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The Existence of Positive Solutions to a Singular Nonlinear Boundary Value Problem

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1. INTRODUCTION

Singular nonlinear boundary value problems occur in a variety of applications and often only their positive solutions are important. Consequently, they have been studied by many authors [1, 3, 4, 6, 8–10].

In a recent paper [6], Gomes and Sprekels proved that the singular nonlinear boundary value problem

$$-u''(t) = k(t)[u(t)]^{-\alpha}[u'(t)]^{\sigma}, \qquad t \in (0, 1), \tag{1}$$

$$u(0) = 0, \qquad u(1) = Du'(1), \tag{2L}$$

has a positive solution, which belongs to $W_B^{2,1}(0, 1)$, under the following hypotheses:

(I) $\alpha > 0$, $\sigma \in [\alpha, 1 + \alpha)$, and $D > 1$ are given real numbers,

(II) $k(t)$ is a nonnegative measurable function defined on $[0, 1]$ such that

$$0 < \int_0^1 k(t)\, t^{-\alpha}\, dt < +\infty. \tag{3}$$

They converted the above problem into an equivalent fixed point equation and then treated the equation by applying a modified version of Krasonselskii's Theorem on operators compressing a cone in a Banach space. Unfortunately, they have not been able to treat the case $\sigma \in (0, \alpha)$. Moreover, they pointed out that the problem (1)–(2L) with $\sigma < 1$ was studied in [2, 5] by using another kind of approach and explained in some

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detail that problems such as the problem (1)–(2L) appear in the study of similarity solutions for the equation

$$\theta_t = \nabla \cdot (K(\theta) \, \|\nabla\theta\| \, \nabla\theta),$$

which arises in different physical situations.

In this paper we consider a more general singular nonlinear boundary value problem of the form

$$-u''(t) = k(t)[u(t)]^{-\alpha} [u'(t)]^{\sigma}, \qquad t \in (0, 1), \tag{1}$$

$$u(0) = 0, \qquad u(1) = \Phi(u'(1)), \tag{2N}$$

and only make the following hypotheses:

(H_1) $\alpha > 0$ and $\sigma < 1 + \alpha$ are given real numbers,

(H_2) $k(t)$ is a nonnegative measurable function defined on $[0, 1]$ such that

$$0 < \int_0^1 k(t)\, dt < +\infty, \tag{4}$$

(H_3) $\Phi(\beta)$ is a positive continuous function defined in $(0, +\infty)$ such that

$$\lim_{\beta \uparrow \infty} \Phi(\beta)\, \beta^{-1} = D > 1, \tag{5}$$

$$\begin{aligned} &\lim_{\beta \downarrow 0} [\Phi(\beta)]^{\alpha} \beta^{1-\sigma} = 0 \qquad \text{if} \quad \sigma > 1 \\ &\lim_{\beta \downarrow 0} [\Phi(\beta)]^{\alpha} \log \beta = 0 \qquad \text{if} \quad \sigma = 1 \\ &\lim_{\beta \downarrow 0} \Phi(\beta) = 0 \qquad \text{if} \quad \sigma < 1. \end{aligned} \tag{6}$$

Here the real number $\sigma \leqslant 0$ can be allowed. It is clear that the problem (1)–(2L) is a special case of the problem (1)–(2N).

In this paper we establish the existence of positive solutions to the problem (1)–(2N), by using the shooting method. All the technical arguments are elementary.

2. Properties of Positive Solutions

A function $u(t)$ is called a positive solution to the problem (1)–(2N), if it satisfies the following conditions:

(a) $u(t)$ is continuous on $[0, 1]$ and positive in $(0, 1]$,

(b) $u'(t)$ exists and is continuous in $(0, 1]$,

(c) $u''(t)$ exists almost everywhere and locally Lebesgue integrable in $(0, 1]$,

(d) $u(0)=0$, $u(1)=\Phi(u'(1))$, and

(e) Equation (1) holds almost everywhere in $(0, 1)$.

Let $u(t)$ be a positive solution to the problem (1)–(2N). Then $u(t)$ is a continuous, increasing, concave function defined on $[0, 1]$ and hence

$$tu(1) \leqslant u(t) \leqslant u(1) \qquad \text{for all} \quad t \in [0, 1], \tag{7}$$

and when

$$u'(0) = \lim_{t \downarrow 0} u'(t)$$

exists,

$$u(t) \leqslant tu'(0) \qquad \text{for all} \quad t \in [0, 1]. \tag{8}$$

Moreover,

$$\begin{aligned} 0 < u(1) &= \int_0^1 u'(t)\, dt \\ &= \int_0^1 \left(u'(1) - \int_t^1 u''(s)\, ds \right) dt \\ &= u'(1) - \int_0^1 su''(s)\, ds, \end{aligned}$$

i.e.,

$$u(1) - u'(1) = \int_0^1 s\, |u''(s)|\, ds. \tag{9}$$

Applying integration by parts, the equality (9) can be written as

$$\begin{aligned} u(1) - u'(t) &= \int_0^t s\, |u''(s)|\, ds - \int_t^1 su''(s)\, ds \\ &= \int_0^t s\, |u''(s)|\, ds + tu'(t) - u(t) + u(1) - u'(1) \end{aligned}$$

for all $t \in [0, 1]$, i.e.,

$$u(t) = tu(t) + \int_0^t s\,|u''(s)|\,ds \qquad \text{for all} \quad t \in [0, 1]. \tag{10}$$

The equalities (9) and (10) show that the function $t\,|u''(t)|$ is Lebesgue integrable on $[0, 1]$ and the condition $D > 1$ is a necessary condition for the problem (1)–(2L) to have positive solutions. Moreover, when $D = 1$ and $k(t) \equiv 0$, the problem (1)–(2L) has exactly a family of positive solutions of the form

$$u(t) = \lambda t, \qquad \lambda > 0.$$

We now ascertain what condition guarantees the existence of the limit

$$u'(0) = \lim_{t \downarrow 0} u'(t).$$

From Eq. (1), we have

$$0 < \int_0^1 k(t)[u(t)]^{-\alpha}\,dt = \begin{cases} \dfrac{1}{(\sigma - 1)}\{[u'(1)]^{1-\sigma} - [u'(0)]^{1-\sigma}\} & \text{if} \quad \sigma \neq 1 \\ \log\left[\dfrac{u'(0)}{u'(1)}\right] & \text{if} \quad \sigma = 1. \end{cases} \tag{11}$$

Assume that $u'(0)$ is finite. Then it follows from (8) that the left-hand side of (11) is greater than or equal to

$$[u'(0)]^{-\alpha} \int_0^1 k(t)\,t^{-\alpha}\,dt,$$

i.e., the condition (3) holds.

Now assume that (3) holds. Then it follows from (7) that the left-hand side of (11) is less than or equal to

$$[u(1)]^{-\alpha} \int_0^1 k(t)\,t^{-\alpha}\,dt.$$

This fact implies that $u'(0)$ is finite. In this case

$$0 < \int_0^1 |u''(t)|\,dt = \int_0^1 k(t)[u(t)]^{-\alpha}\,[u'(t)]^{\sigma}\,dt$$

$$\leqslant \frac{[u'(0)]^{\sigma} + [u'(t)]^{\sigma}}{[u(1)]^{\alpha}} \int_0^1 k(t)\,t^{-\alpha}\,dt < +\infty.$$

Here (7) has been used.

We can summarize the above results in the following statement.

THEOREM 1. *Let $u(t)$ be a positive solution to the problem* (1)–(2N). *Then $u(t)$ is a continuous concave function defined on* $[0, 1]$ *and* (7), (8), (9), (10) *hold. Moreover, a sufficient and necessary condition for $u'(0)$ to be finite is the condition* (3); *when* (3) *holds, $|u''(t)|$ is Lebesgue integrable on* $[0, 1]$, *i.e.,* $u \in W_B^{2,1}(0, 1)$.

3. EXISTENCE OF POSITIVE SOLUTIONS

In order to apply the shooting method, we consider an initial value problem of the form

$$\begin{aligned} &u'(t) = v(t), && t < 1, \\ &v'(t) = -k(t)[u(t)]^{-\alpha}[v(t)]^{\sigma}, && t < 1, \\ &u(1) = \Phi(\beta), && v(1) = \beta, \end{aligned} \tag{12}$$

where β is a positive parameter and the hypotheses (H_1), (H_2), (H_3) are satisfied.

LEMMA 1. *For each fixed $\beta > 0$, the initial value problem* (12) *has a unique solution which is denoted by $(u(t; \beta), v(t; \beta))$ and depends continuously on β. If the maximal interval of existence for the solution is denoted by $(x, 1]$, then either $x = 0$ or $\lim_{t \downarrow x} u(t; \beta) = 0$ or $\lim_{t \downarrow x} v(t; \beta) = +\infty$.*

Proof. The proof can be found in [7].

Evidently, if there exists a $\beta > 0$ such that the maximal interval of existence for the solution $(u(t; \beta), v(t; \beta))$ is the interval $(0, 1]$ and $u(0; \beta) = 0$, then $u(t; \beta)$ is a positive solution to the problem (1)–(2N).

We are now in a position to prove the existence of positive solutions.

LEMMA 2. *There exists a positive number β_1 such that $u(0; \beta) > 0$ whenever $\beta \geq \beta_1$.*

Proof. It is easy to verify that if a function $u(t)$ is a positive solution to the integral equation

$$u(t) = (Tu)(t)$$
$$=: \begin{cases} \Phi(\beta) - \int_t^1 \left(\beta^{1-\sigma} + \int_x^1 (1-\sigma)\, k(s)[u(s)]^{-\alpha}\, ds\right)^{1/(1-\sigma)} dx, & \text{if } \sigma \neq 1 \\ \Phi(\beta) - \int_t^1 \beta \exp\left(\int_x^1 k(s)[u(s)]^{-\alpha}\, ds\right) dx, & \text{if } \sigma = 1 \end{cases} \tag{13}$$

then $(u(t), v(t)) =: (u(t), (Tu)'(t))$ is a solution to the problem (12) and vice versa.

Define

$$u_0(t; \beta) = (D-1)\beta/2,$$
$$u_n(t; \beta) = (Tu_{n-1}(\cdot; \beta))(t), \qquad n = 1, 2, 3, \ldots .$$

By virtue of the hypotheses (H_1), (H_2), (H_3), we have

$$u_1(t; \beta)/\beta \to D - 1 + t, \qquad \text{as} \quad \beta \to +\infty.$$

Consequently, there exists a $\beta_1 > 0$ such that for all $t \in [0, 1]$,

$$u_1(t; \beta) \geqslant (D-1)\beta/2 = u_0(t; \beta)$$

whenever $\beta \geqslant \beta_1$. Further, it is readily verified that

$$u_0(t; \beta) \leqslant u_1(t; \beta) \leqslant \cdots \leqslant u_n(t; \beta) \leqslant \cdots \leqslant \Phi(\beta)$$

and hence, for each fixed $\beta \geqslant \beta_1$

$$u(t; \beta) = \lim_{n \to \infty} u_n(t; \beta) = (Tu)(t; \beta), \qquad t \in [0, 1].$$

Set $v(t; \beta) = u'(t; \beta)$. Then $(u(t; \beta), v(t; \beta))$ is a solution to the problem (12) and $u(0; \beta) > 0$. The lemma is proved.

LEMMA 3. *There exists a positive number β_0 such that the endpoint $t = 0$ is not in the maximal interval of existence for the solution $(u(t; \beta), v(t; \beta))$ whenever $\beta \leqslant \beta_0$.*

Proof. We now distinguish two cases.

Case $\sigma > 1$. Choose a $\beta_0 > 0$ such that

$$[\Phi(\beta)]^{\alpha} \beta^{1-\sigma} < (\sigma - 1) \int_0^1 k(s)\, ds \qquad \textit{for all} \quad \beta \leqslant \beta_0$$

by the hypothesis (H_3). If the endpoint $t = 0$ is in the maximal interval of existence for the solution $(u(t; \beta), v(t; \beta))$ with $\beta \leqslant \beta_0$, then

$$\beta^{1-\sigma} \geqslant \int_0^1 (\sigma - 1)\, k(s)[u(s)]^{-\alpha}\, ds \geqslant [\Phi(\beta)]^{-\alpha} \int_0^1 (\sigma - 1)\, k(s)\, ds.$$

This is impossible.

Case $\sigma \leqslant 1$. Choose first a $\delta > 0$ such that

$$\int_\delta^1 k(s)\, ds > 0.$$

If $u(0;\beta) \geqslant 0$, i.e., $t=0$ is in the maximal interval of existence for the solution, then when $\sigma = 1$

$$\begin{aligned}
0 \leqslant u(0;\beta) &= \Phi(\beta) - \int_t^1 \beta \exp\left(\int_x^1 k(s)[u(s)]^{-\alpha}\, ds\right) dx \\
&\leqslant \Phi(\beta) - \delta\beta \exp\left(\int_\delta^1 k(s)[\Phi(\beta)]^{-\alpha}\, ds\right) \\
&= \Phi(\beta) - \delta \exp\left(\left(\int_\delta^1 k(s)\, ds + [\Phi(\beta)]^{\alpha} \log \beta\right) [\Phi(\beta)]^{-\alpha}\right);
\end{aligned}$$

when $\sigma < 1$

$$\begin{aligned}
0 \leqslant u(0;\beta) &= \Phi(\beta) - \int_0^1 \left(\beta^{1-\sigma} + \int_x^1 (1-\sigma)\, k(s)[u(s)]^{-\alpha}\, ds\right)^{1/(1-\sigma)} dx, \\
&\leqslant \Phi(\beta) - \delta \left[(1-\sigma) \int_\delta^1 k(s)\, ds\right]^{1/(1-\sigma)} [\Phi(\beta)]^{-\alpha/(1-\sigma)}.
\end{aligned}$$

As $\beta \to 0$ the right-hand sides of both inequalities above tend to $-\infty$ by the hypothesis (H_3). Therefore, there is a $\beta_0 > 0$ such that for all $\beta \leqslant \beta_0$ the right-hand sides are less than zero. This is also impossible, and hence the proof is complete.

Define the set

$$E =: \{\beta > 0;\, u(0;\beta) > 0\}.$$

Lemmas 1 and 2 tell us that the set E is open and nonempty, while Lemma 3 asserts that $\beta_0 > 0$ is not in E. Consequently,

$$\beta^* =: \inf E > 0.$$

Choose a sequence $\{\beta_n\} \subset E$ such that $\beta_n \to \beta^*$ as $n \to \infty$. By Lemma 1, we have

$$u(0;\beta^*) = \lim_{n \to \infty} u(0;\beta_n) \geqslant 0.$$

We claim that $u(0;\beta^*) = 0$. If $u(0;\beta^*) > 0$, then $\beta^* \in E$, which contradicts the definition of β^*.

Summarizing, we have the following statement.

THEOREM 2. *Suppose that the hypotheses* (H_1), (H_2), (H_3) *hold. Then the problem* (1)–(2*N*) *has at least one positive solution.*

The condition $\sigma < 1+\alpha$ is the best possible in the following sense. When $\alpha > 1$, $\sigma > 1+\alpha$, and $k(t) \equiv 1$, the problem (1)–(2*L*) has no positive solutions (see [6, Remark 10]). When $\alpha > 0$, $\sigma = 1+\alpha$, and $\Phi(\beta) = D\beta$, $D > 1$, the problem (12) is converted into the equivalent problem

$$\begin{aligned} &U'(t) = V(t), && t<1, \\ &V'(t) = -k(t)[U(t)]^{-\alpha}[V(t)]^{\sigma}, && t<1, \\ &U(1) = D, \quad V(1) = 1, && \end{aligned} \tag{14}$$

by introducing the change of dependent variables

$$u(t) = \beta U(t), \qquad v(t) = \beta V(t), \ \beta > 0.$$

Using the shooting method, we can prove that there is a unique $D_0 > 1$ such that the problem (14) has a unique solution $(U(t), V(t))$ which satisfies $U(0) = 0$, provided $\alpha \int_0^1 k(s)\,ds > 1$. This shows when $D = D_0$ the problem (1)–(2*L*) has a family of solutions $u(t) = \beta U(t)$, $\beta > 0$, and when $D > 1$ and $D \neq D_0$, the problem (1)–(2*L*) has no solutions.

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409 176 2-3