 0$ and $r_i$ odd for each $i$. Recently, finite-sheeted covering maps over toroidal groups $X$ were studied [2]. It turned out that finite-sheeted covering maps over X were determined using finite-index torsion free supergroups of the Pontryagin dual $\hat{X}$ [3]. Moreover, using finite index subgroups of $\hat{X}$ there were also presented finite-sheeted covering maps from $X$ to other compact connected groups. The main step in the investigation was the reduction to the case of finite-sheeted covering homomorphisms $f : X' \to X$ between two toroidal groups. Since $T^2$ is a covering space for Klein bottle $K$, the next step of the investigation was to study finite-sheeted covering maps over Klein bottle weak solenoïdal spaces $\Sigma(p, q, r)$ (see [9]). If a toroidal group $X$ admits a finite-sheeted covering map, then a total space is a toroidal group $X'$, generally non-homeomorphic to $X$. If a Klein bottle weak solenoïdal space $\Sigma(p, q, r)$ admits a finite-sheeted covering map, then a total space is a toroidal group $X$ or a Klein bottle weak solenoïdal space $\Sigma(p', q', r')$ homeomorphic to $\Sigma(p, q, r)$.

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In this note we consider the following questions:

- Is every toroidal group $X$ a covering space over some Klein bottle weak solenoidal space $\Sigma(p, q, r)$?
- If a toroidal group $X$ covers groups, does it cover Klein bottle weak solenoidal spaces?
- If a toroidal group $X$ covers Klein bottle weak solenoidal spaces, does it cover groups?

We answer first two questions in the negative (see Theorem 21, Corollary 22) and the third one in the positive (see Theorem 4, Corollary 5). Moreover, we give an example of a toroidal group $X$, which admits an $s$-sheeted covering map $f : X \rightarrow X'$ to groups for each $s \in \mathbb{N}$, but which does not cover any Klein bottle weak solenoidal space $\Sigma(p, q, r)$ (see Theorem 23, Corollary 24). It turned out that the key object in this investigation was the product $\Sigma(p) \times \Sigma(r)$ of two solenoids $\Sigma(p)$ and $\Sigma(r)$, where at least one of the sequences $p$ and $r$ consists of odd integers.

2. Covering maps from groups to $\Sigma(p, q, r)$

The Klein bottle $K$ can be presented as the quotient manifold $\mathbb{R}^2 / G$, where the group $G = \langle \sigma, \rho \mid \sigma \rho = \rho \sigma^{-1} \rangle$ acts properly discontinuously on $\mathbb{R}^2$ by the affine transformations $\sigma, \rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$:

$$
\sigma(x, y) = (x + 1, y),
\rho(x, y) = (-x, y + 1/2).
$$

Let $y_0 \in K$ be the image of $(0,0) \in \mathbb{R}^2$ under the quotient map. Then $\pi_1(K, y_0)$ can be naturally identified with $G$. Each element of $G$ can be presented as $\sigma^n \rho^m$, $n, m \in \mathbb{Z}$. Note that $G$ can be viewed as the group $(\mathbb{Z}^2, \ast)$, where the group operation $\ast$ is given by $(n, m) \ast (k, l) = (n + (-1)^m k, m + l)$. A subgroup of $(\mathbb{Z}^2, \ast)$ of $(\mathbb{Z}^2, \ast)$ is isomorphic to $(\mathbb{Z}^2, +)$. Since $H = \langle \sigma, \rho \rangle = \langle (\mathbb{Z} \times 2\mathbb{Z}, \ast) \rangle$ is isomorphic to $(\mathbb{Z}^2, +)$, it follows that the quotient manifold $\mathbb{R}^2 / H$ is the 2-torus $\mathbb{T}^2$. Let $x_0 \in \mathbb{T}^2$ be the image of $(0,0) \in \mathbb{R}^2$ under the quotient map. Then $\pi_1(\mathbb{T}^2, x_0) = H$. Since $H$ is a subgroup of index 2 of $G$, the identity map $id_{\mathbb{T}^2} : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ induces a pointed map $\delta : (\mathbb{T}^2, x_0) \rightarrow (K, y_0)$, which is a 2-sheeted covering map, so-called "basic" 2-sheeted covering map of $\mathbb{T}^2$ over $K$.

Each endomorphism of $G$ is of the form $h_{(p, q, r)} : G \rightarrow G$,

$$
h_{(p, q, r)}(\sigma) = \sigma^p,
\quad h_{(p, q, r)}(\rho) = \sigma^q \rho^r,
$$

where $p, q, r \in \mathbb{Z}$ and $r$ is odd whenever $p \neq 0$. Moreover, $h_{(p, q, r)}$ is injective if and only if $p \neq 0$ and $r$ is odd. For each integers $p, q$ and $r$, $p \neq 0$ and $r$ odd, Tezer introduced (see [11, Section 1.5]) pointed covering maps $f_{(p, q, r)} : (K, y_0) \rightarrow (K, y_0)$ in the following way. Let $\theta_{(p, q, r)} : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be a map such that $\theta_{(p, q, r)}(y + 1/2) = -\theta_{(p, q, r)}(y) + q$ and define $F_{(p, q, r)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $F_{(p, q, r)}(x, y) = (px + \Theta_{(p, q, r)}(y), ry)$. $F_{(p, q, r)}$ is the lifting of a map $f_{(p, q, r)} : (K, y_0) \rightarrow (K, y_0)$ such that $f_{(p, q, r)} = h_{(p, q, r)}$. Let $p = (p_i), q = (q_i)$ and $r = (r_i)$ be sequences of integers such that $p_i \neq 0$ and $r_i$ odd for each $i$. The inverse limit of an inverse sequence $\{K_i, f_{i+1, i} : N\}$, where each $K_i = K$ and each bonding map $f_{i+1, i} = f_{(p_i, q_i, r_i)} : K \rightarrow K$, is a Klein bottle weak solenoidal space denoted by $\Sigma(p, q, r)$.

Let $\Pi$ denote the set of prime numbers and let $v = (v_i)$ be a sequence of integers. The *prime profile of $v$* is a function $\xi_v : \Pi \rightarrow \mathbb{N} \cup \{0, \infty\}$ defined by

$$
\xi_v(p) = \sum_{i=1}^{\infty} h_{i},
$$

where

$$
h_i = \max\{k \in \mathbb{N} \cup \{0\} \mid p^k \mid v_i\}.
$$

We say that sequences $v$ and $v'$ have the *same prime profile* if $\xi_v$ and $\xi_{v'}$ differ in at most finitely many arguments and the difference is at most finite.

We say that $\xi_v \leq \xi_{v'}$ if $\xi_v(p) \leq \xi_{v'}(p)$ for almost all $p \in \Pi$ including all $p$ for which $\xi_v(p) = \infty$. Sequences $v$ and $v'$ have *comparable prime profiles* if $\xi_v \leq \xi_{v'}$ or $\xi_{v'} \leq \xi_v$.

Tezer proved that Klein bottle weak solenoidal spaces $\Sigma(p, q, r)$ and $\Sigma(p', q', r')$ are homeomorphic if and only if the sequences $p, r$ and $p', r'$ respectively, have the same prime profiles [11, Proposition 2.5]. In particular, $\Sigma(p, q, r)$ and $\Sigma(p, 0, r)$ are homeomorphic.

Let $\ast \in \Sigma(p, q, r)$ denote a point $\ast = (y_i)$, where each $y_i = y_0 \in K$. Note that every pointed Klein bottle weak solenoidal space $(Y, y)$ is pointed homeomorphic to some $(\Sigma(p, q, r), \ast)$, where $p_i, q_i$ are positive and $0 \leq q_i < p_i$ for each $i$ [9, Proposition 4.1]. In the sequel we consider only spaces $\Sigma(p, q, r)$, where the sequences $p$ and $r$ consist of positive integers.
Recall that every toroidal group $X$ can be presented as the inverse limit of an inverse sequence $\{X_i, g_{i+1}, N_i\}$, where each $X_i = \mathbb{T}^2$ and each bonding map $g_{i+1}: \mathbb{T}^2 \to \mathbb{T}^2$ is a homomorphism represented by an integral matrix $M_i = \begin{bmatrix} v_i & a_i \\ 0 & 1 \end{bmatrix} \in \mathcal{M}_2(\mathbb{Z})$ such that $v_i, a_i > 0$ and $0 < a_i < v_i$. Usually, instead of $g_{i+1}$ we write $g_{i+1} = g_{Mi}$ or just $M_i$ and say that $X$ is obtained by integral matrices $M_i$. Let $* \in X$ denote the identity element of a toroidal group $X$. Note that a pointed space $(X, *) = \lim \{(\mathbb{T}^2, x_0), g_{i+1}, N_i\}$.

**Proposition 1.** Each space $\Sigma(p, q, r)$ admits up to equivalence only one double sheeted covering map $\delta : X \to \Sigma(p, q, r)$, where $X$ is a toroidal group. Moreover, $X$ is homeomorphic to the product $\Sigma(p) \times \Sigma(r)$ of solenoids $\Sigma(p)$ and $\Sigma(r)$, obtained by sequences $p$ and $r$ respectively.

**Proof.** For each $i \in \mathbb{N}$ let $\delta_i = \delta : (\mathbb{T}^2, x_0) \to (K, y_0)$ be the basic double sheeted covering from $\mathbb{T}^2$ to $K$. Since $h_{(p_i, q_i, r_i)}(\sigma) = \sigma^p_i$ and $h_{(p_i, q_i, r_i)}(\rho^2) = (\sigma^p_i)^2 = \rho^{2p_i}$, it follows $h_{(p_i, q_i, r_i)}(H) = h_{(p_i, q_i, r_i)}(\sigma, \rho^2) \subseteq H$. Then, for each $i$, there is a pointed map $\psi_{i+1} : (\mathbb{T}^2, x_0) \to (\mathbb{T}^2, x_0)$ such that $\delta \psi_{i+1} = f_{(p_i, q_i, r_i)} \delta$. Note that $\psi_{i+1}$ is a covering map. Further, $Hh_{(p_i, q_i, r_i)}(\rho) = H\sigma^p_i \rho^{2p_i} = H\rho$ and according to [6, Lemma 10] we conclude that each

$$
\begin{align*}
(T^2, x_0) &\xleftarrow{\psi_{i+1}} (T^2, x_0) \\
\delta &\quad \quad \quad \delta \\
(K, y_0) &\xleftarrow{f_{(p_i, q_i, r_i)}} (K, y_0)
\end{align*}
$$

is a pointed pull-back diagram. $\psi_{i+1} : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a homomorphism represented by the integral matrix $[\begin{smallmatrix} p_i & 0 \\ 0 & r_i \end{smallmatrix}]$. Namely, $\psi_{i+1} : (\mathbb{Z}^2, x_0) \to (\mathbb{Z}^2, x_0)$ is a double sheeted covering from $\mathbb{Z}^2$ to $\mathbb{Z}^2$.

**Corollary 2.** Let $p = (p_i)$ and $r = (r_i)$ be sequences of positive integers, each $r_i$ odd. A toroidal group $X$ double covers $\Sigma(p, q, r)$ if and only if $X$ is homeomorphic to the product $\Sigma(p) \times \Sigma(r)$ of solenoids $\Sigma(p)$ and $\Sigma(r)$.

**Remark 3.** Let $p = (p_i)$ and $r = (r_i)$ be sequences of positive odd integers. If a toroidal group $X$ double covers $\Sigma(p, q, r)$, then, according to Corollary 2, $X$ double covers a Klein bottle weak solenoidal space $\Sigma(r, q, p)$ as well. In general, $\Sigma(p, q, r)$ and $\Sigma(r, q, p)$ are not homeomorphic spaces.
Theorem 4. Let $X$ be a toroidal group and let $f : (X, x) \to (\Sigma(p, q, r), *)$ be a pointed map. $f$ is a covering map if and only if $f = hg$ where $g : (X, x) \to (\Sigma(p) \times \Sigma(r), *)$ is a covering map between toroidal groups and $h : (\Sigma(p) \times \Sigma(r), *) \to (\Sigma(p, q, r), *)$ is a double covering.

Proof. Let $f : (X, x) \to (\Sigma(p, q, r), *)$ be a pointed covering map. Since $X$ is compact, $f$ is $s$-sheeted for some $s \in \mathbb{N}$. According to [8, Theorem 6], $f$ admits a pointed ANR-pull-back expansion, i.e. there exist $i_0 \in \mathbb{N}$ and a pointed map $f_i : (X, x) \to (K, y_{i_0})$ such that $(X, x), \varphi_{i+1}, i \geq i_0 \to (K, y_{i_0}), f_{i+1}$, $i \geq i_0$ is a pointed inverse sequence such that $(X, x) = \lim_i (X, x), \varphi_{i+1}, i \geq i_0$, $f = \lim_i f_i$ for each $(X, x) \to (K, y_{i_0})$ is an $s$-sheeted covering map with a connected total space and pointed maps $f_i, f_{i+1}, \varphi_{i+1}, f_{i+1}$ form pointed pull-back diagrams for each $i$. Note that each $f_i = \delta g_i$, where $g_i : (X, x) \to (\Sigma^2, x_0)$ is an $\frac{s}{2}$-sheeted covering map and $\delta : (\Sigma^2, x_0) \to (\Sigma(p, q, r), *).$ The basic pointed double covering over $(\Sigma(p, q, r), *), i.e. (X, x) = \lim_i (\Sigma^2, x_0), \psi_{i+1}, i \geq i_0, \delta \psi_{i+1}, \delta \varphi_{i+1}$ form pointed pull-back diagrams. First note that $g_{i+1} \psi_{i+1} = \psi_{i+1} g_{i+1}$. Indeed, $g_{i+1} \varphi_{i+1} = \varphi_{i+1} g_{i+1} = \varphi_{i+1} g_{i+1} = \varphi_{i+1} g_{i+1}$ and $g_{i+1} \varphi_{i+1} (X, x) = \varphi_{i+1} g_{i+1} (X, x)$. We claim that $g_i, g_{i+1}, \psi_{i+1}, \psi_{i+1}$ form pull-back diagrams for each $i \geq i_0$. According to [6, Lemma 7] it is sufficient to prove that $\varphi_{i+1}$ induces a bijection between the fibers of $y \in \Sigma^2$ and over $y' = \psi_{i+1} (y) \in \Sigma^2$. Since the fibers are of the same finite cardinality, it is sufficient to prove that $\varphi_{i+1}$ induces injection between the fibers. Let $x_1, x_2 \in g_{i+1}^{-1}((y))$ and $x_1 \neq x_2$. Then $f_{i+1} (x_1) = \delta g_{i+1} (x_1) = \delta g_{i+1} (x_2) = f_{i+1} (x_2)$ and $x_1, x_2 \in (f_{i+1})^{-1}(\delta (y))$. Since $f_i, f_{i+1}, \varphi_{i+1}, f_{i+1}$ form pull-back diagram, [6, Lemma 6] implies that $\varphi_{i+1} (x_1) \neq \varphi_{i+1} (x_2)$. Let $g' = \lim (g_i : (X, x) \to (\Sigma^2, x_0), i \geq i_0) : (X, x) \to (\Sigma^2, x_0).$ According to [8, Theorem 6] $g'$ is a covering map and $f = \delta g'$, since $f (z) = (f_i (z)) = (\delta g_i (z)) = \delta (g' (z))$. Let $\phi : (X, x) \to (\Sigma(p) \times \Sigma(r), *)$ be a pointed homeomorphism and define $h = \delta \phi^{-1}$ and $g = \phi g'$. The converse is obvious since the composition of two finite-sheeted covering map is a covering map. □

Corollary 5. Let $s \in \mathbb{N}$ be an even integer and let $r = (r_1)$ be a sequence of positive odd integers. A toroidal group $X$ admits an $s$-sheeted covering map over $(\Sigma(p, q, r))$ if and only if $X$ admits an $\frac{s}{2}$-sheeted covering map over $(\Sigma(p) \times \Sigma(r))$.

Corollary 6. Let $r = (r_1)$ be a sequence of positive odd integers. If a toroidal group $X$ admits an $s$-sheeted covering map over $(\Sigma(p, q, r))$, then $X$ admits an $s' \cdot 2^{k}$-sheeted covering map over $(\Sigma(p, q, r))$ for each $k \in \mathbb{N}$. In particular, if $X$ is homeomorphic to $(\Sigma(p) \times \Sigma(r))$, then $X$ admits a $2^{k}$-sheeted covering map over $(\Sigma(p, q, r))$ for each $k \in \mathbb{N}$.

Proof. Since the solenoid $(\Sigma, \Sigma)$ admits a $2^{k}$-sheeted covering map $f : \Sigma(r) \to \Sigma(r)$ for each $k \in \mathbb{N}$ (see [8, Theorem 8]), the conclusion follows from Corollary 5. □

We will give examples of toroidal groups which cover Klein bottle weak solenoidal spaces. First we consider simple cases when homeomorphic to the product of two solenoids. Recall that compact connected abelian topological groups $A$ and $B$ are shape equivalent if and only if they are isomorphic as topological groups (see [4, Theorem 1.2 and Corollary 1.3]). Having in mind that every continuous map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is homotopic to a unique homomorphism and every homomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ can be represented by a $2 \times 2$ integral matrix (see [5, §2]), we get the next lemma which is useful for deciding whether two toroidal groups are homeomorphic (see [7, Ch. I, §1.2, Lemma 2 and Ch. II, §2.2, Theorem 5]).

Lemma 7. Let $X$ be a group obtained by integral matrices $M_1 = \left[ \begin{array}{cc} v_1 & a_1 \\ 0 & 1 \end{array} \right]$ and let $Y$ be a group obtained by integral matrices $N_1 = \left[ \begin{array}{cc} v_1 & a_1 \\ 0 & 1 \end{array} \right]$. $X$ is homeomorphic to $Y$ if and only if there exist strictly increasing sequences $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ and integral matrices $F_i = \left[ \begin{array}{cc} v_i & a_i \\ z_i & w_i \end{array} \right]$, $G_i = \left[ \begin{array}{cc} x_i & y_i \\ z_i & w_i \end{array} \right]$ such that the following diagram

\[
\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{M_{i+1} - M_i} & \mathbb{T}^2 \\
F_i & \downarrow & F_{i+1} \\
\mathbb{T}^2 & \xrightarrow{N_{i+1} - N_i} & \mathbb{T}^2 \\
G_i & \nearrow & G_{i+1}
\end{array}
\]

commutes for each $i$.

Let $p = (p_i)$ be a constant sequence, i.e. each $p_i = p \in \mathbb{N}$. Then instead of $p$ we will write $p$, instead of $(\Sigma(p))$ we will write $\Sigma_p$ and instead of $(\Sigma(p, q, r))$ we will write $\Sigma(p, q, r)$. Note that $\Sigma_1$ is the unit circle $S^1$.

Proposition 8. A group $X$ obtained by integral matrices $M_1 = \left[ \begin{array}{cc} v_1 & a_1 \\ 0 & 1 \end{array} \right]$. Then $X$ is homeomorphic to $S^1 \times (\Sigma(p))$. $X$ admits an $s$-sheeted covering map over $(\Sigma(p, q, r))$ for each even $s \in \mathbb{N}$. If each $p_i$ is odd, $X$ covers $\Sigma(1, q, p)$ as well.
Proof. Let \( x_1 = 0, x_{i+1} = \alpha_i + x_ip_i, i > 1 \), and for each \( i \) put \( F_i = \begin{bmatrix} 1 & x_i \\ 0 & 1 \end{bmatrix}, N_i = \begin{bmatrix} 1 & 0 \\ 0 & p_i \end{bmatrix} \). Then
\[
F_iM_i = \begin{bmatrix} 1 & x_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha_i \\ 0 & p_i \end{bmatrix} = \begin{bmatrix} 1 & \alpha_i + x_ip_i \\ 0 & p_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & p_i \end{bmatrix} \begin{bmatrix} 1 & x_{i+1} \\ 0 & 1 \end{bmatrix} = N_iF_{i+1}
\]
and isomorphisms represented by matrices \( F_i \) induce a homeomorphism from \( X \) to \( S^1 \times \Sigma(p) \). Since \( S^1 \times \Sigma(p) \) covers \( S^1 \times \Sigma(p, 1) \) with any number of sheets, it follows that \( X \) covers \( \Sigma(p, q, 1) \) with any even number of sheets. \( \square \)

**Proposition 9.** Let \( p \in \mathbb{N} \) and let \( M_i = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \) for each \( i \). A group \( X \) obtained by integral matrices \( M_i \) is homeomorphic to \( \Sigma_p \times S^1 \). 

\( X \) admits an s-sheeted covering map over \( \Sigma(p, q, 1) \) for each even \( s \in \mathbb{N} \). If \( p \) is odd, \( X \) covers \( \Sigma(1, q, p) \) as well.

**Proof.** Let \( d = \text{GCD}(p-1, \alpha) \). Then \( p-1 = dp' \) and \( \alpha = d\alpha' \) for some integers \( p' \) and \( \alpha' \). Since \( \text{GCD}(p', \alpha') = 1 \), there exist integers \( x_0, y_0 \in \mathbb{Z} \) such that \( x_0p' - y_0\alpha' = 1 \). For each \( i \) put \( F_i = \begin{bmatrix} p' & y_0p'y_0\alpha' \sum_{j=0}^{i-1} p^{j} + x_0 \\ 0 & 1 \end{bmatrix} \) and \( N_i = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \). Note that
\[
\det F_i = y_0\alpha'(p-1) \sum_{j=0}^{i-1} p^j + p'x_0 - y_0p'^i\alpha' = y_0\alpha'(p^i - 1) + p'x_0 - y_0p'^i\alpha' = 1
\]
for each \( i \). Furthermore,
\[
F_iM_i = \begin{bmatrix} p' & y_0\alpha' \sum_{j=0}^{i-1} p^j + x_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p' & y_0p'y_0\alpha' \sum_{j=0}^{i-1} p^{j} + x_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p' & y_0p'y_0\alpha' \sum_{j=0}^{i-1} p^{j} + x_0 \\ 0 & 1 \end{bmatrix},
\]
which shows \( F_iM_i = N_iF_{i+1} \) for each \( i \). Matrices \( F_i \) induce a homeomorphism from \( X \) to \( \Sigma_p \times S^1 \). Since \( \Sigma_p \times S^1 \) covers \( \Sigma \times S^1 \) with any number of sheets, it follows that \( X \) covers \( \Sigma(p, q, 1) \) with any even number of sheets. \( \square \)

**Remark 10.** The statement that the group \( X \) obtained by matrices \( M_i = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \) is homeomorphic to \( \Sigma_p \times S^1 \) also follows from [3, Theorem 6.2], where Pontryagin dual \( \hat{X} \) of \( X \) was considered.

**Proposition 11.** Let \( X \) be a group obtained by integral matrices \( M_i = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}, \alpha \), \( \lambda_i, \mu_i \in \mathbb{N} \). Then \( X \) is homeomorphic to \( \Sigma_p \times \Sigma_p \). If \( p \) is odd, \( X \) admits an s-sheeted covering map over \( \Sigma(p, q, p) \) for each even \( s \), which is relatively prime with \( p \).

**Proof.** First we show that \( X \) is homeomorphic to \( \Sigma_p \times \Sigma_p \). Let \( \varphi(1) = 1 \) and \( F_1 = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \). By induction, for each \( i \in \mathbb{N} \), we will define integral matrices \( N_i = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}, G_i = \begin{bmatrix} 1 & 0 \\ 0 & p^{\frac{1}{i}} \end{bmatrix} \), \( F_i = \begin{bmatrix} p \alpha \sum_{j=0}^{i-1} p^j + x_0 \\ 0 \end{bmatrix} \) and an integer \( \varphi(i+1) > \varphi(i), G_iF_i = M_{\varphi(i)} \cdots M_{\varphi(i+1)} \) and \( N_i = F_iG_i \).

For \( i = 1 \) put \( \varphi(2) = 2 \), \( N_1 = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & 0 \\ 0 & p^{\frac{1}{1}} \end{bmatrix} \). Then \( G_1F_2 = M_1 \) and \( N_1 = F_1G_1 \).

Assume that \( \varphi(i) > \varphi(i+1), N_{i+1}, G_{i+1}, 1 \leq i \leq n, \) with requested properties are already defined. Let \( k \geq \varphi(n+1) \) be the first integer such that \( \mu = \varphi(n+1) + \mu_1 + \cdots + \mu_k > u_{n+1} \) and denote \( M_{\varphi(n+1)} \cdots M_k = \begin{bmatrix} p^k & 0 \\ 0 & 1 \end{bmatrix} \). Then \( \lambda = \varphi(n+1) + \lambda_{n+1} + \cdots + \lambda_k \) and \( \mu_k \). Put \( \varphi(n+2) = k+1 \), \( u_{n+2} = \lambda \), \( v_{n+2} = p - u_{n+1} \), \( x_{n+2} = \alpha + x_{n+1}p^{\mu - u_{n+1}} \), and \( N_{n+1} = \begin{bmatrix} p & 0 \\ 0 & p^{\frac{1}{k+1}} \end{bmatrix} \).

We get
\[
G_{n+1}F_{n+2} = \begin{bmatrix} 1 & -x_{n+1} \\ 0 & p^{u_{n+1}} \end{bmatrix} \begin{bmatrix} p^\lambda & \alpha + x_{n+1}p^\mu - u_{n+1} \\ 0 & p^\mu - u_{n+1} \end{bmatrix} = \begin{bmatrix} p^\lambda & \alpha \\ 0 & p^\mu \end{bmatrix}
\]
and the inductive step is done. Note that all \( u_n \) and \( v_n \) are positive integers. Let \( Y \) be a toroidal group obtained by matrices \( N_i = \begin{bmatrix} p & 0 \\ 0 & p^{\frac{1}{i+1}} \end{bmatrix} \). \( Y \) is homeomorphic to \( \Sigma_p \times \Sigma_p \). By the construction each diagram.
Corollary 6.

Proof. Assume that a toroidal group $X$ admits an $s$-sheeted covering map over $\Sigma(\mathbf{p}, \mathbf{q}, \mathbf{r})$. Then $X$ admits an $\frac{s}{2}$-sheeted covering map over $\Sigma(\mathbf{p}) \times \Sigma(\mathbf{r})$. Let $A$ and $A_p \oplus A_r$ be Pontryagin duals of $X$ and $\Sigma(\mathbf{p}) \times \Sigma(\mathbf{r})$ respectively. Here $A_p = \{x/j \mid j \in \mathbb{Z}, n \in \mathbb{N}\}$ and $A_r = \{x/j \mid j \in \mathbb{Z}, n \in \mathbb{N}\}$. Since $X$ admits a finite-sheeted covering map over $\Sigma(\mathbf{p}) \times \Sigma(\mathbf{r})$, $A$ and $A_p \oplus A_r$ are quasi isomorphic torsion free groups of rank two. According to [1, Theorem 9.6] $A$ and $A_p \oplus A_r$ are isomorphic groups, which implies that $X$ is homeomorphic to $\Sigma(\mathbf{p}) \times \Sigma(\mathbf{r})$. Converse follows from Corollary 6. \qed

A sequence $(x_n)$ of integers is said to be semi-periodic, if there exist positive integers $n_0$ and $k$ such that $x_n = x_{n+k}$ for each $n \geq n_0$.

Corollary 13. Let $p$ be a prime number and let $X$ be a toroidal group obtained by matrices $M_i = \begin{bmatrix} p & \alpha_i \\ 0 & 1 \end{bmatrix}$, $0 \leq \alpha_i < p$. Then the following statements are equivalent.

(i) $X$ covers $\Sigma(\mathbf{p}, \mathbf{q}, 1)$ and $\Sigma(1, \mathbf{q}, \mathbf{p})$;
(ii) $X$ is homeomorphic to $\Sigma_p \times S^1$;
(iii) $(\alpha_i)$ is semi-periodic.

Proof. According to Theorem 12 $X$ covers $\Sigma(\mathbf{p}, 1)$ and $\Sigma(1, \mathbf{q}, \mathbf{p})$ if and only if $X$ is homeomorphic to $\Sigma_p \times S^1$. On the other hand, $X$ is homeomorphic to $\Sigma_p \times S^1$ if and only if $(\alpha_i)$ is semi-periodic (see [3, Corollary 6.3]). \qed

In the sequel we will give an example of group which is not homeomorphic to the product of two solenoids, but covers Klein bottle weak solenoidal spaces.

Proposition 14. Let $X$ be a group obtained by integral matrices $M_i = \begin{bmatrix} v & 0 \\ 0 & t \end{bmatrix}$, $v, t \in \mathbb{N}$, $v \neq t$. If $\alpha \equiv \beta \pmod{(v-t)}$, then $X$ is homeomorphic to a group $Y$ obtained by matrices $N_i = \begin{bmatrix} v & \beta \\ 0 & t \end{bmatrix}$. In particular, if $v-t$ divides $\alpha$, then $X$ is homeomorphic to $\Sigma_v \times \Sigma_t$.

Proof. Let $x_0$ be integer such that $\alpha - \beta = x_0(v-t)$. For each $i$ put $F_i = \begin{bmatrix} 1 & x_0 \\ 0 & 1 \end{bmatrix}$.

Then $F_i M_i = \begin{bmatrix} 1 & x_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v & \alpha \\ 0 & t \end{bmatrix} = \begin{bmatrix} v & \alpha + tx_0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} v & \beta + vx_0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} v & \beta \\ 0 & t \end{bmatrix} \begin{bmatrix} 1 & x_0 \\ 0 & 1 \end{bmatrix}$ for each $i$. Matrices $F_i$ induce homeomorphism from $X$ to $Y$. If $(v-t) | \alpha$, then $\alpha \equiv 0 \pmod{(v-t)}$ and $X$ is homeomorphic to $\Sigma_v \times \Sigma_t$. \qed

Lemma 15. Let $x, y \in \mathbb{N}\setminus\{1\}$ be relatively prime integers, $\alpha \in \mathbb{Z}$, and let $(n_i)$ be a sequence in $\mathbb{N}$. If $(x-y) \nmid \alpha$, then there does not exist a sequence $(\varepsilon_i)$ of integers such that $\alpha \sum_{j=0}^{n_i-1} x^{n_i-1-j} y^j = \varepsilon_{i+1} x^{n_i} - \varepsilon_i y^{n_i}$ for each $i$.
**Proof.** Assume the contrary, i.e. there is a sequence \((e_i)\) of integers such that \(\alpha \sum_{j=0}^{n_i-1} x^{n_i-1-j}y^j = e_i x^{n_i} - e_i y^{n_i}\) for each \(i\). Let \(\delta_i\) denote \(\alpha \sum_{j=0}^{n_i-1} x^{n_i-1-j}y^j\). Then

\[
\alpha x^{n_i} - \alpha \delta_i = \alpha x^{n_i} - \alpha \sum_{j=0}^{n_i-1} x^{n_i-1-j}y^j = -\alpha \sum_{j=0}^{n_i-1} x^{n_i-1-j}y^j,
\]

(a)

\[
y^{n_i} - \delta_i = \alpha y^{n_i} - \alpha \sum_{j=0}^{n_i-1} x^{n_i-1-j}y^j + 1 = -\alpha \sum_{j=0}^{n_i-2} x^{n_i-1-j}y^j + 1 = -\alpha \sum_{j=0}^{n_i-1} x^{n_i-1-j}y^j.
\]

(b)

(a) and (b) imply \(\alpha x^{n_i} - \delta_i = \alpha y^{n_i} - \delta_i\) and

\[
\alpha x^{n_i} - x(e_{i+1} x^{n_i} - e_i y^{n_i}) = \alpha y^{n_i} - y(e_{i+1} x^{n_i} - e_i y^{n_i}),
\]

\[
x^{n_i}(\alpha - xe_{i+1} + ye_{i+1}) = y^{n_i}(\alpha - xe_i + ye_i).
\]

(c)

Multiplying (c) for \(i = 1, \ldots, k\) we get

\[
x^{n_1 + n_2 + \cdots + n_k}(\alpha - xe_{i+1} + ye_{i+1}) = y^{n_1 + n_2 + \cdots + n_k}(\alpha - xe_1 + ye_1),
\]

The integers \(x, y\) are relatively prime, which implies \(x^{n_1 + n_2 + \cdots + n_k}|(\alpha - xe_1 + ye_1)\) for each \(k\). Since sums \(n_1 + n_2 + \cdots + n_k \to \infty\), an expression \(\alpha - xe_1 + ye_1\) has to be 0, i.e. \(\alpha = e_1(x - y)\). This contradicts the assumption \((x - y)|\alpha\). □

**Theorem 16.** Let \(v, t \in \mathbb{P} \cup \{1\}, \lambda, \mu \in \mathbb{N}\), and let \(M_i = \begin{bmatrix} v^\lambda & \alpha \\ 0 & t^\mu \end{bmatrix}\) for each \(i\). A group \(X\) obtained by integral matrices \(M_i\) is homeomorphic to \(\Sigma_v \times \Sigma_t\) if and only if either

1. \(v = t, 0\)
2. one of \(v\) and \(t\) equals 1, or
3. \(v^\delta - t^\mu\) divides \(\alpha\).

**Proof.** Let \(X\) be homeomorphic to \(\Sigma_v \times \Sigma_t\). Assume the contrary, i.e. \(v, t \geq 2, v \neq t\) and \((v^\delta - t^\mu)|\alpha\). Since \(X\) is homeomorphic to \(\Sigma_v \times \Sigma_t\), by Lemma 7 there exist strictly increasing sequences \(\varphi, \psi: \mathbb{N} \to \mathbb{N}\) and integral matrices \(F_i = \begin{bmatrix} X_i \ y_i \\ Z_i \ W_i \end{bmatrix}\), \(G_i = \begin{bmatrix} x'_i \ y'_i \\ z'_i \ w'_i \end{bmatrix}\), such that the following diagram

\[
\begin{array}{c}
\mathbb{T}^2 \ \\
F_1 \ \\
\mathbb{T}^2
\end{array}
\begin{array}{c}
\xrightarrow{M_{\beta(i)+}} \ \\
\xrightarrow{G_i} \ \\
\xleftarrow{N_{\gamma(i)+}} \ \\
\mathbb{T}^2
\end{array}
\begin{array}{c}
\mathbb{T}^2 \ \\
F_{i+1} \ \\
\mathbb{T}^2
\end{array}
\]

commutes for each \(i\). Here \(N_i\) denotes a matrix \(\begin{bmatrix} v^\delta & 0 \\ 0 & t^\mu \end{bmatrix}\) for each \(i\).

Let \(n_i = \varphi(i+1) - \varphi(i)\) and \(k_i = \psi(i+1) - \psi(i)\). Then

\[
\begin{bmatrix} x'_i & y'_i \\ Z_i & W_i \end{bmatrix} = \begin{bmatrix} X_{i+1} \ y_{i+1} \\ Z_{i+1} \ W_{i+1} \end{bmatrix} = \begin{bmatrix} v^\delta n_i \alpha \sum_{j=0}^{n_i-1} (v^\lambda)^{n_i-1-j}t^\mu j \\
0 \\ 0 \ t^\mu k_i \end{bmatrix}
\]

and

\[
\begin{bmatrix} x_i & y_i \\ Z_i & W_i \end{bmatrix} = \begin{bmatrix} x'_j & y'_j \\ Z'_j \ W'_j \end{bmatrix} = \begin{bmatrix} v^{jk_i} & 0 \\ 0 \ t^\mu k_i \end{bmatrix}
\]
for each $i$. Note that
\[
\begin{bmatrix}
  x_i & y_i \\
  z_i & w_i
\end{bmatrix}
= \begin{bmatrix}
  \alpha \sum_{j=0}^{n_i-1} (\lambda^j \eta_i - \lambda^j \mu_i) \\
  0
\end{bmatrix}
\begin{bmatrix}
  x_{i+1} & y_{i+1} \\
  z_{i+1} & w_{i+1}
\end{bmatrix}
\]
and $z_i \eta_i = t^{\lambda_i} z_{i+1}$ for each $i$. First we prove that all $z_i = 0$. Assume the contrary, i.e. let $z_i \neq 0$ for some $i$. Then $z_i \neq 0$ for all $i > i$. Choose $j > i$ such that $\mu(k_j + k_{i+1} + \cdots + k_j) > \sigma \geq 0$, where $z_i = z_0 t^\sigma$ and $\text{GCD}(z_0, t) = 1$. Then $z_i \eta_i = t^{\lambda_i} \eta_i$, which implies that $t$ divides $z_0 t^{\lambda_i(n_i + \cdots + n_j)}$ and we get a contradiction. This proves all $z_i$ are 0 and consequently $x_i \neq 0 = w_i$ for each $i$. Then $w_i t^{\mu_i} = w_i t^{\mu_i} w_{i+1}$ and $z_i' = 0$ for each $i$. Equalities $x' = \lambda^j \lambda_i w_i$, $w_i' = t^{\mu_i} w_i$ and $x' = -y_i \lambda_i - \lambda_i$, $w_i = t^{\mu_i}$ imply $x_i' = \lambda^j \lambda_i$, $x_i' = -y_i \lambda_i - \lambda_i$, $w_i = t^{\mu_i}$ and $x_i' = -y_i \lambda_i - \lambda_i$, $w_i = t^{\mu_i}$ for some integers $\lambda_i, \mu_i$. Note that $0 \leq \lambda_i \leq \lambda_i, 0 \leq \mu_i \leq \mu_i, \lambda_i + \lambda_i = \lambda_i + \lambda_i + 1$ and $\lambda_i + \lambda_i = \lambda_i + \lambda_i + 1$. Since $v$ and $t$ are different prime numbers, it follows $\lambda^j \lambda_i = -e_1 t^{\mu_i} - e_i v_i^j$ for some integer $e_i$. Now, an equality $ae_i \sum_{j=0}^{n_i-1} (\lambda^j \eta_i - \lambda^j \mu_i) + y_i t^{\mu_i} = \lambda^j \lambda_i \eta_i$ implies $ae_i \sum_{j=0}^{n_i-1} (\lambda^j \eta_i - \lambda^j \mu_i) + e_1 t^{\mu_i} + \lambda^j \lambda_i \eta_i = e_i t^{\mu_i}$, i.e. $ae_i \sum_{j=0}^{n_i-1} (\lambda^j \eta_i - \lambda^j \mu_i) = e_i t^{\mu_i}$. This means that for relatively prime integers $\lambda^j$, $t^\mu > 1, a \in \mathbb{Z}, (\lambda^j - t^\mu) | a$ and a sequence $(n_i)$ in $\mathbb{N}$, there exists a sequence $(e_i)$ of integers such that $ae_i \sum_{j=0}^{n_i-1} (\lambda^j \eta_i - \lambda^j \mu_i) = e_i t^{\mu_i}$ for each $i$, which contradicts Lemma 15.

Conversely, if either (1) or (2) or (3) is fulfilled, the conclusion follows from Propositions 11, 8, 9 and 14. □

Let $X$ be a toroidal group obtained by integral matrices $M_i = \begin{bmatrix}
  v_i & a_i \\
  0 & t_i
\end{bmatrix}$, $v_i, t_i > 0, 0 \leq a_i < v_i$, and let $a, c \in \mathbb{N}$ be positive integers such that $\text{GCD}(a, v_i) = \text{GCD}(c, t_i) = 1$ for all $i \geq i_0$.

A sequence $b_{ac} : \{i \in \mathbb{N} : i \geq i_0\} \to \{0, 1, \ldots, a - 1\}$ is said to be super-admissible for $X$ if $v_i b_{ac}(i + 1) \equiv t_i b_{ac}(i) (a c)$ (mod $a$) for each $i \geq i_0$.

A sequence $d_{ac} : \{i \in \mathbb{N} : i \geq i_0\} \to \{0, 1, \ldots, a - 1\}$ is said to be sub-admissible for $X$ if $v_i d_{ac}(i + 1) \equiv t_i d_{ac}(i) + a c (\mod c)$ for each $i \geq i_0$.

Lemma 17. Let $X$ be a toroidal group obtained by integral matrices $M_i = \begin{bmatrix}
  v_i & a_i \\
  0 & t_i
\end{bmatrix}$, $v_i, t_i > 0, 0 \leq a_i < v_i$. Then the following claims are equivalent.

(i) $X$ admits an $s$-sheeted covering map over $\Sigma(p, q, r)$;
(ii) There is a super-admissible sequence $b_{ac} : \{i \in \mathbb{N} : i \geq i_0\} \to \{0, 1, \ldots, a - 1\}$ for $\Sigma(p) \times \Sigma(q) \times \Sigma(r)$, $ac = \frac{s}{2}$, such that $X$ is homeomorphic to a toroidal group $Y$ obtained by matrices $N_i = \begin{bmatrix}
  v_i (p_i b_{ac}(i + 1) - t_i b_{ac}(i) (a c)) (a c) \end{bmatrix}$, $i \geq i_0$;
(iii) There is a sub-admissible sequence $d_{ac} : \{i \in \mathbb{N} : i \geq i_0\} \to \{0, 1, \ldots, a - 1\}$ for $\Sigma(p) \times \Sigma(q) \times \Sigma(r)$, $ac = \frac{s}{2}$, such that a toroidal group $Y$ obtained by matrices $N_i = \begin{bmatrix}
  v_i (-v_i d_{ac}(i + 1) + t_i d_{ac}(i) (a c)) (a c) \end{bmatrix}$, $i \geq i_0$, is homeomorphic to $\Sigma(p) \times \Sigma(q) \times \Sigma(r)$.

Proof. (i) $\Rightarrow$ (ii). Assume that $X$ admits an $s$-sheeted covering map over $\Sigma(p, q, r)$. According to Corollary 5 $X$ admits an $\frac{s}{2}$-sheeted covering map over $\Sigma(p) \times \Sigma(q) \times \Sigma(r)$. Then there exists a super-admissible sequence $b_{ac} : \{i \in \mathbb{N} : i \geq i_0\} \to \{0, 1, \ldots, a - 1\}$ for $\Sigma(p) \times \Sigma(q) \times \Sigma(r)$ and $X$ is homeomorphic to a toroidal group $Y$ obtained by matrices $N_i = \begin{bmatrix}
  v_i (p_i b_{ac}(i + 1) - t_i b_{ac}(i) (a c)) (a c) \end{bmatrix}$, $i \geq i_0$ (see [2, Appendix A] and [3, §3]).

(ii) $\Rightarrow$ (iii). By (ii) there is an $\frac{s}{2}$-sheeted covering map $f : X \to \Sigma(p) \times \Sigma(q) \times \Sigma(r)$. Then there is a sub-admissible sequence $d_{ac} : \{i \in \mathbb{N} : i \geq i_0\} \to \{0, 1, \ldots, a - 1\}$ for $\Sigma(p) \times \Sigma(q) \times \Sigma(r)$ and $X$ is homeomorphic to a toroidal group $Y$ obtained by matrices $N_i = \begin{bmatrix}
  v_i (-v_i d_{ac}(i + 1) + t_i d_{ac}(i) (a c)) (a c) \end{bmatrix}$, $i \geq i_0$ (see [2, Appendix A and [3, §3]).

(iii) $\Rightarrow$ (i). By (iii) there is an $\frac{s}{2}$-sheeted covering map from $X$ to $\Sigma(p) \times \Sigma(q) \times \Sigma(r)$. Then $X$ admits an $s$-sheeted covering map over $\Sigma(p, q, r)$ by Corollary 5. □

Theorem 18. Let $p, r \in \mathbb{N}$ be relatively prime positive integers and let $X$ be a toroidal group obtained by integral matrices $M_i = \begin{bmatrix}
  0 & a \\
  v & 0
\end{bmatrix}$. If $r$ is odd, $X$ covers $\Sigma(p, q, r)$. If $p$ is odd, $X$ covers $\Sigma(r, q, p)$.

Proof. If either $p = 1$ or $r = 1$ or $(p - r) \mid \alpha$, $X$ is homeomorphic to $\Sigma(p) \times \Sigma(r)$ and the conclusion is obtained. Assume $p, r \geq 2$ and $(p - r) \not\mid \alpha$. Then $|p - r| \geq 2$ and without loss of generality we may assume $0 < \alpha < |p - r|$. We distinguish two cases.

1. $|p - r| = p - r$. 

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Let $r$ be odd. Since $p - r$ and $p$ are relatively prime and $p\alpha \equiv r\alpha \pmod{(p - r)}$ we may define a super-admissible sequence $b_{i(p-r)_1}: \mathbb{N} \to \{0, 1, \ldots, p - r - 1\}$ for $\Sigma_p \times \Sigma_r$ in the following way $b_{i(p-r)_1}(i) = \alpha$ for each $i$. Moreover, for each $i$ $N_i = [p/p_{b_{i(p-r)_1}(i+1) - r b_{i(p-r)_1}(i)/(p-r)}] = [p^{\alpha}]$ and $X$ covers $\Sigma(p, q, r)$ according to Lemma 17(ii).

Let $p$ be odd. Since $p - r$ and $p$ are relatively prime we may define a sub-admissible sequence $d_{i(p-r)_1}: \mathbb{N} \to \{0, 1, \ldots, p - r - 1\}$ for $X$ putting $d_{i(p-r)_1}(i) = 0$ for each $i$. Then $\Sigma_r \times \Sigma_p$ is homeomorphic to a group $Y$ obtained by matrices $N_i = [p/p_{b_{i(p-r)_1}(i+1) + r d_{i(p-r)_1}(i)+(p-r)}] = [p^{\alpha}]$ by Proposition 14. According to Lemma 17(iii) $X$ covers $\Sigma(r, q, p)$.

2. $|p - r| = r - p$.

Let $r$ be odd. Since $p$ and $p - r$ are relatively prime and $p\alpha \equiv r\alpha \pmod{(r-p)}$ we may define a super-admissible sequence $b_{i(p-r)_1}: \mathbb{N} \to \{0, 1, \ldots, r - p - 1\}$ for $\Sigma_p \times \Sigma_r$ putting $b_{i(p-r)_1}(i) = \alpha$ for each $i$. Let $Y$ be a toroidal group obtained by matrices $N_i = [p/p_{b_{i(p-r)_1}(i+1) - r b_{i(p-r)_1}(i)/(r-p)}] = [p^{\alpha}]$. Note that $Y$ is homeomorphic to $X$ since $[1 \ 0] [p^{\alpha}] = [1 \ 0]$ and $X$ covers $\Sigma(p, q, r)$ according to Lemma 17(ii).

Let $p$ be odd. Since $r - p$ and $p$ are relatively prime we may define a sub-admissible sequence $d_{i(p-r)_1}: \mathbb{N} \to \{0, 1, \ldots, r - p - 1\}$ for $X$ putting $d_{i(p-r)_1}(i) = 0$ for each $i$. Then $\Sigma_r \times \Sigma_p$ is homeomorphic to a group $Y$ obtained by matrices $N_i = [p/p_{b_{i(p-r)_1}(i+1) + r d_{i(p-r)_1}(i)+(p-r)}] = [p^{\alpha}]$ by Proposition 14. $X$ covers $\Sigma(r, q, p)$ by Lemma 17(iii).

Corollary 19. Let $p, r \in \mathbb{P}$ be different odd primes, $\lambda, \mu \in \mathbb{N}$, $|p^x - r^y| > 1$, and let $X$ be a group obtained by integral matrices $M_i = [p^{\alpha}]$, where $0 < \alpha < |p^x - r^y|$. Then $X$ is not homeomorphic to $\Sigma_p \times \Sigma_r$, but covers spaces $\Sigma(p, q, r)$ and $\Sigma(r, q, p)$.

3. Groups which cover only groups

In Section 2 it is shown that whenever a toroidal group $X$ covers Klein bottle weak solenoidal spaces, it covers groups as well. Here we will give an example of groups which weak cover groups but do not cover any Klein bottle weak solenoidal space.

Claim 20. Let $B = \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix}$, $N = \begin{bmatrix} v & 0 \\ 0 & t \end{bmatrix}$, $F = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, $F' = \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}$, $G = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$ be integral matrices, such that $v, t, p, r \in \mathbb{N}$, and the following diagram

\[
\begin{array}{ccc}
\mathbb{Z}^2 & \xleftarrow{B} & \mathbb{Z}^2 \\
\downarrow{F} & & \downarrow{F'} \\
\mathbb{Z}^2 & \xrightarrow{G} & \mathbb{Z}^2 \\
\downarrow{N} & & \downarrow{N} \\
\mathbb{Z}^2 & \xrightarrow{G} & \mathbb{Z}^2 \\
\end{array}
\]

commutes.

1. If $r$ is odd and $t$ is even, then $w$ is even, at least one of $y'$ and $w'$ is odd and $yr = vy' + aw'$.
2. If $t = 1$, then $\text{GCD}(z, w) = 1$ and $z' = zp$ and $w' = wzr$.

Proof. (1) Since $wr = tw'$ and $r = my' + nw'$, it follows that $w$ is even and at least one of $y'$ and $w'$ is odd.
(2) If $t = 1$ then $1 = zl + wn$ and $\text{GCD}(z, w) = 1$.

Theorem 21. Let $X$ be a toroidal group obtained by integral matrices $M_i = [v_{i \ t_{i}}]$, $v_i, t_i \in \mathbb{N}$. If even integers occur infinitely often in both sequences $(v_i)$ and $(t_i)$, then $X$ does not cover Klein bottle weak solenoidal spaces.

Proof. Assume the contrary, i.e. let $X$ admit an $s$-sheeted covering map over a Klein bottle weak solenoidal space $\Sigma(p, q, r)$. Without loss of generality we may assume that each $v_i$ and $t_i$ is even. According to Lemma 17(iii) there is a sub-admissible sequence $d_{ac}: \{i \in \mathbb{N}: i \geq i_0\} \to \{0, 1, \ldots, a - 1\}$ for $X$, $ac = \frac{r}{z}$, such that a toroidal group $Y$ obtained by matrices $N_i = \begin{bmatrix} v_{i - v_i d_{ac}(i+1) - t_i d_{ac}(i)+a_{i_0}} \\ t_{i_0} \end{bmatrix}$, $i_0 = i_{i_0}$, is homeomorphic to $\Sigma(p) \times \Sigma(r)$. Now we can find strictly increasing sequences may $\varphi, \psi: \{i \in \mathbb{N}: i \geq 0\} \to \{i \in \mathbb{N}: i \geq 0\}$ and integral matrices $B_i = \begin{bmatrix} p_i & 0 \\ 0 & r_i \end{bmatrix}$, $F_i = \begin{bmatrix} x_i & y_i \\ z_i & w_i \end{bmatrix}$, $G_i = \begin{bmatrix} k_i & l_i \\ m_i & n_i \end{bmatrix}$ such that diagrams
commute.

Note that

$$B_{\psi(i)}B_{\psi(i+1)} \cdots B_{\psi(i+1)-1} = \begin{bmatrix} P_{\psi(i)} \cdots P_{\psi(i+1)-1} & 0 \\ 0 & r_{\psi(i)} \cdots r_{\psi(i+1)-1} \end{bmatrix}$$

and

$$N_{\psi(i)}N_{\psi(i+1)} \cdots N_{\psi(i+1)-1} = \begin{bmatrix} V_{\psi(i)} \cdots V_{\psi(i+1)-1} & 0 \\ 0 & t_{\psi(i)} \cdots t_{\psi(i+1)-1} \end{bmatrix}$$

for some integer $v'_{\psi(i)}$. According to Claim 20 all $w_i$ are even and all $y_i$ are odd. This leads to the contradiction since $y_i r_{\psi(i)} \cdots r_{\psi(i+1)-1}$ is odd, $v_{\psi(i)} \cdots v_{\psi(i+1)-1} y_{i+1} + v''_{\psi(i)} w_{i+1}$ is even and $y_i t_{\psi(i)} \cdots t_{\psi(i+1)-1} = v_{\psi(i)} \cdots v_{\psi(i+1)-1} y_{i+1} + v''_{\psi(i)} w_{i+1}$.

**Corollary 22.** Let $X$ be a toroidal group obtained by matrices $M_i = \begin{bmatrix} a_i & \alpha_i \\ 0 & 2p_i \end{bmatrix}$, $\lambda_i, \mu_i \in \mathbb{N}$. Then $X$ is homeomorphic to $\Sigma_2 \times \Sigma_2$ and covers groups with any odd number of sheets, but does not cover Klein bottle weak solenoidal spaces.

A natural question arises:

Let $X$ be a toroidal group obtained by integral matrices $M_i = \begin{bmatrix} p_i & \alpha_i \\ 0 & 1 \end{bmatrix}$, where at least one of sequences $p = (p_i)$ and $r = (r_i)$ consists of positive odd integers. Does $X$ cover Klein bottle weak solenoidal spaces?

We answer this question in the negative.

**Theorem 23.** Let $p$ be a prime number and let $X$ be a toroidal group obtained by integral matrices $M_i = \begin{bmatrix} p & \alpha_i \\ 0 & 1 \end{bmatrix}$. $X$ covers Klein bottle weak solenoidal spaces if and only if $X$ is homeomorphic to $\Sigma_p \times S^1$.

**Proof.** Sufficiency follows from Theorem 13 and we need to prove only necessity. Assume $X$ covers a Klein bottle weak solenoidal space $\Sigma(p, q, r)$. Reasoning as in the proof of Theorem 21 we can find strictly increasing sequences $\psi, \psi': i \in \mathbb{N} \mapsto \{i \in \mathbb{N} : \psi(i) \geq i_0 \}$ and integral matrices $M_i = \begin{bmatrix} p & \alpha_i \\ 0 & 1 \end{bmatrix}$, $B_i = \begin{bmatrix} p & \beta_i \\ 0 & 1 \end{bmatrix}$, $F_i = \begin{bmatrix} \beta_i & y_i \\ 0 & w_i \end{bmatrix}$, $G_i = \begin{bmatrix} h_i & h_i \end{bmatrix}$ such that diagrams

commute for each $i \geq i_0$.

Note that $N_{\psi(i)}N_{\psi(i)+1} \cdots N_{\psi(i+1)-1} = \begin{bmatrix} \psi_{\psi(i+1)-\psi(i)} & \psi_{\psi(i)} \\ 0 & 1 \end{bmatrix}$ for some integer $\psi_{\psi(i)}$. Since $\psi(i+1) - \psi(i) > 0$ and det $F_i$ det $G_i = p^{\psi(i) + 1 - \psi(i)}$ we conclude that $p_{\psi(i)} \cdots p_{\psi(i+1)-1} t_{\psi(i)} \cdots t_{\psi(i+1)-1} = p^{\lambda_i}$ for some $\lambda_i, 0 < \lambda_i \leq \psi(i+1) - \psi(i)$. We see that all terms in both sequences $(p_i)$ and $(r_i)$ are powers of $p$, i.e. $p_i = p^{m_i}, r_i = p^{m_i}$, $n_i, m_i \geq 0$. First note that it is not possible $p_i = r_i = 1$ for almost all $i$. We claim that $p$ divides infinitely many terms in only one of the sequences $(p_i)$ and $(r_i)$. Assume the contrary, i.e. $p$ divides infinitely many terms in each of the sequences $(p_i)$ and $(r_i)$. Then for each $i$ we can find $i$ and $k, k > i > i_1$, such that $\text{GCD}(p_{\psi(i)}, \cdots p_{\psi(k)-1}, t_{\psi(i)} \cdots t_{\psi(k)-1}) > 1$. Consider diagrams
Then $\gcd(z_k, w_k) = \gcd(z_1 \cdot p_{\psi(i)} \cdots p_{\psi(k)}^{-1}, w_1 \cdot r_{\psi(i)} \cdots r_{\psi(k)}^{-1}) > 1$, which contradicts Claim 20. Hence $X$ covers $\Sigma_p \times S^1$, which implies that $X$ covers $\Sigma(p, q, 1)$ and $\Sigma(1, q, p)$. According to Corollary 13 $X$ is homeomorphic to $\Sigma_p \times S^1$.

**Corollary 24.** Let $p$ be a prime number and let $X$ be a toroidal group obtained by matrices $M_i = \begin{bmatrix} p & \alpha_i \\ 0 & 1 \end{bmatrix}$, where $(\alpha_i)$ is not a semi-periodic sequence. Then $X$ is not homeomorphic to $\Sigma_p \times S^1$ and covers groups with any finite number of sheets, but does not cover Klein bottle weak solenoidal spaces.

**References**