Matrices with Permanent Equal to One

Victor A. Nicholson
Department of Mathematics
Kent State University
Kent, Ohio 44242

Submitted by Richard S. Varga

ABSTRACT

We show that a nonnegative square matrix $M$ is nilpotent if and only if the permanent of $M + I$ is one. We also show that a 2-complex obtained by sewing disks to a wedge of circles is collapsible if and only if its incidence matrix has permanent one.

1. INTRODUCTION

We show in Theorem 1 that a nonnegative square matrix $M$ is nilpotent if and only if the permanent of $M + I$ is one. We consider the geometry underlying this result in Corollary 1. Corollary 2 characterizes the square matrices with integer entries that have permanent equal to one. We use this result to characterize the collapsible 2-complexes obtained by sewing disks to a wedge of circles (Theorem 2).

2. MAIN RESULTS

Let $M$ be a nonnegative square matrix and $r$ a positive integer. If $r$ positive elements $m_{ij}$ of $M$ can be arranged to have the form $m_{i_1 j_1}, m_{i_2 j_2}, \ldots, m_{i_r j_r}$, they will be called a positive cycle (of length $r$) of elements in $M$. The permanent of an $n \times n$ matrix $M$ with entries $m_{ij}$ is defined by

$$\text{per}(M) = \sum_{\sigma} \prod_{j=1}^{n} m_{i_\sigma(j)}$$

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where the sum extends over all \( n! \) permutations \( \sigma \) of the first \( n \) positive integers. We use \( I \) to denote the \( n \times n \) identity matrix. We say an \( n \times n \) matrix \( M \) is upper triangular if \( m_{ij} = 0 \) for all \( i > j \), and strictly upper triangular if \( m_{ij} = 0 \) for all \( i > j \).

**Theorem 1.** Let \( M \) be a nonnegative \( n \times n \) matrix. Then the following are equivalent:

1. there exists a permutation matrix \( P \) such that \( PMP^T \) is strictly upper triangular,
2. there is no positive cycle of elements in \( M \),
3. \( \text{per}(M + I) = 1 \),
4. \( M \) is nilpotent.

**Proof.** (1)\( \Rightarrow \)(2)\( \Rightarrow \)(3). This is immediate.

(3)\( \Rightarrow \)(1). Since \( M \) is nonnegative, \( \prod_{i=1}^{n} (m_{ii} + 1) \geq 1 \). Thus, the permanent of \( M + I \) is equal to the product of its diagonal elements. By Lemma 2 of [1], there is a permutation matrix \( P \) such that \( P(M + I)P^T = PMP^T + I \) is upper triangular. The diagonal elements of \( PMP^T + I \) are all ones because each diagonal element is \( \geq 1 \), \( \text{per}(PMP^T + I) \) is the product of the diagonal elements, and \( \text{per}(PMP^T + I) = 1 \). Thus \( PMP^T \) is strictly upper triangular.

(2)\( \Leftrightarrow \)(4). The matrix \( M \) is nilpotent if and only if all of the eigenvalues of \( M \) are zero. A nonnegative square matrix has a real eigenvalue equal to its spectral radius [5, Theorem 2.7]. Thus \( M \) is nilpotent if and only if \( M \) has no positive eigenvalues. By Theorem 1 of [4], \( M \) has a positive eigenvalue if and only if there is a positive cycle of elements in \( M \).

The following corollary and Fig. 1 make clear the geometry underlying Theorem 1. If \( G \) is a loopless directed graph (we allow \( G \) to have multiple lines) with vertices \( v_1, \ldots, v_n \), then the adjacency matrix \( M = (m_{ij}) \) of \( G \) is given by \( m_{ij} \) the number of arrows from \( v_i \) to \( v_j \). The bipartite graph of a nonnegative square matrix \( M \) is the bipartite graph \( G(M) \) whose points

![Fig. 1.](image-url)
permaments of matrices consist of the two sets $R = \text{(the rows of } M\text{)}$ and $C = \text{(the columns of } M\text{)}$, and whose lines are the ordered pairs $(r_i, c_j)$, where $m_{ij} \neq 0$. The bipartite graph $G(M)$ does not have multiple lines. A 1-factor of a graph is a family $F$ of lines of the graph such that every point of the graph is incident with exactly one line in $F$.

**Corollary 1.** Let $G$ be a loopless directed graph and $M$ its adjacency matrix. Then $G$ is acyclic if and only if the bipartite graph $G(M + I)$ has a unique 1-factor.

**Proof.** The graph $G$ is acyclic if and only if $M$ has no positive cycles. By Theorem 1, $M$ has no positive cycles if and only if the permanent of $M + I$ is one. It is easy to see that the permanent of $M + I$ is one if and only if $G(M + I)$ has a unique 1-factor. 

**Corollary 2.** Suppose $M$ is an $n \times n$ matrix with nonnegative integer entries. Then the permanent of $M$ is one if and only if there exist $n \times n$ permutation matrices $P$ and $Q$ such that $PMQ$ is upper triangular with all ones on the main diagonal.

**Proof.** Suppose the permanent of $M$ is one. Since $M$ is nonnegative, there is a permutation $\sigma$ such that $\prod_{i=1}^{n} M_{\sigma(i)i} = 1$. Since each entry is an integer, $M_{\sigma(i)i} = 1$ for each $i = 1, \ldots, n$. Let $R = (r_{ij})$ be the $n \times n$ permutation matrix with $r_{\sigma(i)j} = 1$ for each $j = 1, \ldots, n$. Then $RM = N + I$ for some nonnegative matrix $N$. Since $\text{per}(N + I) = 1$, Theorem 1 implies that there exists a permutation matrix $S$ such that $SNS^T$ is strictly upper triangular. Let $P = RS$ and $Q = S^T$. Then $PMQ = SRMST = S(N + I)S^T = SNS^T + I$, which is upper triangular with ones down the main diagonal. The converse is immediate.

3. **Application**

A 2-complex $K$ obtained by sewing disks to a wedge of circles is collapsible if it is possible to order the disks $D_1, D_2, \ldots, D_n$ so that for each $i = 1, \ldots, n$ there is a circle $S_i$ that $D_i$ is sewn onto exactly once but that $D_j$ is not sewn onto for all $j > i$. Intuitively, we are able to grasp $D_1$ at the part of its edge that is sewn to $S_1$, pluck $D_1$ from $K$ as one plucks a petal from a flower, and continue to pluck the remaining disks. If $K$ is not collapsible, there will be a stage at which no disk has an edge that we can grasp. For a discussion of collapsing see [3, p. 42].
THEOREM 2. Let $\langle a_1, a_2, \ldots, a_n : w_1 = w_2 = \cdots = w_n = 1 \rangle$ be a free group with $n$ generators and $n$ relations. Let $K$ be the 2-complex formed by sewing $n$ disks to a wedge of $n$ simple closed curves by the words $w_1, \ldots, w_n$. Let $M = (m_{ij})$ be the $n$-square nonnegative matrix formed by $m_{ij} = \text{the sum of the absolute values of the exponents on } a_i \text{ in } w_j$. Then $K$ collapses to a point if and only if the permanent of $M$ is one.

Proof. Let $D_i$ denote the disk corresponding to the word $w_i$, and let $S_i$ be the curve corresponding to generator $a_i$, for each $i = 1, 2, \ldots, n$. The complex $K$ collapses to a point if and only if there is an ordering of the disks $D_{a(1)}, D_{a(2)}, \ldots, D_{a(n)}$ and a permutation $\beta$ such that, for each $i = 1, 2, \ldots, n$, $S_{\beta(i)}$ is sewn to $D_{a(i)}$ exactly once and is not sewn to $D_{a(j)}$ for any $j > i$. Suppose the permanent of $M$ is one. Then, by Corollary 2, there exist permutation matrices $P$ and $Q$ such that $PMQ = (x_{ij})$ is upper triangular with ones down the main diagonal. Since $P$ interchanges the rows of $M$ and $Q$ interchanges the columns, there exist permutations $\alpha$ and $\beta$ such that $x_{ij} = M_{\alpha(i)\beta(j)}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Since $x_{ii} = 1$, $S_{\beta(i)}$ is sewn to $D_{a(i)}$ exactly once for each $i = 1, \ldots, n$. Since $x_{ij} = 0$ whenever $i > j$, $S_{\beta(i)}$ is not sewn to $D_{a(j)}$ whenever $j > i$ for each $i = 1, \ldots, n$. Thus $K$ collapses to a point. Conversely, if $K$ collapses to a point, then the two permutations $\alpha$ and $\beta$ give rise to permutation matrices $P$ and $Q$, so that $PMQ$ is upper triangular with ones on the diagonal. By Corollary 2, the permanent of $M$ is one.

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