Journal of Combinatorial Theory, Series A 117 (2010) 683–703

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Cyclic generators for irreducible representations of affine Hecke algebras

Valentina Guizzi[a], Maxim Nazarov[b], Paolo Papi[c]

[a] Dipartimento di Economia, Università Roma Tre, Via Silvio D'Amico 77, 00145 Roma, Italy
[b] Department of Mathematics, University of York, York YO10 5DD, United Kingdom
[c] Dipartimento di Matematica, Sapienza Università di Roma, Piazzale Aldo Moro 2, 00185 Roma, Italy

A R T I C L E   I N F O

Article history:
Received 30 August 2008
Communicated by Christian Krattenthaler

Keywords:
Affine Hecke algebras
Young diagrams
Fusion procedure

A B S T R A C T

We give a detailed account of a combinatorial construction, due to Cherednik, of cyclic generators for irreducible modules of the affine Hecke algebra of the general linear group with generic parameter q.

0. Introduction

Let F be a non-Archimedean local field. Denote by $\mathcal{O}$ the ring of integers in F. Let P be the maximal ideal of the ring $\mathcal{O}$. Take the general linear group $GL_n(F)$. The group $GL_n(\mathcal{O})$ is a maximal compact subgroup of $GL_n(F)$. Now consider the *Iwahori subgroup* $J \subset GL_n(F)$. It consists of all matrices from $GL_n(\mathcal{O})$ whose entries below the main diagonal belong to P. By definition, the *affine Hecke algebra* $\widetilde{\mathcal{H}}_n$ consists of J-biinvariant compactly supported functions on the group $GL_n(F)$ with complex values. The multiplication on $\widetilde{\mathcal{H}}_n$ is the convolution of functions.

The complex associative algebra $\widetilde{\mathcal{H}}_n$ admits a remarkable presentation, due to Bernstein. It is generated by the elements $T_1, \dots, T_{n-1}$ and invertible elements $X_1, \dots, X_n$ subject to relations (1.1)–(1.6). Here q is the cardinality of the residue field $\mathcal{O}/P$. The subalgebra of $\widetilde{\mathcal{H}}_n$ generated solely by the elements $T_1, \dots, T_{n-1}$ can be then identified as the subalgebra of functions supported on $GL_n(\mathcal{O})$. It is sometimes called the *finite Hecke algebra*; in this article we denote it by $\mathcal{H}_n$.

For any representation W of the group $GL_n(F)$ consider the subspace W^J in W consisting of the vectors fixed by the action of the Iwahori subgroup J. The algebra $\widetilde{\mathcal{H}}_n$ acts on the subspace W^J by definition. The correspondence $W \mapsto W^J$ is an equivalence between the category of representa-

E-mail addresses: guizzi@uniroma3.it (V. Guizzi), mln1@york.ac.uk (M. Nazarov), papi@mat.uniroma1.it (P. Papi).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2009.10.003

tions of $GL_n(F)$ generated by their subspaces of J-fixed vectors, and the category of all $\widetilde{\mathcal{H}}_n$-modules. Furthermore, all irreducible $\widetilde{\mathcal{H}}_n$-modules are finite-dimensional, see for instance [12] and [24].

Results of Bernstein and Zelevinsky [2,29] provide a classification of irreducible $\widetilde{\mathcal{H}}_n$-modules. As is explained for instance in [15], it suffices to classify only those irreducible $\widetilde{\mathcal{H}}_n$-modules where all the eigenvalues of $X_1, \dots, X_n$ belong to $q^{\mathbb{Z}}$. The latter $\widetilde{\mathcal{H}}_n$-modules are labeled by the combinatorial objects called multisegments.

A *multisegment* is a formal finite unordered sum of intervals in $\mathbb{Z}$,

$$M = \sum_{i \leqslant j} m_{ij}[i, j] \tag{0.1}$$

where the coefficients m_{ij} are non-negative integers. Zelevinsky's construction is as follows. To a segment $[i, j]$ one associates the 1-dimensional $\widetilde{\mathcal{H}}_{j-i+1}$-module $V_{[i,j]}$ where the generator T_k acts as q whereas X_l acts as q^{i+l-1}. Given a multisegment M, fix any order on it and consider the tensor product

$$\bigotimes_{i \leqslant j} V_{[i,j]}^{\otimes m_{ij}}. \tag{0.2}$$

This is a module over the tensor product of the algebras $\widetilde{\mathcal{H}}_{j-i+1}^{\otimes m_{ij}}$ which can be naturally identified with a subalgebra in $\widetilde{\mathcal{H}}_n$. Now induce (0.2) to $\widetilde{\mathcal{H}}_n$. For a certain ordering of M, the induced module has a unique non-zero irreducible submodule. The irreducible $\widetilde{\mathcal{H}}_n$-modules obtained in this way are non-equivalent for different multisegments M and form a complete set with all eigenvalues of $X_1, \dots, X_n$ in $q^{\mathbb{Z}}$.

Using the above mentioned ordering of M, we describe the unique non-zero irreducible submodule in the $\widetilde{\mathcal{H}}_n$-module induced from (0.2). Our description is more explicit than that given by Rogawski [24], and follows the works of Cherednik [4,5]. We employ combinatorial objects which are in a bijection with Zelevinsky multisegments, and which we call Cherednik diagrams. They are certain subsets of $\mathbb{Z}^2$ similar to Young diagrams, see Definition 1.1. Moreover, both usual and skew Young diagrams are particular cases of Cherednik diagrams.

Let $\mathcal{A}_n$ be the subalgebra of $\widetilde{\mathcal{H}}_n$ generated by $X_1^{\pm 1}, \dots, X_n^{\pm 1}$. This is a maximal commutative subalgebra of $\widetilde{\mathcal{H}}_n$. For each Cherednik diagram λ we produce a pair (E, χ) where $E \in \mathcal{H}_n$ and $\chi : \mathcal{A}_n \to \mathbb{C}$ is a character of the algebra $\mathcal{A}_n$, such that:

(i) E is an eigenvector for $\mathcal{A}_n$ inside $\operatorname{Ind}_{\mathcal{A}_n}^{\widetilde{\mathcal{H}}_n} \chi = \mathcal{H}_n$;
(ii) the space $\mathcal{H}_n \cdot E$ which is an $\widetilde{\mathcal{H}}_n$-module by (i), is irreducible.

Our E will be the element E_λ defined by (1.13), and χ will be $w_0 \cdot \chi_\lambda$ where the character χ_λ is defined by (1.20). Here w_0 is the longest element of $\mathcal{S}_n$, and we use the natural action of the group $\mathcal{S}_n$ on the characters of $\mathcal{A}_n$. The elements E_λ where introduced by Cherednik in [4] for the degenerate affine Hecke algebra and then in [5] for $\widetilde{\mathcal{H}}_n$. But proofs are not given in [4,5] and the purpose of our paper is to provide them. The notion of a Cherednik diagram is also taken from [4,5].

From now on we will regard q as a formal parameter. Thus $\widetilde{\mathcal{H}}_n$ will be defined as an algebra over the field $\mathbb{C}(q)$ with generators $T_1, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and relations (1.1)–(1.6). The subalgebras $\mathcal{H}_n$ and $\mathcal{A}_n$ of $\widetilde{\mathcal{H}}_n$ then become $\mathbb{C}(q)$-algebras too. It is known that when the parameter q specializes to a non-zero complex number of infinite multiplicative order, the parametrization of the irreducible $\widetilde{\mathcal{H}}_n$-modules is the same for any such specialization. Our construction of the element $E_\lambda \in \mathcal{H}_n$ also allows any such specialization of q. Moreover, the corresponding specialization of the $\widetilde{\mathcal{H}}_n$-module $V_\lambda = \mathcal{H}_n \cdot E_\lambda$ remains irreducible; cf. [16,27,28].

Our construction of the element E_λ is based on the *fusion procedure* due to Cherednik [4,5]. Up to normalization, here E_λ is obtained as a limit of certain $\mathcal{H}_n$-valued function $\varphi_0(x_1, \dots, x_n)$ of n variables from $\mathbb{C}(q)$. This function is a product of elementary factors (1.9) corresponding to simple transpositions in a reduced decomposition of the element $w_0 \in \mathcal{S}_n$. The factors satisfy the Yang–Baxter relations, so that the function $\varphi_0(x_1, \ldots, x_n)$ does not depend on the choice of the reduced decomposition of w_0; see Lemma 1.4.

Fusion procedure is a method that was initially used to reproduce the Young symmetrizers in the group ring of $\mathcal{S}_n$. Nazarov used it in his works on projective representations of the symmetric group [19] and on their q-analogues [13]. In the case when λ is a usual Young diagram, a detailed construction of the element $E_\lambda \in \mathcal{H}_n$ by this method has been given in [20]. The results of [20] are easy to generalize to skew Young diagrams. But here we have to extend the method to those Cherednik diagrams λ, which are neither usual nor skew Young diagrams. In the corresponding $\widetilde{\mathcal{H}}_n$-modules, the action of the subalgebra $\mathcal{A}_n$ is not semisimple. Indeed, ours seems to be the first instance of a combinatorial treatment for these modules. By contrast, the irreducible $\widetilde{\mathcal{H}}_n$-modules with a semisimple action of $\mathcal{A}_n$ are well understood; they correspond to skew Young diagrams λ. A thoroughful treatment of them can be found in the work of Ram [23]. Moreover, by using the fusion procedure, in each of these modules one can construct a basis of eigenvectors of $\mathcal{A}_n$, not only an $\mathcal{H}_n$-cyclic vector. This result was also stated by Cherednik [4,5].

The emphasis in our paper is on the combinatorial aspects of fusion procedure; Sections 2 and 3 are devoted to this. The key technical difference comparing to the case of a skew Young diagram λ is that to remove the singularity, the function $\varphi_0(x_1, \ldots, x_n)$ has to be multiplied by a correction factor $\delta_\lambda(x_1, \ldots, x_n)$; the limit is taken afterwards. Further, for any skew Young diagram λ one has the equality

$$E_\lambda = T_{w_0} + \sum_{\ell(w) < \ell(w_0)} a_w T_w \tag{0.3}$$

for some coefficients $a_w \in \mathbb{C}(q)$; here T_w are the standard basis elements of $\mathcal{H}_n$ and $\ell(w)$ is the length function on $\mathcal{S}_n$. But for an arbitrary Cherednik diagram λ we have the equality (3.8) where the element $w_\lambda \in \mathcal{S}_n$ may differ from the longest element w_0. We would also like to emphasize that our construction of the element $E_\lambda \in \mathcal{H}_n$ is completely explicit, see Corollaries 2.9 and 3.5.

In Section 1 of our paper we fix the notation, state the main theorems and prove the irreducibility of the $\widetilde{\mathcal{H}}_n$-module V_λ. The irreducibility is proved by a rather indirect approach. Using the q-analogue of Drinfeld functor [8] due to Cherednik [6], we reduce the argument to irreducibility of certain finite-dimensional modules of quantum affine algebras, which has been proved by Akasaka and Kashiwara [1]. The results of [1] also imply that the $\widetilde{\mathcal{H}}_n$-modules V_λ for different Cherednik diagrams λ are pairwise non-equivalent. Note that the irreducibility and pairwise non-equivalence of the $\widetilde{\mathcal{H}}_n$-modules V_λ can also be proved using the methods of [7,14], see for instance [26].

1. Fusion procedure

1.1. Hecke algebras

Let $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$ denote respectively the finite and affine Hecke algebras of GL_n. We define $\mathcal{H}_n$ as the associative algebra over the field $\mathbb{C}(q)$ with generators $T_1, \ldots, T_{n-1}$ and relations

$$T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \quad 1 \leqslant k \leqslant n-2, \tag{1.1}$$

$$T_k T_l = T_l T_k, \quad 1 < |k-l|, \tag{1.2}$$

$$(T_k - q)(T_k + 1) = 0, \quad 1 \leqslant k \leqslant n-1. \tag{1.3}$$

Here q is a formal parameter. It is well known [11] that $\mathcal{H}_n$ is a q-deformation of the group algebra of the symmetric group $\mathcal{S}_n$, the generator T_k corresponding to the simple transposition $s_k = (k, k+1)$. Let $w = s_{k_1} \ldots s_{k_m}$ be a reduced decomposition of an element $w \in \mathcal{S}_n$. By (1.1), (1.2) the element $T_w = T_{k_1} \ldots T_{k_m}$ does not depend on the choice of the reduced decomposition of w.

We define $\widetilde{\mathcal{H}}_n$ as the associative algebra over $\mathbb{C}(q)$ generated by $T_1, \ldots, T_{n-1}$ subject to the relations above, and by the invertible elements $X_1, \ldots, X_n$ such that

$$X_k X_l = X_l X_k, \qquad 1 \leqslant k, l \leqslant n, \tag{1.4}$$

$$X_l T_k = T_k X_l, \qquad l \neq k, k+1, \tag{1.5}$$

$$T_k X_k T_k = q X_{k+1}, \quad 1 \leqslant k \leqslant n-1. \tag{1.6}$$

Let $\mathcal{A}_n$ be the subalgebra of $\widetilde{\mathcal{H}}_n$ generated by the elements $X_1, \ldots, X_n$ and by their inverses. Then $\{T_w \mid w \in \mathcal{S}_n\}$ is a basis of $\widetilde{\mathcal{H}}_n$ both as a right and as a left $\mathcal{A}_n$-module. Moreover, $\mathcal{A}_n$ is a maximal commutative subalgebra of $\widetilde{\mathcal{H}}_n$. The centre of $\widetilde{\mathcal{H}}_n$ consists of those elements of $\mathcal{A}_n$ which are invariant under the action of the group $\mathcal{S}_n$ by permutations of $X_1, \ldots, X_n$.

An easy way to produce an $\widetilde{\mathcal{H}}_n$-module is to induce from an algebra character $\chi : \mathcal{A}_n \to \mathbb{C}(q)$. Then the one-dimensional space $\mathbb{C}(q)$ can be regarded as an $\mathcal{A}_n$-module via χ and we can form the induced module I_χ. As a vector space I_χ can be identified with $\mathcal{H}_n$ where the algebra $\mathcal{H}_n$ acts via left multiplication. Hence all I_χ are isomorphic to each other as $\mathcal{H}_n$-modules. In I_χ we also have $X_k \cdot 1 = \chi(X_k)$ for $i = 1, \ldots, n$. We shall realize our irreducible $\widetilde{\mathcal{H}}_n$-modules as cyclic submodules of certain induced modules I_χ. When the parameter q specializes to any non-zero complex number of infinite multiplicative order, our submodules will remain irreducible. Moreover, then they will make a complete set of irreducible pairwise non-equivalent modules over any such specialization of $\widetilde{\mathcal{H}}_n$. To produce appropriate characters χ we need combinatorial tools, which we introduce next.

1.2. Cherednik diagrams

Zelevinsky [29] employed the multisegments (0.1) to parametrize the irreducible representations of the group $GL_n(F)$ generated by their subspaces of the vectors fixed by the action of the Iwahori subgroup J. It is useful to introduce an ordering on a multisegment. This leads to considering the following combinatorial object, which we call a *Cherednik diagram*.

Let $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_r)$ be sequences of integers with the same number of terms, such that $a_i \leqslant b_i$ for each index i. Consider the set

$$\left\{(i, j) \in \mathbb{Z}^2 \;\middle|\; 1 \leqslant i \leqslant r,\ a_i \leqslant j \leqslant b_i\right\}. \tag{1.7}$$

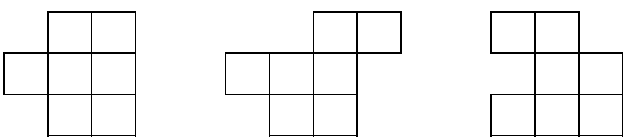
The total number of elements in the set (1.7) is called its *degree* and denoted by n,

$$\sum_{i=1}^{r} (b_i - a_i + 1) = n.$$

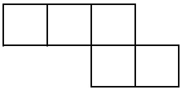
Then we can graphically represent the set (1.7) by a diagram with n boxes arranged in r rows, consisting respectively of $b_1 - a_1 + 1, \ldots, b_r - a_r + 1$ boxes. Using the matrix-style coordinates on the plane $\mathbb{R}^2$, the element (i, j) of (1.7) is represented by a unit box on $\mathbb{R}^2$ with the lower right corner placed at the point (i, j). Let $\mathcal{C}_n$ denote the collection of sets (1.7) of degree n, represented by their diagrams.

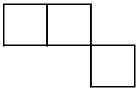
Definition 1.1. The set (1.7) is a Cherednik diagram if for each $i = 1, \ldots, r-1$ either $b_{i+1} \leqslant b_i$, or $b_{i+1} = b_i + 1$ and $a_{i+1} \leqslant a_i + 1$.

Let $\mathcal{C}_n$ denote the collection of Cherednik diagrams of degree n. It is clear that $\mathcal{C}_n$ contains all ordinary and skew Young diagrams of degree n. But the set $\mathcal{C}_n$ also contains other diagrams. For instance, the set $\mathcal{C}_7$ contains the three diagrams

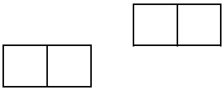


Here and in what follows we identify diagrams with their graphical representations. The next two diagrams do *not* belong to $\mathcal{C}_5$ and $\mathcal{C}_3$ respectively:





We do allow “disconnected” diagrams. For instance, $\mathcal{C}_4$ contains the diagram



For the set (1.7), we will use the partition-like notation $\lambda = (\lambda_1, \ldots, \lambda_r)$ where $\lambda_i = [a_i, b_i]$ and $1 \leqslant i \leqslant r$. Moreover, we will then write $|\lambda| = n$ if λ has degree n.

Let us denote by $\mathcal{M}_n$ the set of multisegments (0.1) such that

$$\sum_{i \leqslant j} m_{ij}(j - i + 1) = n.$$

There is a natural bijection $\mathcal{M}_n \to \mathcal{C}_n$. Indeed, given a multisegment $M \in \mathcal{M}_n$, consider the multiset

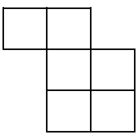
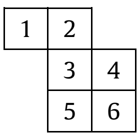
$$\bigl\{\ldots, \underbrace{[i, j], \ldots, [i, j]}_{m_{ij}}, \ldots\bigr\}$$

and endow it with the reverse lexicographical order, hence obtaining an ordered collection of segments $([i_1, j_1], \ldots, [i_r, j_r])$. Form a diagram λ by setting

$$\lambda_k = [a_k, b_k] = [i_k + k, j_k + k]$$

for each $k = 1, \ldots, r$. We claim that $\lambda \in \mathcal{C}_n$. Suppose that $b_k < b_{k+1}$. This means that $j_k \leqslant j_{k+1}$. Either $[i_k, j_k]$ precedes $[i_{k+1}, j_{k+1}]$ in the reverse lexicographical order, or these two segments are equal to each other. Hence $j_k \geqslant j_{k+1}$, so that $j_k = j_{k+1}$ and $b_k + 1 = b_{k+1}$. In that case $i_k \geqslant i_{k+1}$, so that $a_k + 1 \geqslant a_{k+1}$. Thus the conditions of Definition 1.1 are satisfied. The map $M \mapsto \lambda$ is clearly invertible.

For a diagram $\lambda \in \mathcal{C}_n$, we denote by the corresponding upper case letter Λ the *row filling* of λ. This is the tableau obtained by filling λ with the numbers $1, \ldots, n$ first from left to right and then from top to bottom. For instance,

$\lambda =$  $\Lambda =$ 

(1.8)

Definition 1.2. Two rows $\lambda_i = [a_i, b_i]$ and $\lambda_j = [a_j, b_j]$ of a diagram $\lambda \in \mathcal{C}_n$ are said to be *parallel* if $a_i - i = a_j - j$ and $b_i - i = b_j - j$.

For instance, the first two rows in the diagram λ in the display (1.8) are parallel. Note that under the map $M \mapsto \lambda$ defined above, parallel rows of the Cherednik diagram λ correspond to identical segments of the Zelevinsky multisegment M. The following lemma follows directly from the conditions of Definition 1.1.

Lemma 1.3. *Let $\lambda \in \mathcal{C}_n$. Suppose the rows λ_i and λ_j with $i < j$ end on the same diagonal. Then the same happens for all rows λ_k with $i \leqslant k \leqslant j$. If moreover λ_i is parallel to λ_j, then all rows λ_k with $i \leqslant k \leqslant j$ are parallel to each other.*

1.3. The function φ_λ

For any $x \in \mathbb{C}(q)$ put $\langle x\rangle = (1-q)/(1-x)$. For each $k = 1, \ldots, n-1$ introduce the rational function of the variables $x_1, \ldots, x_n \in \mathbb{C}(q)$,

$$\varphi_k(x_1, \ldots, x_n) = T_k + \langle x_{k+1}/x_k\rangle \tag{1.9}$$

with values in the algebra $\mathcal{H}_n$. For each permutation $w \in \mathcal{S}_n$ and any rational function ψ of $x_1, \ldots, x_n$ with values in $\mathcal{H}_n$ we will write

$${}^{w}\psi(x_1, \ldots, x_n) = \psi(x_{w(1)}, \ldots, x_{w(n)}).$$

Using any reduced decomposition $w = s_{k_1} \ldots s_{k_m}$ in $\mathcal{S}_n$ define the rational function

$$\varphi_w = \varphi_{k_1}\left({}^{s_{k_1}}\varphi_{k_2}\right)\left({}^{s_{k_1}s_{k_2}}\varphi_{k_3}\right)\ldots\left({}^{s_{k_1}\ldots s_{k_{m-1}}}\varphi_{k_m}\right). \tag{1.10}$$

For instance, if $w = s_1s_2s_3$, we have

$$\varphi_w = \big(T_1 + \langle x_2/x_1\rangle\big)\big(T_2 + \langle x_3/x_1\rangle\big)\big(T_3 + \langle x_4/x_1\rangle\big).$$

The function φ_w does not depend on the choice of a reduced decomposition of w. The independence follows from the next lemma, proved by a direct computation.

Lemma 1.4. *We have equality of rational functions in $x, y, z, x,' y', z' \in \mathbb{C}(q)$,*

$$\big(T_k + \langle x\rangle\big)\big(T_{k+1} + \langle y\rangle\big)\big(T_k + \langle z\rangle\big) = \big(T_{k+1} + \langle z'\rangle\big)\big(T_k + \langle y'\rangle\big)\big(T_{k+1} + \langle x'\rangle\big)$$

if and only if $x = x'$, $z = z'$ and $y = y' = xz$.

Now let $\lambda \in \mathcal{C}_n$ and Λ be the row filling of λ. For every box (i, j) of λ the difference $j - i$ is called the *content* of this box. For $k = 1, \ldots, n$ denote by c_k the content of the box which is filled with the number k in Λ. Denote by $\delta_\lambda(x_1, \ldots, x_n)$ the product of the differences $1 - x_l/x_k$ taken over all pairs (k,l) such that $k < l$ while in Λ the numbers k, l occur in the leftmost boxes of two parallel rows of λ. We assume that $\delta_\lambda(x_1, \ldots, x_n) = 1$ if λ does not have distinct parallel rows.

Let $\mathcal{F}_\lambda$ be the affine subspace in $\mathbb{C}(q)^{\times n}$ consisting of all points $(x_1, \ldots, x_n)$ such that $x_k q^{c_l} = q^{c_k} x_l$ whenever k and l are in the same row of Λ. Consider the rational function (1.10) corresponding to the element $w_0 \in \mathcal{S}_n$ of maximal length

$$\varphi_0(x_1, \ldots, x_n) = \varphi_{w_0}(x_1, \ldots, x_n). \tag{1.11}$$

The following theorem strengthens a classical result of Cherednik [5, Theorem 1].

Theorem 1.5. *For any $\lambda \in \mathcal{C}_n$ the restriction of the rational function $\delta_\lambda\varphi_0$ to the subspace $\mathcal{F}_\lambda$ is regular and non-zero at the point*

$$(x_1, \ldots, x_n) = \left(q^{c_1}, \ldots, q^{c_n}\right). \tag{1.12}$$

So we can take the value of the restriction of $\delta_\lambda\varphi_0$ to $\mathcal{F}_\lambda$ at the point (1.12),

$$E_\lambda = (\delta_\lambda\varphi_0)|_{\mathcal{F}_\lambda}\left(q^{c_1}, \ldots, q^{c_n}\right). \tag{1.13}$$

Theorem 1.5 is proved in Sections 2 and 3. We first show that the restriction to $\mathcal{F}_\lambda$ of the function $\delta_\lambda\varphi_0$ is regular at the point (1.12), see Proposition 2.8. Then we show that the corresponding value is non-zero, see Proposition 3.4. An important role of the non-zero element $E_\lambda \in \mathcal{H}_n$ is explained next.

1.4. Intertwining operators

For each index $k = 1, 2, \ldots, n-1$, denote

$$\Phi_k = T_k + (1-q)/\bigl(1 - X_k X_{k+1}^{-1}\bigr).$$

Then $\Phi_1, \ldots, \Phi_{n-1}$ lie in the localization of the algebra $\widetilde{\mathcal{H}}_n$ relative to the set of denominators

$$\bigl\{1 - X_k X_l^{-1} \bigm| 1 \leqslant k, l \leqslant n,\ k \neq l\bigr\}.$$

The following relations imply, in particular, that the Ore conditions are satisfied:

$$\Phi_k X_k = X_{k+1} \Phi_k, \tag{1.14}$$

$$\Phi_k X_{k+1} = X_k \Phi_k, \tag{1.15}$$

$$\Phi_k X_l = X_l \Phi_k, \quad l \neq k, k+1; \tag{1.16}$$

see for instance [17]. By (1.1) to (1.6) we also have relations in the ring of fractions,

$$\Phi_k \Phi_{k+1} \Phi_k = \Phi_{k+1} \Phi_k \Phi_{k+1}, \quad 1 \leqslant k \leqslant n-2, \tag{1.17}$$

$$\Phi_k \Phi_l = \Phi_l \Phi_k, \quad 1 < |k-l|, \tag{1.18}$$

$$\Phi_k^2 = \bigl(q - X_k X_{k+1}^{-1}\bigr)\bigl(1 - q X_k X_{k+1}^{-1}\bigr)/\bigl(1 - X_k X_{k+1}^{-1}\bigr)^2, \quad 1 \leqslant k \leqslant n-1.$$

By using any reduced recomposition $w = s_{k_1} \ldots s_{k_m}$ in $\mathcal{S}_n$, we can define an element $\Phi_w = \Phi_{k_1} \ldots \Phi_{k_m}$ of the ring of fractions. Due to (1.17), (1.18) this element does not depend on the choice of the reduced decomposition of w. By (1.14) to (1.16), for all $w \in \mathcal{S}_n$ and $k = 1, \ldots, n$ we than have an equality

$$\Phi_w X_k = X_{w(k)} \Phi_w. \tag{1.19}$$

The symmetric group $\mathcal{S}_n$ naturally acts on any character χ of the subalgebra $\mathcal{A}_n \subset \widetilde{\mathcal{H}}_n$ so that

$$(w \cdot \chi)(X_k) = \chi(X_{w^{-1}(k)}).$$

Let $\pi_\chi : \widetilde{\mathcal{H}}_n \to \mathrm{End}(\mathcal{H}_n)$ be the defining homomorphism of the $\widetilde{\mathcal{H}}_n$-module I_χ. Note that $\chi(X_k) \neq 0$ for any character χ and index k, because $X_k^{-1} \in \mathcal{A}_n$. The character χ is called *regular* if $\chi(X_k) \neq \chi(X_l)$ for $k \neq l$. For a regular character χ, the action of the algebra $\widetilde{\mathcal{H}}_n$ on I_χ extends to each element Φ_w of the ring of fractions. This extended action is also denoted by π_χ.

Proposition 1.6. *For any regular χ, the operator of right multiplication in $\mathcal{H}_n$ by the element $\pi_\chi(\Phi_w)(1)$ is an intertwining operator $I_{w\cdot\chi} \to I_\chi$ of $\widetilde{\mathcal{H}}_n$-modules.*

Proof. Denote by μ this operator. The action of the generators $T_1, \ldots, T_{n-1}$ on the representation space $\mathcal{H}_n$ of I_χ and $I_{w\cdot\chi}$ is through left multiplication and commutes with the operator μ. It therefore remains for us to verify that the action of the elements $X_1, \ldots, X_n$ commutes with μ as well. Since the vector $1 \in \mathcal{H}_n$ is cyclic for the actions of the subalgebra $\mathcal{H}_n \subset \widetilde{\mathcal{H}}_n$ on I_χ and $I_{w\cdot\chi}$, it is sufficient to demonstrate that the composition operators $\pi_\chi(X_k)\mu$ and $\mu\pi_{w\cdot\chi}(X_k)$ coincide on the identity vector for each $k = 1, 2, \ldots, n$. But this follows from (1.19):

$$\begin{aligned}
\pi_\chi(X_k)\bigl(\pi_\chi(\Phi_w)(1)\bigr) &= \pi_\chi(X_k \Phi_w)(1) \\
&= \pi_\chi(\Phi_w X_{w^{-1}(k)})(1) \\
&= \pi_\chi(\Phi_w)\bigl(\pi_{w\cdot\chi}(X_k)(1)\bigr). \qquad \square
\end{aligned}$$

Let us now fix an element $w \in \mathcal{S}_n$ and a point $(x_1, \ldots, x_n) \in \mathbb{C}(q)^{\times n}$ such that $x_k \neq x_l$ for $k \neq l$, and $x_k \neq 0$ for all k. A regular character χ can then be determined by setting $\chi(X_k) = x_{w(k)}$ for $k = 1, \ldots, n$. We have $(w \cdot \chi)(X_k) = x_k$.

Proposition 1.7. *We have the equality $\pi_\chi(\Phi_w)(1) = \varphi_w(x_1,\dots,x_n)$ in $\mathcal{H}_n$.*

Proof. We will use the induction on the length $\ell(w)$ of the element $w \in \mathcal{S}_n$. By definition, $\ell(w)$ is the number of factors in any reduced decomposition of w. If $\ell(w)=0$ then w is the identity element of $\mathcal{S}_n$, and Proposition 1.7 is trivial.

Now suppose that Proposition 1.7 is valid for some element $w \in \mathcal{S}_n$ and every regular character χ. Take any index $l \in \{1,\dots,n-1\}$ such that $\ell(s_l w) > \ell(w)$. Determine a character χ' of $\mathcal{A}_n$ by setting $\chi'(X_k) = x_{s_l w(k)}$ for $k=1,\dots,n$. This character is regular. We shall make the induction step by showing that

$$\pi_{\chi'}(\Phi_{s_l w})(1) = \varphi_{s_l w}(x_1,\dots,x_n).$$

Let $(x'_1,\dots,x'_n) \in \mathbb{C}(q)^{\times n}$ be the point obtained from $(x_1,\dots,x_n)$ by swapping the coordinates x_l and x_{l+1}. Then $\chi'(X_k) = x'_{w(k)}$ for $k=1,\dots,n$ so that

$$\pi_{\chi'}(\Phi_w)(1) = \varphi_w\big(x'_1,\dots,x'_n\big) = {}^{s_l}\varphi_w(x_1,\dots,x_n)$$

by the induction assumption. Further, by the definition of the character χ', we get

$$\chi'(X_{w^{-1}(l)}) = x_{l+1} \quad \text{and} \quad \chi'(X_{w^{-1}(l+1)}) = x_l.$$

Hence by using (1.19)

$$\pi_{\chi'}(X_l\Phi_w)(1) = x_{l+1}\pi_{\chi'}(\Phi_w)(1),$$
$$\pi_{\chi'}(X_{l+1}\Phi_w)(1) = x_l\pi_{\chi'}(\Phi_w)(1).$$

Therefore

$$\begin{aligned}\pi_{\chi'}(\Phi_{s_l w})(1) &= \pi_{\chi'}(\Phi_l\Phi_w)(1)\\ &= \varphi_l(x_1,\dots,x_n)\pi_{\chi'}(\Phi_w)(1)\\ &= \varphi_l(x_1,\dots,x_n)^{s_l}\varphi_w(x_1,\dots,x_n)\\ &= \varphi_{s_l w}(x_1,\dots,x_n). \qquad \square\end{aligned}$$

1.5. Cyclic generators for irreducible $\widetilde{\mathcal{H}}_n$-modules

For any $\lambda \in \mathcal{C}_n$ define a character χ_λ of $\mathcal{A}_n$ by setting

$$\chi_\lambda(X_k) = q^{c_k} \tag{1.20}$$

for each $k=1,\dots,n$. This character is regular, if and only if no diagonal of the diagram λ contains more than one box. However, using Propositions 1.6 and 1.7 we obtain the following corollary to Theorem 1.5.

Corollary 1.8. *If $\lambda \in \mathcal{C}_n$ then the operator of right multiplication in $\mathcal{H}_n$ by the element E_λ is an intertwining operator $I_{\chi_\lambda} \to I_{w_0\cdot\chi_\lambda}$ of $\widetilde{\mathcal{H}}_n$-modules.*

Proof. Take any point $(x_1,\dots,x_n) \in \mathcal{F}_\lambda$ such that $x_k \neq x_l$ for $k \neq l$, and $x_k \neq 0$ for all k. A regular character χ can then be determined by setting

$$(w_0\cdot\chi)(X_k) = x_k \tag{1.21}$$

for each $k=1,\dots,n$. By choosing $w=w_0$ in Propositions 1.6 and 1.7, we obtain that the operator of right multiplication in $\mathcal{H}_n$ by $\varphi_0(x_1,\dots,x_n)$ is an intertwining operator $I_{w_0\cdot\chi} \to I_\chi$. So is the operator of right multiplication by the product

$$\delta_\lambda(x_1,\dots,x_n)\varphi_0(x_1,\dots,x_n).$$

On the other hand, at the point (1.12) the character $w_0 \cdot \chi$ defined by the equalities (1.21) specializes to χ_λ, see (1.20). The character χ then specializes to $w_0 \cdot \chi_\lambda$. We now get Corollary 1.8 by the definition (1.13) of the element E_λ. □

Consider the left ideal in $\mathcal{H}_n$ generated by the element E_λ. Corollary 1.8 shows that this left ideal is a submodule of the induced $\widetilde{\mathcal{H}}_n$-module $I_{w_0 \cdot \chi_\lambda}$. Let us denote by V_λ this submodule. Note that $V_\lambda \neq \{0\}$ because $E_\lambda \neq 0$. The following theorem has been stated in [5] without proof.

Theorem 1.9. *The $\widetilde{\mathcal{H}}_n$-module V_λ is irreducible.*

Proof. Using the notation of Section 1.2, write $\lambda = (\lambda_1, \ldots, \lambda_r)$ where $\lambda_i = [a_i, b_i]$ for each $i = 1, \ldots, r$. Then consider another diagram $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_r)$ with the same number r of rows, such that for each $i = 1, \ldots, r$,

$$\bar{\lambda}_i = [r - b_{r-i+1} + 1, r - a_{r-i+1} + 1].$$

Then $\bar{\lambda}$ has degree n, but is not necessarily a Cherednik diagram. If $\bar{c}_1, \ldots, \bar{c}_n$ are the contents corresponding to $\bar{\lambda}$ then $\bar{c}_k = -c_{n-k+1}$ for $k = 1, \ldots, n$.

We claim that an analogue of Theorem 1.5 holds for the diagram $\bar{\lambda}$ instead of λ. Namely, the restriction of the rational function $\delta_{\bar{\lambda}} \varphi_0$ to the subspace $\mathcal{F}_{\bar{\lambda}}$ is regular and non-zero at the point

$$(x_1, \ldots, x_n) = \left(q^{\bar{c}_1}, \ldots, q^{\bar{c}_n}\right). \tag{1.22}$$

Further, let ω_n be the involutive automorphism of the $\mathbb{C}(q)$-algebra $\mathcal{H}_n$ defined by setting $\omega_n(T_k) = T_{n-k}$ for each $k = 1, \ldots, n-1$. We also claim that the value of that restriction at the point (1.22) equals $\omega_n(E_\lambda)$.

To verify these two claims, consider the transformation of $(\mathbb{C}(q) \setminus \{0\})^{\times n}$,

$$(x_1, \ldots, x_n) \mapsto \left(x_n^{-1}, \ldots, x_1^{-1}\right).$$

This transformation maps the point (1.12) to the point (1.22), and also maps $\mathcal{F}_\lambda$ to $\mathcal{F}_{\bar{\lambda}}$. But by using Lemma 1.4, we get the relation

$$\varphi_0\left(x_n^{-1}, \ldots, x_1^{-1}\right) = \omega_n\left(\varphi_0(x_1, \ldots, x_n)\right).$$

Moreover, by the definition of function $\delta_{\bar{\lambda}}$ we have the relation

$$\delta_{\bar{\lambda}}\left(x_n^{-1}, \ldots, x_1^{-1}\right) = \delta_\lambda(x_1, \ldots, x_n).$$

Our two claims now follow from Theorem 1.5 and from the definition of E_λ.

Further, we have an analogue of Corollary 1.8 for the diagram $\bar{\lambda}$ instead of λ. Consider the character $\chi_{\bar{\lambda}}$ of $\mathcal{A}_n$ corresponding to $\bar{\lambda}$. The operator of right multiplication in $\mathcal{H}_n$ by the element $\omega_n(E_\lambda)$ is then an intertwiner $I_{\chi_{\bar{\lambda}}} \to I_{w_0 \cdot \chi_{\bar{\lambda}}}$. Let $V_{\bar{\lambda}}$ be the left ideal in $\mathcal{H}_n$ generated by the element $\omega_n(E_\lambda)$. This left ideal is a submodule of the induced $\widetilde{\mathcal{H}}_n$-module $I_{w_0 \cdot \chi_{\bar{\lambda}}}$. We claim that the irreducibility of the $\widetilde{\mathcal{H}}_n$-module $V_{\bar{\lambda}}$ is equivalent to that of V_λ.

Indeed, the automorphism ω_n of $\mathcal{H}_n$ extends to an involutive automorphism of the $\mathbb{C}(q)$-algebra $\widetilde{\mathcal{H}}_n$ by setting $\omega_n(X_k) = X_{n-k+1}^{-1}$ for each index $k = 1, \ldots, n$. Consider the $\widetilde{\mathcal{H}}_n$-module $I_{w_0 \cdot \chi_\lambda}^{\omega_n}$ obtained from $I_{w_0 \cdot \chi_\lambda}$ by twisting the latter module with the automorphism ω_n of $\widetilde{\mathcal{H}}_n$. The twisted $\widetilde{\mathcal{H}}_n$-module is equivalent to $I_{w_0 \cdot \chi_{\bar{\lambda}}}$: the underlying vector space of the two modules is $\mathcal{H}_n$, and the equivalence map

$$I_{w_0 \cdot \chi_\lambda}^{\omega_n} \to I_{w_0 \cdot \chi_{\bar{\lambda}}} \tag{1.23}$$

can be chosen as $\omega_n : \mathcal{H}_n \to \mathcal{H}_n$. Here we also use the equalities for $k = 1, \ldots, n$,

$$(w_0 \cdot \chi_\lambda)\left(\omega_n(X_k)\right) = q^{-c_k} = (w_0 \cdot \chi_{\bar{\lambda}})(X_k).$$

The image of the submodule $V_\lambda^{\omega_n} \subset I_{w_0 \cdot \chi_\lambda}^{\omega_n}$ under the map (1.23) is $V_{\bar{\lambda}}$. Therefore the irreducibility of $V_{\bar{\lambda}}$ is equivalent to that of $V_\lambda^{\omega_n}$, and hence to that of V_λ.

There is an involutive automorphism of $\widetilde{\mathcal{H}}_n$ as $\mathbb{C}$-algebra, defined by mapping

$$q \mapsto q^{-1}, \qquad T_k \mapsto -q^{-1}T_k, \qquad X_k \mapsto X_k$$

for all possible indices k. Denote by V'_λ the $\widetilde{\mathcal{H}}_n$-module obtained by twisting $V_{\bar\lambda}$ with this automorphism. We shall establish the irreducibility of V'_λ under the conditions of Definition 1.1 on the diagram λ. The irreducibility of V_λ will then follow. We will use the representation theory of the quantum enveloping algebra $U_\upsilon(\widehat{\mathfrak{sl}}_N)$ of the Kac–Moody Lie algebra $\widehat{\mathfrak{sl}}_N$, with the parameter $\upsilon = q^{1/2}$.

A link between the representation theories of the affine Hecke algebras and quantum affine algebras was discovered by Drinfeld [8]. For the algebras $\widetilde{\mathcal{H}}_n$ and $U_q(\widehat{\mathfrak{sl}}_N)$ this link was established by Cherednik [5,6]. We will employ a version of this link due to Chari and Pressley [3]. However unlike in [3], here q is a formal parameter, not a complex number. Hence we regard $U_\upsilon(\widehat{\mathfrak{sl}}_N)$ as a $\mathbb{C}(\upsilon)$-algebra.

There is a functor $\mathcal{J}$ from the category of all finite-dimensional $\widetilde{\mathcal{H}}_n$-modules to the category of finite-dimensional $U_\upsilon(\widehat{\mathfrak{sl}}_N)$-modules [3, Theorem 4.2]. If $N > n$, then the $U_\upsilon(\widehat{\mathfrak{sl}}_N)$-module $\mathcal{J}(V)$ is non-zero for each non-zero $\widetilde{\mathcal{H}}_n$-module V. But under the conditions on the diagram $\bar\lambda$ implied by Definition 1.1, the $U_\upsilon(\widehat{\mathfrak{sl}}_N)$-module $\mathcal{J}(V'_\lambda)$ is irreducible [1, Corollary 2.3, Proposition 3.5 and Theorem 4.1]. Hence V_λ is also irreducible. Note that the irreducibility of the $U_\upsilon(\widehat{\mathfrak{sl}}_N)$-module $\mathcal{J}(V'_\lambda)$ can also be derived from [21, Proposition 3.1]. $\square$

2. Beginning of the proof of Theorem 1.5

2.1. Combinatorial preliminaries

Let $u_1, \dots, u_n$ be the standard basis in the Euclidean vector space $\mathbb{R}^n$. Take the root system in $\mathbb{R}^n$ of type A_{n-1}. A choice of the set $\mathcal{P}$ positive roots is made as

$$\mathcal{P} = \{u_i - u_j \mid 1 \leqslant i < j \leqslant n\}.$$

With this choice, the simple roots are $\alpha_i = u_i - u_{i+1}$ for $i = 1, \dots, n-1$. We will be interested into certain subsets of $\mathcal{P}$ and certain total orders on them. They have been studied independently in [9,22,25] where they appear under different names: total reflection orders, normal orders, compatible orders respectively.

Definition 2.1.

(a) A subset $\mathcal{L} \subset \mathcal{P}$ is called biconvex if both $\mathcal{L}$ and $\mathcal{P} \setminus \mathcal{L}$ are closed under root addition.
(b) A total order $<$ on a biconvex set $\mathcal{L}$ is said to be a convex order if it satisfies the following conditions:
  (i) if $\alpha, \beta \in \mathcal{L}$, $\alpha + \beta \in \mathcal{P}$ and $\alpha < \beta$, then $\alpha < \alpha + \beta < \beta$;
  (ii) if $\alpha + \beta \in \mathcal{L}$ and $\alpha \notin \mathcal{L}$, then $\beta < \alpha + \beta$.

Note that $\mathcal{P}$ is in canonical bjiection with the set $\{(i, j) \mid 1 \leqslant i < j \leqslant n\}$ by $u_i - u_j \mapsto (i, j)$. We shall tacitly use this identification in the following, speaking of *pairs* rather than of roots. We say that two pairs α, β are orthogonal if the corresponding roots are orthogonal in $\mathbb{R}^n$. Then we write $\alpha \perp \beta$.

We will be interested into two particular convex orderings. The lexicographic order on $\mathcal{P}$ will be denoted by $<_1$:

$$(i, j) <_1 (k, l) \quad \text{if and only if} \quad i < k \text{ or } i = k \text{ and } j < l. \tag{2.1}$$

We will also use another useful order which will be denoted by $<_2$:

$$(i, j) <_2 (k, l) \quad \text{if and only if} \quad j < l \text{ or } j = l \text{ and } i < k. \tag{2.2}$$

The symmetric group $\mathcal{S}_n$ acts on the root system, via permutations of the basis vectors $u_1, \dots, u_n \in \mathbb{R}^n$. For any $w \in \mathcal{S}_n$ take the set

$$\mathcal{I}_w = \{\alpha \in \mathcal{P} \mid w^{-1}(\alpha) \notin \mathcal{P}\}.$$

This is the set of *inversions* for w^{-1}. It is well known that, if $w = s_{i_1} \dots s_{i_m}$ is a reduced decomposition, then $\mathcal{I}_w = \{\beta_1, \dots, \beta_m\}$ where $\beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$ for $k = 1, \dots, m$. Here $m = \ell(w)$. The choice of the reduced decomposition for w provides a total ordering of $\mathcal{I}_w$: here $\beta_i < \beta_k$ if and only if $i < k$. Furthermore, a subset $\mathcal{L} \subset \mathcal{P}$ is biconvex if and only if it is of the form $\mathcal{I}_w$ for some $w \in \mathcal{S}_n$, see [22]. The convex orders on $\mathcal{I}_w$ are exactly the ones provided by the reduced decompositions of w. The order $<_1$ on $\mathcal{P}$ is provided by the decomposition

$$w_0 = (s_1 \dots s_{n-1}) \dots (s_1 s_2) s_1, \tag{2.3}$$

while the order $<_2$ on $\mathcal{P}$ is provided by the decomposition

$$w_0 = s_1(s_2 s_1) \dots (s_{n-1} \dots s_1).$$

Finally, we recall the following technical result [10, Proposition 1.9]. Let $<$ be any total order on $\mathcal{P}$. For any $\alpha \in \mathcal{P}$ define the set $\alpha^{\leqslant}$ as $\{\beta \in \mathcal{P} \mid \beta \leqslant \alpha\}$. Then define the set $\alpha^{\geqslant}$ in the obvious way.

Lemma 2.2. *Suppose that in a certain convex order $<'$ on $\mathcal{P}$ the pair (i, j) is covered by $(i+1, j)$ or covers $(i, j-1)$. Then there exists a convex order $<$ on $\mathcal{P}$ in which (i, j) covers $(i, i+1)$ or is covered by $(j-1, j)$ respectively, and which restricts to the order $<'$ on $(i, j)^{\geqslant}$ or on $(i, j)^{\leqslant}$ respectively.*

2.2. Yang–Baxter relations

With a slight abuse of notation, set

$$\langle \beta \rangle = \langle x_j / x_i \rangle \tag{2.4}$$

for each $\beta = (i, j) \in \mathcal{P}$, see Section 1.3. Then our basic function (1.9) can be written as $\varphi_k = T_k + \langle \alpha_k \rangle$. If $w = s_{i_1} \dots s_{i_m}$ is a reduced decomposition, then

$$\begin{aligned} \varphi_w &= (\varphi_{i_1})\big({}^{s_{i_1}}\varphi_{i_2}\big)\big({}^{s_{i_1}s_{i_2}}\varphi_{i_3}\big) \dots \big({}^{s_{i_1} \dots s_{i_{m-1}}}\varphi_{i_m}\big) \\ &= \big(T_{i_1} + \langle \beta_1 \rangle\big)\big(T_{i_2} + \langle \beta_2 \rangle\big)\big(T_{i_3} + \langle \beta_3 \rangle\big) \dots \big(T_{i_n} + \langle \beta_n \rangle\big) \end{aligned}$$

where as above $\beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$ for each $k = 1, \dots, m$. Hence, if we denote

$$\varphi^l_\beta = T_l + \langle \beta \rangle,$$

then we get

$$\varphi_w = \varphi^{i_1}_{\beta_1} \varphi^{i_2}_{\beta_2} \dots \varphi^{i_m}_{\beta_m}. \tag{2.5}$$

To simplify our notation, we will write φ_{β_j} for $\varphi^{i_j}_{\beta_j}$. Also, if $\beta = (i, j) \in \mathcal{P}$ we will sometimes write $\varphi_\beta = \varphi_{ij}$. A direct consequence of Lemma 1.4 is the following.

Lemma 2.3. *The function φ_w is invariant under the following moves of adjacent factors in the product* (2.5):

$$\varphi_\alpha \varphi_\beta = \varphi_\beta \varphi_\alpha \quad \text{if } \alpha \perp \beta, \tag{2.6}$$

$$\varphi_\alpha \varphi_{\alpha+\beta} \varphi_\beta = \varphi_\beta \varphi_{\alpha+\beta} \varphi_\alpha \quad \text{if } \alpha + \beta \in \mathcal{P}. \tag{2.7}$$

Another easy observation which will be used later is the following.

Lemma 2.4. *If $\alpha, \beta \in \mathcal{P}$ are adjacent in a convex order then $\alpha \perp \beta$ implies that $\varphi_\alpha = \varphi^k_\alpha$, $\varphi_\beta = \varphi^l_\beta$ for some k, l with $|k - l| > 1$. If $\alpha, \alpha + \beta, \beta \in \mathcal{P}$ are adjacent in a convex order then $\varphi_\alpha = \varphi^k_\alpha$, $\varphi_{\alpha+\beta} = \varphi^{k\pm1}_{\alpha+\beta}$, $\varphi_\beta = \varphi^k_\beta$ for some k.*

2.3. Singular pairs

We now begin working towards our proof of Theorem 1.5. Fix a diagram $\lambda \in \mathcal{C}_n$. For any numbers i and j such that $1 \leqslant i < j \leqslant n$, the pair (i, j) will be called *singular* if $c_i = c_j$. So the singularity of (i, j) means that i and j occur in the same diagonal of Λ. If $\beta = (i, j)$ is a singular pair then the function (2.4) is singular at the point (1.12), hence the terminology.

The next lemma is the core of the fusion procedure. It appeared in [19] in the symmetric group setting, and can be proved by a direct computation using (1.3). Note that the middle factors of the products appearing at the left-hand sides of (2.8), (2.9) are singular at (1.12) but the products, upon restricting to $\mathcal{F}_\lambda$, are not.

Lemma 2.5. *Suppose that the pair (i, j) is singular.*

(a) *If $i, i+1$ belong to the same row of Λ, then for any $k = 1, \dots, n-2$,*

$$\big(\varphi^k_{i,i+1}\varphi^{k+1}_{i,j}\varphi^k_{i+1,j}\big)\big|_{\mathcal{F}_\lambda}\big(q^{c_1}, \dots, q^{c_n}\big) = (1 + T_k)(T_{k+1}T_k - qT_{k+1} - q). \tag{2.8}$$

(b) *If $j-1, j$ belong to the same row of Λ, then for any $k = 1, \dots, n-2$,*

$$\big(\varphi^k_{i,j-1}\varphi^{k+1}_{i,j}\varphi^k_{j-1,j}\big)\big|_{\mathcal{F}_\lambda}\big(q^{c_1}, \dots, q^{c_n}\big) = (T_kT_{k+1} - qT_{k+1} - q)(1 + T_k). \tag{2.9}$$

Write $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\lambda_k = [a_k, b_k]$ for $k = 1, \dots, r$. For $1 \leqslant k \leqslant l \leqslant r$ put

$$\mathcal{P}_{kl} = \big\{(i, j) \bigm| 1 \leqslant i < j \leqslant n \text{ and } i, j \text{ are in rows } k, l \text{ of } \Lambda \text{ respectively}\big\}. \tag{2.10}$$

Then the set $\mathcal{P}$ becomes a disjoint union of the subsets (2.10). Define a total order $<$ on the set $\mathcal{P}$ by first defining an order on every subset (2.10): the restriction of the order $<$ to $\mathcal{P}_{kl}$ is $<_1$ if $b_k - k = b_l - l$, and is $<_2$ otherwise; see (2.1) and (2.2). Note that if the row k or l of λ has length one, then the restrictions of $<_1$ and $<_2$ to $\mathcal{P}_{kl}$ are the same. Now order the subsets (2.10) relative to each other:

$$\mathcal{P}_{11} < \mathcal{P}_{12} < \mathcal{P}_{22} < \cdots < \mathcal{P}_{1r} < \mathcal{P}_{2r} < \cdots < \mathcal{P}_{rr}. \tag{2.11}$$

The so defined order $<$ on $\mathcal{P}$ will be called *special*. A straightforward verification of the conditions (i), (ii) in Definition 2.1 shows that the special order $<$ is convex. Let $w_0 = s_{i_1} \dots s_{i_m}$ be the corresponding reduced decomposition. As before, let $\beta_1, \dots, \beta_m$ be the elements of $\mathcal{P}$ written in this order. Here $m = n(n-1)/2$.

Lemma 2.6. *If $\beta_k = (i, j)$ then $i_k = j - i$.*

Proof. Recall that the order $<_1$ on $\mathcal{P}$ corresponds to the reduced decomposition (2.3). For this reduced decomposition, the statement of the lemma is clearly true. On the other hand, it is easy to see that the special order can be obtained from $<_1$ by a sequence of switches of adjacent orthogonal pairs. Switching two orthogonal pairs corresponds to switching two adjacent simple transpositions from $\mathcal{S}_n$ in a reduced decomposition of w_0. Hence the statement is true for the order $<$ too. □

Denote by d_λ the number of singular pairs for λ. Denote by p_λ the number of pairs of distinct non-empty parallel rows of λ. Any pair of distinct parallel rows of length l gives rise to l singular pairs. In particular, we have $d_\lambda \geqslant p_\lambda$.

Further, denote by $\mathcal{R}$ the collection of all singular pairs for λ except the pairs (i, j) where i and j are the first numbers in two parallel rows of Λ. The set $\mathcal{R}$ consists of exactly $d_\lambda - p_\lambda$ elements.

Let $\xi = (i, j) = \alpha_i + \dots + \alpha_{j-1}$ be a singular pair for λ. If $i, i+1$ belong to the same row of Λ, then denote $\xi^+ = (i+1, j)$ so that $\xi = \alpha_i + \xi^+$. Similarly, if $j-1, j$ belong to the same row of Λ, then denote $\xi^- = (i, j-1)$ so that $\xi = \alpha_{j-1} + \xi^-$.

Lemma 2.7. *For each $\xi \in \mathcal{R}$ one can choose $e(\xi) \in \{\pm\}$ such that ξ and $\xi^{e(\xi)}$ are adjacent in the special order, while for all $\xi \in \mathcal{R}$ the elements $\xi^{e(\xi)}$ are distinct.*

Proof. For any $\xi \in \mathcal{P}$ there is a unique subset $\mathcal{P}_{kl} \subset \mathcal{P}$ containing ξ; here $k \leqslant l$. Write $\xi = (i, j)$; the numbers i and j occur in the rows k and l of Λ respectively. Define the function $\xi \mapsto e(\xi)$ so that $e(\xi) = -$ if $b_k - k = b_l - l$, and $e(\xi) = +$ otherwise. This is a function on the set $\mathcal{P}$. Let us show that the restriction of this function to the subset $\mathcal{R} \subset \mathcal{P}$ has all the required properties.

Let $\xi \in \mathcal{R}$. Then $k < l$; the numbers i and j occur in the same diagonal of Λ.

First suppose that $b_k - k = b_l - l$. If $a_k - k = a_l - l$ then the rows k and l of Λ are parallel. Then the number $j - 1$ belongs to the same row l of Λ as j, because j cannot be the first number in its row. If $a_k - k \neq a_l - l$ then $a_k - k > a_l - l$ by Definition 1.1, and again $j - 1$ belongs to the same row l as j. Hence the pair $\xi^- = (i, j-1) \in \mathcal{P}_{kl}$ is defined, and the two pairs ξ^-, ξ are adjacent in the order $<_1$. The latter order coincides with $<$ on the subset $\mathcal{P}_{kl} \subset \mathcal{P}$.

Now suppose that $b_k - k \neq b_l - l$. Then $b_k - k > b_l - l$ by Definition 1.1. Then the number $i + 1$ belongs to the same row k of Λ as i, because the numbers i and j occur in the same diagonal of Λ. Hence the pair $\xi^+ = (i+1, j) \in \mathcal{P}_{kl}$ is defined, and the two pairs ξ, ξ^+ are adjacent in the order $<_2$. The latter order coincides with $<$ on the subset $\mathcal{P}_{kl} \subset \mathcal{P}$.

Now let ξ run through the set $\mathcal{R}$. The pairs of the form ξ^- are different from each other, and so are the pairs of the form ξ^+. Moreover, the pairs of the form ξ^- belong to the subsets (2.10) with $b_k - k = b_l - l$, while the pairs of the form ξ^+ belong to the subsets (2.10) with $b_k - k > b_l - l$. Since all the subsets (2.10) are disjoint, all pairs of the form ξ^- are different from all those of the form ξ^+. $\square$

For example, consider the Cherednik diagram $\lambda = ([1,2],[2,3],[2,3])$; see (1.8). Here $d_\lambda = 4$ and $p_\lambda = 1$. The special order on $\mathcal{P}$ is given by the following table:

$$
\begin{aligned}
&\mathcal{P}_{11}\colon\quad (1,2),\\
&\mathcal{P}_{12}\colon\quad (1,3),(1,4),(2,3),(2,4),\\
&\mathcal{P}_{22}\colon\quad (3,4),\\
&\mathcal{P}_{13}\colon\quad (1,5),(2,5),(1,6),(2,6),\\
&\mathcal{P}_{23}\colon\quad (3,5),(4,5),(3,6),(4,6),\\
&\mathcal{P}_{33}\colon\quad (5,6).
\end{aligned}
$$

Here the collection $\mathcal{R}$ consists of all singular pairs except the pair $(1,3)$. In the next table all singular pairs are set in bold; the pairs $\xi \in \mathcal{R}$ are underlined together with their corresponding pairs $\xi^{e(\xi)}$:

$$
\begin{aligned}
&\mathcal{P}_{11}\colon\quad (1,2),\\
&\mathcal{P}_{12}\colon\quad (\mathbf{1},\mathbf{3}),(1,4),\underline{(2,3),(\mathbf{2},\mathbf{4})},\\
&\mathcal{P}_{22}\colon\quad (3,4),\\
&\mathcal{P}_{13}\colon\quad (1,5),(2,5),\underline{(\mathbf{1},\mathbf{6}),(2,6)},\\
&\mathcal{P}_{23}\colon\quad (3,5),(4,5),\underline{(\mathbf{3},\mathbf{6}),(4,6)},\\
&\mathcal{P}_{33}\colon\quad (5,6).
\end{aligned}
$$

In the proof of Lemma 2.7, the value $e(\beta)$ was defined for any $\beta \in \mathcal{P}$. Hence we can uniquely divide $\mathcal{P}$ into new ordered subsets $\mathcal{P}_1, \dots, \mathcal{P}_s$ so that:

(i) $\mathcal{P}_1 < \cdots < \mathcal{P}_s$;
(ii) the function $e(\beta)$ is constant on each of the subsets $\mathcal{P}_1, \dots, \mathcal{P}_s$;
(iii) the subsets $\mathcal{P}_1, \dots, \mathcal{P}_s$ are maximal with the properties (i), (ii).

Each of the subsets $\mathcal{P}_1, \ldots, \mathcal{P}_s$ is a union of certain subsets (2.10). For the diagram $\lambda = ([1,2],[2,3],[2,3])$ from the previous example, we have $s = 3$ and

$$\mathcal{P}_1 = \mathcal{P}_{11} \sqcup \mathcal{P}_{12} \sqcup \mathcal{P}_{22}, \qquad \mathcal{P}_2 = \mathcal{P}_{13} \sqcup \mathcal{P}_{23}, \qquad \mathcal{P}_3 = \mathcal{P}_{33}.$$

For any Cherednik diagram λ and $t = 1, \ldots, s$ the subset $\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_t$ is biconvex.

Proposition 2.8. *Restriction of the function $\delta_\lambda \varphi_0$ to $\mathcal{F}_\lambda$ is regular at* (1.12)*.*

Proof. We have already observed that the restriction to $\mathcal{F}_\lambda$ of any factor φ_β of φ_0 with a non-singular β is regular at the point (1.12). Let us explain the idea of the proof in the particular case $d_\lambda = 1$. First suppose that $p_\lambda = 1$. Then it suffices to consider the diagram $\lambda = \{[1,1],[2,2]\}$. Here

$$\delta_\lambda \varphi_0(x_1, x_2) = (1 - x_2/x_1)\left(T_1 + \frac{1-q}{1 - x_2/x_1}\right) = (1 - x_2/x_1)T_1 + 1 - q,$$

which is regular and has the value $1 - q$ at the point $(x_1, x_2) = (1, 1)$.

Now let $p_\lambda = 0$, then $\delta_\lambda = 1$. The set $\mathcal{R}$ consists of a singular pair ξ. Write $\xi = (i, j)$ and suppose that $e(\xi) = -$; the case of $e(\xi) = +$ will be similar. By Lemmas 2.6 and 2.7, the product φ_0 written using the order $<$ on $\mathcal{P}$ has the form

$$\cdots \underline{\varphi^{k}_{i,j-1}\varphi^{k+1}_{i,j}} \cdots \varphi^{1}_{j-1,j} \cdots \tag{2.12}$$

where $k = j - i - 1$ and we underlined the factors corresponding to $\xi^- = (i, j-1)$ and ξ. Using Lemmas 2.2 and 2.3, modify the order of pairs $(i, j), \ldots, (j-1, j)$ and the order of the corresponding factors of the product φ_0 to write this product as

$$\cdots \varphi^{k}_{i,j-1}\varphi^{k+1}_{i,j}\varphi^{k}_{j-1,j} \cdots \tag{2.13}$$

where we also used Lemma 2.4. Restrict the three factors in (2.13) to $\mathcal{F}_\lambda$ and note that the last one becomes $1 + T_k$ upon restriction. Now use Lemma 2.5(b) and get, by evaluating at (1.12) the restriction to $\mathcal{F}_\lambda$ of the product of the three factors,

$$(T_k T_{k+1} - qT_{k+1} - q)(1 + T_k).$$

We can then take the factor $\varphi_{j-1,j}$ back to its original position and evaluate the restrictions to $\mathcal{F}_\lambda$ of other elementary factors. In this way we obtain

$$\varphi_0|_{\mathcal{F}_\lambda}\big(q^{c_1}, \ldots, q^{c_n}\big) = A(T_k T_{k+1} - qT_{k+1} - q)B,$$
$$A = \prod_{(k,l)<(i,j-1)} \varphi_{kl}\Big|_{\mathcal{F}_\lambda}\big(q^{c_1}, \ldots, q^{c_n}\big), \qquad B = \prod_{(k,l)>(i,j)} \varphi_{kl}\Big|_{\mathcal{F}_\lambda}\big(q^{c_1}, \ldots, q^{c_n}\big).$$

So the final effect of our procedure has been to make a "fusion" of the two adjacent elementary factors underlined in (2.12) into the three-term factor

$$F_{k,k+1} = T_k T_{k+1} - qT_{k+1} - q, \tag{2.14}$$

and to evaluate all other elementary factors at (1.12) straightforwardly.

Now suppose that the set $\mathcal{R}$ consists of a singular pair $\xi = (i, j)$ with $e(\xi) = +$; we still assume that $p_\lambda = 0$. By Lemmas 2.6 and 2.7, the product φ_0 has the form

$$\cdots \varphi^{1}_{i,i+1} \cdots \underline{\varphi^{k+1}_{ij}\varphi^{k}_{i+1,j}} \cdots \tag{2.15}$$

where we underlined the factors corresponding to ξ and $\xi^+ = (i+1, j)$. Here $k = j - i - 1$ again. Modify the order of pairs $(i, i+1), \ldots, (i, j)$ and the order of the corresponding factors of the product φ_0 to write this product as

$$\ldots \varphi^{k}_{i,i+1} \varphi^{k+1}_{ij} \varphi^{k}_{i+1,j} \cdots \tag{2.16}$$

where we used Lemma 2.4. Restrict the three factors in (2.16) to $\mathcal{F}_\lambda$ and note that the first one becomes $1+T_k$ upon restriction. Now use Lemma 2.5(a) and get, by evaluating at (1.12) the restriction to $\mathcal{F}_\lambda$ of the product of the three factors,

$$(1+T_k)(T_{k+1}T_k - qT_{k+1} - q).$$

Take the factor $\varphi_{i,i+1}$ back to its original position and evaluate the restrictions to $\mathcal{F}_\lambda$ of other elementary factors. In this way we obtain

$$\varphi_0|_{\mathcal{F}_\lambda}\big(q^{c_1},\ldots,q^{c_n}\big) = C(T_{k+1}T_k - qT_{k+1} - q)D,$$
$$C = \prod_{(k,l)<(i,j)} \varphi_{kl}\Big|_{\mathcal{F}_\lambda}\big(q^{c_1},\ldots,q^{c_n}\big), \qquad D = \prod_{(k,l)>(i+1,j)} \varphi_{kl}\Big|_{\mathcal{F}_\lambda}\big(q^{c_1},\ldots,q^{c_n}\big).$$

Here the final effect of our procedure has been to make a “fusion” of two adjacent elementary factors underlined in (2.15) to the three-term factor

$$T_{k+1}T_k - qT_{k+1} - q,$$

and to evaluate all other elementary factors at (1.12) straightly.

To deal with the general case, we first observe that the factor δ_λ takes care of the singularities coming from the singular pairs which are not in $\mathcal{R}$, as in the case $p_\lambda = 1$ above. Hence we have only to prove that we can perform the procedure used in the case $p_\lambda = 0$ in a coherent way for all singular pairs from the set $\mathcal{R}$. By Lemma 2.7 the pairs $\xi^{e(\xi)}$ are all different when ξ ranges over $\mathcal{R}$. Hence it suffices to prescribe the order on $\mathcal{R}$ in which the previous procedure should be performed. Once this order is given, one can just repeat the argument for each pair in $\xi \in \mathcal{R}$.

Take any pair $\xi \in \mathcal{R}$ and write $\xi = (i,j)$. There is a unique subset $\mathcal{P}_t \subset \mathcal{P}$ containing ξ. Suppose that $e(\xi) = -$. The key combinatorial fact is that then all $\beta \in \mathcal{P}$ with $\xi \leqslant \beta \leqslant (j-1,j)$ belong to the same subset $\mathcal{P}_t \subset \mathcal{P}$. Here we use the special order on $\mathcal{P}$. This key fact follows from the first claim of Lemma 1.3.

Write $\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_s$ and suppose that $e(\mathcal{P}_s) = \{-\}$. Then perform the previous procedure for all singular pairs in $\mathcal{P}_s$ starting from the first to the last; here we refer to our special order on $\mathcal{P}_s$. Each time we modify the order of the pairs following the singular pair which we are dealing with, and then restore the original order of these following pairs. In particular, each time we do not affect the singular pairs we dealt with previously.

Here we have $e(\mathcal{P}_{s-1}) = \{+\}$. Let us now deal with the singular pairs in $\mathcal{P}_{s-1}$ starting from the last to the first singular pair in $\mathcal{P}_{s-1}$, this procedure takes place in $\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_{s-1}$. After that we can deal with the singular pairs contained in $\mathcal{P}_{s-2}$ starting as above from the first to the last singular pair. By the key fact, the latter procedure can performed within the subset $\mathcal{P}_{s-2}$ so that the pairs from $\mathcal{P}_{s-1} \sqcup \mathcal{P}_s$ are not affected by this procedure. Now we can proceed inductively. An obvious modification of the previous argument works in the case $e(\mathcal{P}_s) = \{+\}$. □

Corollary 2.9. *The element $E_\lambda \in \mathcal{H}_n$ can be calculated as follows*:

(1) *arrange the elementary factors φ_β in* (2.5) *according to the special order*;
(2) *replace every two adjacent factors corresponding to $\beta \in \mathcal{R}$ by a three-term factor*;
(3) *replace each factor indexed by a singular pair $\beta \notin \mathcal{R}$ by the scalar $1-q$*;
(4) *evaluate the factors corresponding to the remaining non-singular pairs.*

For example, consider again the diagram $\lambda = ([1,2],[2,3],[2,3])$; see (1.8). Here

$$\begin{aligned} E_\lambda &= (\delta_\lambda \varphi_0)|_{\mathcal{F}_\lambda}\big(1,q,1,q,q^{-1},1\big) \\ &= (T_1+1)(1-q)(T_3+1)(T_1T_2 - qT_2 - q)(T_1+1)(T_4-q)\big(T_3 - q^2(q+1)^{-1}\big) \\ &\quad \times (T_5T_4 - qT_5 - q)(T_2-q)\big(T_1 - q^2(q+1)^{-1}\big)(T_3T_2 - qT_3 - q)(T_1+1). \end{aligned}$$

3. End of the proof of Theorem 1.5

Now we complete the proof of Theorem 1.5 by showing that $E_\lambda \neq 0$. If the Cherednik diagram λ has no parallel rows, then Corollary 2.9 immediately shows that the equality (0.3) holds for some coefficients $a_w \in \mathbb{C}(q)$. This equality implies that $E_\lambda \neq 0$. However, the equality (0.3) does not hold always. For instance, let us again consider the diagram $\lambda = ([1,2],[2,3],[2,3])$; see (1.8). The first two rows of this diagram are parallel. A direct computation gives

$$E_\lambda = q\bigl(q^2-1\bigr)T_1T_3T_4T_3T_5T_4T_2T_1T_3T_2T_1 + \sum_{\ell(w)<11} a_w T_w$$

while $\ell(w_0)=15$. We shall give a similar presentation of $E_\lambda \in \mathcal{H}_n$ for any $\lambda \in \mathcal{C}_n$; see (3.8). Like (0.3), this presentation immediately shows that $E_\lambda \neq 0$.

We start with giving another expression for the element E_λ when the diagram $\lambda \in \mathcal{C}_n$ consists of one row only. Then we will employ the notation E_n instead of E_λ. As usual, denote $[m]_q = (1-q^m)/(1-q)$ for each positive integer m.

Lemma 3.1. *We have*

$$E_n = \sum_{w\in\mathcal{S}_n} T_w, \tag{3.1}$$

$$E_n^2 = [1]_q \dots [n]_q E_n. \tag{3.2}$$

Proof. By definition,

$$E_n = \prod_{(i,j)\in\mathcal{P}} \left(T_{j-i} + \frac{1-q}{1-q^{j-i}} \right)$$

where the pairs $(i,j)\in\mathcal{P}$ are ordered lexicographically. Using Lemma 2.3, for any $k=1,\dots,n-1$ we can write the product E_n as $(T_k+1)E$ for some $E\in\mathcal{H}_n$. By (1.3), we then have $T_kE_n = qE_n$ for every $k=1,\dots,n-1$. By [18, Lemma 2.4], E_n equals the right-hand side of (3.1) multiplied by a scalar from $\mathbb{C}(q)$. This scalar is actually 1 because the coefficient at T_{w_0} is 1 for both sides of (3.1). In view of (3.1), the equality (3.2) is well known; see for instance [18] once again. □

Using the natural embedding $\mathcal{H}_m \to \mathcal{H}_n$, we will regard E_m as an element of $\mathcal{H}_n$ whenever $1 \leqslant m \leqslant n$. More generally, for any non-negative integer h such that $h+m \leqslant n$, we have an embedding $\iota_h : \mathcal{H}_m \to \mathcal{H}_n$ such that $\iota_h(T_k) = T_{k+h}$ for any $k=1,\dots,m-1$. Denote

$$E_m^{(h)} = \iota_h(E_m).$$

Lemma 3.2. *Suppose that the diagram $\lambda\in\mathcal{C}_n$ consists of r parallel rows of length m each. Then*

$$E_\lambda = f_m^{r(r-1)/2} E_m^{(0)} E_m^{(m)} \dots E_m^{(n-m)},$$

where

$$f_m = (-1)^m q^{m(m-1)/2}\bigl(q^m - 1\bigr).$$

Proof. If $r=1$, the lemma is tautological. Suppose that $r=2$, so that $n=2m$. Recall the notation (2.14). By Corollary 2.9,

$$E_\lambda = E_m(1-q)ZE_m$$

where Z stands for the product

$$\begin{aligned}
&\varphi_{m+1}\big(q^0,q^1\big)\varphi_{m+2}\big(q^0,q^2\big)\varphi_{m+3}\big(q^0,q^3\big)\varphi_{m+4}\big(q^0,q^4\big)\ldots\varphi_{2m-1}\big(q^0,q^{m-1}\big)\\
&\quad\times F_{m-1,m}\varphi_{m+1}\big(q^1,q^2\big)\varphi_{m+2}\big(q^1,q^3\big)\varphi_{m+3}\big(q^1,q^4\big)\ldots\varphi_{2m-2}\big(q^1,q^{m-1}\big)\\
&\quad\times \varphi_{m-2}\big(q^2,q^0\big)F_{m-1,m}\varphi_{m+1}\big(q^2,q^3\big)\varphi_{m+2}\big(q^2,q^4\big)\ldots\varphi_{2m-3}\big(q^2,q^{m-1}\big)\\
&\quad\times \varphi_{m-3}\big(q^3,q^0\big)\varphi_{m-2}\big(q^3,q^1\big)F_{m-1,m}\varphi_{m+1}\big(q^3,q^4\big)\ldots\varphi_{2m-4}\big(q^3,q^{m-1}\big)\\
&\quad\vdots\\
&\quad\times \varphi_2\big(q^{m-2},q^0\big)\ldots\varphi_{m-2}\big(q^{m-2},q^{m-4}\big)F_{m-1,m}\varphi_{m+1}\big(q^{m-2},q^{m-1}\big)\\
&\quad\times \varphi_1\big(q^{m-1},q^0\big)\ldots\varphi_{m-3}\big(q^{m-1},q^{m-4}\big)\varphi_{m-2}\big(q^{m-1},q^{m-3}\big)F_{m-1,m}.
\end{aligned}$$

Note that there are exactly m lines in the above display. If $m=1$ then $E_m=1$ and $Z=1$, so that $E_\lambda=(1-q)$ as required.

If $m\geqslant 2$, consider the second line in the display. The element E_m is divisible on the right by $T_{m-1}+1$, while the product in the first line of the display commutes with $T_{m-1}+1$. But $(T_{m-1}+1)T_{m-1}=q(T_{m-1}+1)$ and therefore

$$(T_{m-1}+1)F_{m-1,m}=-q(T_{m-1}+1).$$

Hence in the second line of the display, we can replace the factor $F_{m-1,m}$ by the scalar factor $-q$, without changing the value of the product E_mZ.

If $m\geqslant 3$, also consider the third line in the display. The element E_m is divisible on the right by $T_{m-2}+1$, while the product in the first line of the display commutes with $T_{m-2}+1$. After replacing $F_{m-1,m}$ by $-q$ in the second line, all factors in that line then commute with $T_{m-2}+1$. But

$$\begin{aligned}
(T_{m-2}+1)\varphi_{m-2}\big(q^2,q^0\big)&=(T_{m-2}+1)\left(T_{m-2}+\frac{1-q}{1-q^{-2}}\right)\\
&=(T_{m-2}+1)\left(q+\frac{1-q}{1-q^{-2}}\right)=\frac{q-q^2}{1-q^2}(T_{m-2}+1).
\end{aligned}$$

Hence in the third line of the display, we can now replace the factor $\varphi_{m-2}(q^2,q^0)$ by the scalar $(q-q^2)/(1-q^2)$, without changing the value of the product E_mZ. After this replacement, we apply the arguments like those already applied to the second line, and replace $F_{m-1,m}$ in the third line by $-q$, without changing E_mZ.

By continuing these arguments for all m lines of the display, we show that

$$E_\lambda=E_m(1-q)YE_m \tag{3.3}$$

where Y stands for the product

$$\begin{aligned}
&\varphi_{m+1}\big(q^0,q^1\big)\varphi_{m+2}\big(q^0,q^2\big)\varphi_{m+3}\big(q^0,q^3\big)\varphi_{m+4}\big(q^0,q^4\big)\ldots\varphi_{2m-1}\big(q^0,q^{m-1}\big)\\
&\quad\times(-q)\varphi_{m+1}\big(q^1,q^2\big)\varphi_{m+2}\big(q^1,q^3\big)\varphi_{m+3}\big(q^1,q^4\big)\ldots\varphi_{2m-2}\big(q^1,q^{m-1}\big)\\
&\quad\times\frac{q-q^2}{1-q^2}(-q)\varphi_{m+1}\big(q^2,q^3\big)\varphi_{m+2}\big(q^2,q^4\big)\ldots\varphi_{2m-3}\big(q^2,q^{m-1}\big)\\
&\quad\times\frac{q-q^3}{1-q^3}\frac{q-q^2}{1-q^2}(-q)\varphi_{m+1}\big(q^3,q^4\big)\ldots\varphi_{2m-4}\big(q^3,q^{m-1}\big)\\
&\quad\vdots\\
&\quad\times\frac{q-q^{m-2}}{1-q^{m-2}}\cdots\frac{q-q^2}{1-q^2}(-q)\varphi_{m+1}\big(q^{m-2},q^{m-1}\big)\\
&\quad\times\frac{q-q^{m-1}}{1-q^{m-1}}\cdots\frac{q-q^3}{1-q^3}\frac{q-q^2}{1-q^2}(-q).
\end{aligned}$$

By collecting the scalar factors here and by performing cancellations,

$$Y = (-1)^{m-1} q^{m(m-1)/2} [1]_q^{-1} \ldots [m-1]_q^{-1} E_m^{(m)}.$$

Since the elements E_m and $E_m^{(m)}$ of the algebra $\mathcal{H}_{2m}$ commute, (3.3) implies that

$$E_\lambda = (-1)^{m-1} q^{m(m-1)/2} [1]_q^{-1} \ldots [m-1]_q^{-1} E_m^2 (1-q) E_m^{(m)} = f_m E_m E_m^{(m)}$$

as required when $r = 2$. Here we used the relation (3.2) for m instead of n.

It remains to observe that for $r > 2$, Lemma 3.2 follows from the case $r = 2$. Consider the expression which Corollary 2.9 provides for E_λ. This is an ordered product of factors corresponding to all pairs $(i, j) \in \mathcal{P}$. According to step (1) of the corollary we use the special order on $\mathcal{P}$, so that the ordered set $\mathcal{P}$ is divided into the subsets (2.11). Then we also perform steps (2)–(4) which do not affect division into the subsets. The last two subsets in (2.11) are $\mathcal{P}_{r-1,r}$ and $\mathcal{P}_{rr}$. The product of the factors of E_λ corresponding to $\mathcal{P}_{rr}$ equals E_m. Further, consider the factors in our expression for E_λ that precede the factors corresponding to $\mathcal{P}_{r-1,r}$. By considering the subset $\mathcal{P}_{r-1,r-1}$ and by using Lemma 2.3, one can show that the product of these preceding factors is divisible by E_m on the right. By applying to the last two rows of λ the arguments used in the case $r = 2$, we can modify our expression for E_λ without changing the value of that expression. Namely, we replace the factors corresponding to $\mathcal{P}_{r-1,r}$ and $\mathcal{P}_{rr}$ by f_m and $E_m^{(m)}$ respectively.

Next we replace the factors corresponding to the subset $\mathcal{P}_{r-2,r}$ by f_m and change to $E_m^{(2m)}$ the element $E_m^{(m)}$ previously appeared on the right of our expression. By continuing this process, we replace by $f_m^{r-1} E_m^{(n-m)}$ all factors corresponding to $\mathcal{P}_{1r}, \ldots, \mathcal{P}_{rr}$ in the expression for E_λ, initially provided by Corollary 2.9. An induction on the number r of parallel rows completes the proof. $\square$

Let us now deal with an arbitrary Cherednik diagram λ. Take any $(i, j) \in \mathcal{P}$ and suppose that the numbers i and j occur in the rows k and l of Λ respectively. Here $k \leqslant l$. If $k = l$, denote by h the quantity of numbers in Λ occurring in those rows which precede the row k and are parallel to it. Denote

$$t_{ij} = \begin{cases} s_{j-i} & \text{if the rows } k \text{ and } l \text{ of } \lambda \text{ are not parallel;} \\ 1 & \text{if the rows } k \text{ and } l \text{ of } \lambda \text{ are parallel but } k < l; \\ s_{j-i+h} & \text{if } k = l. \end{cases} \tag{3.4}$$

Take the ordered product

$$w_\lambda = \prod_{(i,j) \in \mathcal{P}} t_{ij} \tag{3.5}$$

where we use our special order on the set $\mathcal{P}$. For the diagram $\lambda = ([1, 2], [2, 3], [2, 3])$ considered before,

$$w_\lambda = s_1 s_3 s_4 s_3 s_5 s_4 s_2 s_1 s_3 s_2 s_1.$$

Lemma 3.3. *For any $\lambda \in \mathcal{C}_n$ the decomposition* (3.5) *in $\mathcal{S}_n$ is reduced.*

Proof. Take $(i, j) \in \mathcal{P}$ and suppose that the numbers i and j occur in the rows k and l of Λ respectively. For the purpose of this proof, we will say that the pair (i, j) is of type I, II or III if the rows k and l satisfy the conditions in the first, second or third line of (3.4) respectively. A direct check of the conditions of Definition 2.1(a) shows that the set $\mathcal{L}$ of all pairs of types I, III is biconvex. Hence there are $w, w' \in \mathcal{S}_n$ such that $w_0 = ww'$ and $\ell(w_0) = \ell(w) + \ell(w')$ while $\mathcal{I}_w = \mathcal{L}$. This implies that by using only the relations $s_i s_j = s_j s_i$ for $|i - j| > 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leqslant i \leqslant n-1$, we can modify the reduced decomposition

$$w_0 = \prod_{(i,j) \in \mathcal{P}} s_{j-i} \tag{3.6}$$

to another one such the first $\ell(w)$ pairs associated with the new decomposition are exactly those of types I and III. Note that in (3.6) we used our special order on $\mathcal{P}$. There is a procedure which changes (3.6) to a reduced decomposition whose first $\ell(w)$ elements are exactly the simple transpositions from (3.5), taken in the same order. It implies at once that (3.5) is a reduced decomposition of $w_\lambda = w$.

To show how this procedure works, consider a particular case when the diagram λ consists of four rows, of which the second and third make a pair of parallel rows, while the first and the fourth are not parallel to any other rows. Our arguments will be easy to extend to an arbitrary diagram λ. In this case the set $\mathcal{P}$, endowed with the special order, is uniquely divided into three subsets $A < B < C$ where B consists of pairs of type II while each of the subsets A and C consist of pairs of types I, III. Let us write $w_0 = w_A w_B w_C$ accordingly. We have to move the factor w_B to the right. In our case $B = \mathcal{P}_{23}$ so that

$$w_B = \prod_{(i,j)\in\mathcal{P}_{23}} s_{j-i}.$$

Further divide C as $\mathcal{P}_{33} < D$, here $\mathcal{P}_{33}$ consists of pairs of type III. Accordingly,

$$w_C = \left(\prod_{(i,j)\in\mathcal{P}_{33}} s_{j-i}\right) w_D.$$

We have the following equalities of products of sequences of simple transpositions:

$$\begin{aligned} w_0 &= w_A w_B \left(\prod_{(i,j)\in\mathcal{P}_{33}} s_{j-i}\right) w_D \\ &= w_A \left(\prod_{(i,j)\in\mathcal{P}_{33}} s_{j-i+h}\right) w_B w_D \\ &= w_A \left(\prod_{(i,j)\in\mathcal{P}_{33}} s_{j-i+h}\right) w_D \left(\prod_{(i,j)\in\mathcal{P}_{23}} s_{j-i+m}\right) \end{aligned} \tag{3.7}$$

where $m = \lambda_4$ and $h = \lambda_2$ as required in (3.4). For the pairs (i, j) from the subsets A and D we have $t_{ij} = s_{j-i}$. Hence the simple transpositions in (3.7) multiplied over the subsets $A, \mathcal{P}_{33}, D$ make precisely the right-hand side of (3.5). $\square$

For each $m = 1, 2, \ldots$ denote by p_m the number of pairs of distinct parallel rows of λ of length m. In particular, if $\lambda \in \mathcal{C}_n$ consists of r parallel rows of length m each, then $p_m = r(r-1)/2$. We have $p_\lambda = p_1 + p_2 + \cdots$ in general.

Proposition 3.4. *For some coefficients $a_w \in \mathbb{C}(q)$, we have the equality in $\mathcal{H}_n$,*

$$E_\lambda = a T_{w_\lambda} + \sum_{\ell(w)<\ell(w_\lambda)} a_w T_w \tag{3.8}$$

where

$$a = \prod_{m\geqslant 1} f_m^{p_m}.$$

Proof. Take any maximal subsequence of parallel rows of λ. In particular, there is no row before or after these rows, parallel to them. By Lemma 1.3, there is no row of λ occuring in between of these parallel rows. Suppose that this subsequence consists of more than one row. Let m the length of any row in this subsequence.

Consider the expression which Corollary 2.9 initially provides for E_λ. By using Lemma 3.2 we can modify this expression, without affecting the value of E_λ. Let k and l be any two rows from our sequence of parallel rows. In the case $k < l$ we replace by the scalar factor f_m the product of all factors in our expression for E_λ corresponding to the pairs $(i, j) \in \mathcal{P}_{kl}$. Note that $t_{ij} = 1$ for these pairs, see (3.4).

In the case $k = l$ the factor corresponding to $(i, j) \in \mathcal{P}_{kk}$ in our initial expression for E_λ is $\varphi_{ij}^{j-i}(q^{c_i}, q^{c_j})$. We replace this factor by $\varphi_{ij}^{j-i+h}(q^{c_i}, q^{c_j})$ where h is the quantity of numbers in Λ occurring in the rows which precede the row k and are parallel to it. Here we use the maximality of our sequence of parallel rows. In this case $t_{ij} = s_{j-i+h}$, see again the definition (3.4). Using the so modified expression for E_λ along with Lemma 3.3, we get Proposition 3.4. □

Here is a corollary to our proof of Proposition 3.4; it refines Corollary 2.9.

Corollary 3.5. *The element $E_\lambda \in \mathcal{H}_n$ can be calculated as follows*:

(1) *arrange the elementary factors in (2.5) according to the special order*;
(2) *for every row of Λ of length $m = 1, 2, \ldots$ with $h = 1, 2, \ldots$ entries in the rows preceding and parallel to that row, replace by $E_m^{(h)}$ the product E_m of factors corresponding to all pairs (i, j) where both i and j are in that row*;
(3) *for every two distinct parallel rows of λ of length $m = 1, 2, \ldots$ replace by the scalar factor f_m the product of factors corresponding to all (i, j) where i and j are respectively in the first and second of the two parallel rows*;
(4) *replace the two adjacent factors corresponding to any remaining singular pair (i, j) by a three-term factor*;
(5) *evaluate the factors corresponding to all remaining non-singular pairs.*

For example, consider once again the diagram $\lambda = ([1, 2], [2, 3], [2, 3])$. Here

$$E_\lambda = q\big(q^2 - 1\big)(T_1 + 1)(T_3 + 1)(T_4 - q)\big(T_3 - q^2(q + 1)^{-1}\big)(T_5T_4 - qT_5 - q)(T_2 - q) \times \big(T_1 - q^2(q + 1)^{-1}\big)(T_3T_2 - qT_3 - q)(T_1 + 1).$$

Acknowledgments

We are grateful to Ivan Cherednik for illuminating conversations, and to the anonymous referee for helpful remarks. We are also grateful to George Lusztig and to Nanhua Xi for drawing our attention to their works [16] and [27,28]. Our work has been supported by the EPSRC and by the London Mathematical Society.

References

[1] T. Akasaka, M. Kashiwara, Finite-dimensional representations of quantum affine algebras, Publ. Res. Inst. Math. Sci. 33 (1997) 839–867.
[2] I. Bernstein, A. Zelevinsky, Induced representations of reductive p-adic groups I, Ann. Sci. École Norm. Sup. 10 (1977) 441–472.
[3] V. Chari, A. Pressley, Quantum affine algebras and affine Hecke algebras, Pacific J. Math. 174 (1996) 295–326.
[4] I. Cherednik, On special bases of irreducible finite-dimensional representations of the degenerate affine Hecke algebra, Funct. Anal. Appl. 20 (1986) 87–89.
[5] I. Cherednik, On R-matrix quantization of formal loop groups, in: Group Theoretical Methods in Physics, VNU Publ., 1986, pp. 218–232.
[6] I. Cherednik, A new interpretation of Gelfand–Tzetlin bases, Duke Math. J. 54 (1987) 563–577.
[7] I. Cherednik, A unification of Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras, Invent. Math. 106 (1991) 411–431.
[8] V. Drinfeld, Degenerate affine Hecke algebras and Yangians, Funct. Anal. Appl. 20 (1986) 56–58.
[9] M. Dyer, Hecke algebras and shellings of Bruhat intervals, Compos. Math. 89 (1993) 91–115.
[10] V. Guizzi, P. Papi, A combinatorial approach to the fusion process for the symmetric group, European J. Combin. 19 (1998) 835–845.
[11] A. Gyoja, K. Uno, On the semisimplicity of Hecke algebras, J. Math. Soc. Japan 41 (1989) 75–79.
[12] R. Howe, Hecke algebras and p-adic GL_n, Contemp. Math. 177 (1994) 65–100.
[13] A. Jones, M. Nazarov, Affine Sergeev algebra and q-analogues of Young symmetrizers for the projective representations of the symmetric group, Proc. London Math. Soc. 78 (1999) 481–512.

[14] S. Kato, Irreducibility of principal series representations for Hecke algebras of affine type, J. Fac. Sci. Univ. Tokyo Sect. Math. 28 (1981) 929–943.
[15] B. Leclerc, M. Nazarov, J.-Y. Thibon, Induced representations of affine Hecke algebras and canonical bases of quantum groups, Progr. Math. 210 (2003) 115–153.
[16] G. Lusztig, Cells in affine Weyl groups III, J. Fac. Sci. Univ. Tokyo Sect. Math. 34 (1987) 223–243.
[17] G. Lusztig, Affine Hecke algebras and their graded version, J. Amer. Math. Soc. 2 (1989) 599–685.
[18] G. Murphy, The representations of Hecke algebras of type A_n, J. Algebra 173 (1989) 97–121.
[19] M. Nazarov, Projective representations of the symmetric group, Adv. Math. 127 (1997) 190–257.
[20] M. Nazarov, A mixed hook-length formula for affine Hecke algebras, European J. Combin. 25 (2004) 1345–1376.
[21] M. Nazarov, V. Tarasov, On irreducibility of tensor products of Yangian modules, Int. Math. Res. Not. (1998) 125–150.
[22] P. Papi, A characterization of a special ordering in a root system, Proc. Amer. Math. Soc. 120 (1994) 661–665.
[23] A. Ram, Affine Hecke algebras and generalized Young tableaux, J. Algebra 260 (2003) 367–415.
[24] J. Rogawski, On modules over the Hecke algebras of a p-adic group, Invent. Math. 79 (1985) 443–465.
[25] V. Tolstoy, Extremal projectors for reductive classical Lie superalgebras with nondegenerate generalized Killing form, Russian Math. Surveys 40 (1985) 241–242.
[26] M. Vazirani, Parameterizing Hecke algebra modules: Bernstein–Zelevinsky multisegments, Kleshchev multipartitions, and crystal graphs, Transform. Groups 7 (2002) 267–303.
[27] N. Xi, The based ring of two-sided cells of affine Weyl groups of type $\widetilde{A}_{n-1}$, Mem. Amer. Math. Soc. 157 (2002) 1–95.
[28] N. Xi, Representations of affine Hecke algebras and based rings of affine Weyl groups, J. Amer. Math. Soc. 20 (2007) 211–217.
[29] A. Zelevinsky, Induced representation of reductive p-adic groups II. On irreducble representations of $GL(n)$, Ann. Sci. École Norm. Sup. 13 (1980) 165–210.