Computation of $K_2 \mathbb{Z}[\sqrt{-6}]$

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Abstract

We show that $K_2 \mathbb{Z}[\sqrt{-6}]$ is trivial (order one). The method used can also be applied to other imaginary quadratic fields.

1. Introduction

Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field, and let $O_F$ denote its ring of integers. Tate [7] developed a method by which he showed that $K_2 O_F$ is trivial for $d = -1, -2, -3, -11,$ and $K_2 O_F \cong \mathbb{Z}/2\mathbb{Z}$ for $d = -7, -15$. We know that M. Skalba has shown that $K_2 O_F$ is also trivial for $d = 5, -19$ (see [3]). In this paper, we propose a method by which we can determine the structure of $K_2 O_F$ for an imaginary quadratic field $F$. We show that $\#(K_2 O_F) = 1$ for $d = -6$.

2. Notations and some facts

Let $F$ be a number field, $O_F$ be the ring of integers of $F$ and $S_F$ be the set of Archimedean places of $F$. We know that a finite place can be identified with a discrete valuation $v$ of $F$. If $S \supset S_F$ then $S$ is a non-empty set of places. We put

$$O_S = \{a \in F | v(a) \geq 0 \text{ for all } v \notin S\}$$

and call $O_F$ the ring of "$S$-integers". For any $v \notin S$, $k(v) = O_S/P$, where $P$ is the maximal ideal corresponding to the place $v$. As in [1], we put

$$K_2^S F = \text{the subgroup of } K_2 F \text{ generated by } \{x, y\}, \text{ where } x, y \in O_S = U.$$

We list the finite places of $F$

$v_1, v_2, \ldots, v_n, \ldots$
so that $N(r_i) \leq N(r_{i+1})$ for all $i$, where $N(r)$ is the norm of the finite place $r$ which is defined to be $\#(k(r))$. Let $S_m = \{r_1, r_2, \ldots, r_m\} \cup S_r$. Bass and Tate [1] show that there exists a positive integer $m$ such that

$$K_2 O_F = \text{Ker} \left( K_2^{\pm} F \xrightarrow{\tau_i} \bigcup_{r \in S_r} k'(r) \right)$$

For any $\{x, y\} \in K_2^{\pm} F$ and any $r \in S_m$,

$$\tau_r \{x, y\} = (\pm 1)^{\nu(x, y)} \frac{y^{\nu(x)} x^{\nu(y)}}{y^{\nu(y)} x^{\nu(x)}} \pmod{P}.$$ 

Therefore we aim at seeking for a positive integer $m$ (not too large) such that $\tau_m$ is bijective if $m > m$.

Suppose that the ideal $P$ (corresponding to $r$) is principal, say $P = \pi O_S$. Let $\beta$ be the map from $U$ to $k'$ defined by $\beta(u) = u \pmod{\pi}$. Denote by $U_1$ the subgroup of $U$ generated by $1 + \pi U \cap U$. Tate [7] gives the following result.

Lemma 2.1. Suppose that $W, C$ and $G$ are subsets of $U$ such that

1. $W \subset C U_1$ and $W$ generates $U$.
2. $CG \subset C U_1$ and $\beta(G)$ generates $k'$.
3. $1 \in C \cap \ker \beta \subset U_1$.

Then $\tau_r$ is bijective. $\Box$.

The following two lemmas are also useful to our computation.

Lemma 2.2 [1, Chapter II, Lemma 3.4]. Suppose we are given subsets $D \subset O_F$ and $W \subset O_F \cap U$. Put

$$E = \{d - d' | d, d' \in D, d \neq d'\}.$$

Then $\tau_r$ is bijective provided $D$ and $W$ satisfy the following conditions:

1. $\#(D)^2 > N(r)^2$.
2. $E \subset U$.
3. $1 \in W$ and $W$ generates $U$.
4. If $e_1, e_2, e_3, e_4 \in E$ and $w \in W$ then
   i. $N(e_1, e_2 - e_3 e_4) < N(r)^2$.
   ii. $N(e_1 w - e_2) < N(r)^2$.

Together with that the ideal $P$ (corresponding to $r$) is principal. $\Box$

Lemma 2.3 [7, Lemma M1]. Let $F$ be an imaginary quadratic field. Let $M$ be an ideal in $O_F$, the prime factorization of which involves only primes in $S$. Suppose $a, b \in U \cap M$ and $|a| + |b| < N \sqrt{N M}$. If $\beta(a) = \beta(b)$, then $a \in b U_1$. Especially, if $a, b \in U \cap O_F$, $|a| + |b| < N \sqrt{N}$ and $\beta(a) = \beta(b)$, then $a \in b U_1$. $\Box$
Remark. Lemma 3.4 in Chapter II in [1] has another condition, that is, \( N(c_1 + c_2 + c_3) < N(r) \). We do not need this one because we only discuss \( K_2 \). Now we turn to computation of \( K_2[\sqrt{6}] \).

3. Case 1: \( N_r > 293 \)

From now on, \( F = Q(\sqrt{6}) \), we know that \( h(F) = 2 \) and 2 is ramified in \( O_F = Z[\sqrt{6}] \). Suppose \( Q^2 = 2O_F \), then \( Q = 2O_F + \sqrt{6}O_F = 2Z + \sqrt{6}Z \).

By a discussion similar to that in [7], we can easily show the following.

Lemma 3.1. Let \( W = \{ u \in O_F \cap U \ | \ |u|^2 \leq 2Nv \} \); then \( W \) generates \( U \).  

Lemma 3.2. Pick \( d \) such that \( d^2 = Nr/9 \). Let \( D = \{ x \in O_F \ | \ |x| \leq d \} \) and \( E = \{ d - d' \ | \ d, d' \in D, d \neq d' \} \). Then \( E \) satisfies (2) and (4) of Lemma 2.2, if \( Nr \geq 137 \).

Proof. First, we have \( E \subseteq U \). In other words, (2) of Lemma 2.2 is satisfied. In fact, for any \( e \in E \), there exist \( d, d' \in D \) such that \( e = d - d' \). Hence \( N(e) \leq (|d| + |d'|)^2 \leq 4/9Nr < Nr \), so \( e \in U \). On the other hand, for \( e_1, e_2, e_3, e_4 \in E, w \in W \), \( N(e_1e_2 - e_3e_4) \leq (|e_1e_2| + |e_3e_4|)^2 \leq (4/9Nr + 4/9Nr)^2 = (8/9Nr)^2 < Nr^2 \). \( N(e_1w - e_2) \leq (|e_1w| + |e_2|)^2 \leq 4/9Nr(|w| + 1)^2 \leq 4/9Nr(\sqrt{2Nr} + 1)^2 \). If \( Nr > 136 \), then \((\sqrt{2Nr} + 1)^2 < 9/4Nr \). Therefore, \( N(e_1w - e_2) < Nr^2 \). Thus (4) of Lemma 2.2 is also satisfied. □

Lemma 3.3. Suppose that \( D = \{ x \in O_F \ | \ |x| \leq d \} \), then

\[
\#(D) = 1 + 2[d] + 2 \left[ \frac{d}{\sqrt{6}} \right] + 4 \left( \left[ \sqrt{d^2 - 6} \right]^2 + \left[ \sqrt{d^2 - 6} \right]^2 + \ldots \right),
\]

where \( [x] \) denotes the greatest integer \( \leq x \).

Proof. The rational integers of \( D \) arc: \( 0, 1, 2, \ldots, \lfloor [d] \rfloor \).

The elements of \( D \) having the form \( \pm \sqrt{6} \) are: \( \pm \sqrt{6} \).

The elements of \( D \) having the form \( \pm 2\sqrt{6}, \ldots, \pm \lfloor [d\sqrt{6}] \rfloor \sqrt{6} \) are: \( \pm 2\sqrt{6}, \ldots, \pm \lfloor [d\sqrt{6}] \rfloor \sqrt{6} \).

Obviously, by this process we can get all elements of \( D \). □

Lemma 3.4. If \( Nr > 900 \), then \( \tau, \) is bijective.
Proof. Take $d^2 = Nv/9$; by Lemma 3.2, we only need to prove that $(#(D))^3 > Nv^2$ if $Nv > 900$.

By Lemma 3.3,

$$\#(D) = 1 + 2[d] + \frac{d}{\sqrt{6}} + 4\left(\sqrt{d^2 - 6.1^2} + \cdots + \sqrt{d^2 - 6\left(\frac{d}{\sqrt{6}}\right)^2}\right).$$

Therefore,

$$\#(D) > 1 + 2[d] + 2\left(\frac{d}{\sqrt{6}}\right) - 4\left(\frac{d}{\sqrt{6}}\right)$$

$$+ 4\left(\sqrt{d^2 - 6.1^2} + \cdots + \sqrt{d^2 - 6\left(\frac{d}{\sqrt{6}}\right)^2}\right)$$

$$= 1 + 2[d] - 2\left(\frac{d}{\sqrt{6}}\right) + 4\left(\sqrt{d^2 - 6.1^2} + \cdots + \sqrt{d^2 - 6\left(\frac{d}{\sqrt{6}}\right)^2}\right).$$

Let $f(x) = \sqrt{d^2 - 6x^2}$. Then $f(x)$ is a strictly decreasing function when $x \in [0, d/6]$.

$$\sum_{n=1}^{[d/6]} \sqrt{d^2 - 6x^2} > \int_1^{d/6} f(x) \, dx$$

$$= \left(\frac{\sqrt{6}x}{2} \sqrt{\frac{d^2}{6} - x^2} + \frac{d^2}{2\sqrt{6}} \arcsin\frac{\sqrt{6}x}{d}\right)^{d/6}$$

$$= \frac{d^2}{2\sqrt{6}} \arcsin 1 - \frac{1}{2} \sqrt{d^2 - 6} - \frac{d^2}{2\sqrt{6}} \arcsin\frac{\sqrt{6}}{d}.$$

We know that if $|x| \leq 1$ then

$$\arcsin x = x + \sum_{n=1}^{+\infty} \frac{(2n-1)!!}{2n} \frac{x^{2n+1}}{(2n)!!}.$$

Hence, if $\sqrt{6}/d < 1$ then

$$\frac{4}{2\sqrt{6}} \frac{d^2}{2\sqrt{6}} = \frac{2d^2}{\sqrt{6}} \left(\left(\frac{\sqrt{6}}{d}\right)^3 + \frac{3 \cdot 1}{4 \cdot 2} \left(\frac{\sqrt{6}}{d}\right)^5 + \cdots\right)$$

$$= 2d + \frac{2}{\sqrt{6}} d^2 \left(\frac{1}{6} \left(\frac{\sqrt{6}}{d}\right)^3 + \frac{3 \cdot 1}{4 \cdot 2} \left(\frac{\sqrt{6}}{d}\right)^5 + \cdots\right).$$
When $d > 1 + \sqrt{7}$, $d - 6/d > 2$, hence $2/(d - 6/d) < 1$.

Therefore,

$$\#(D) > 1 + 2[d] - 2\left[\frac{\sqrt{d}}{6}\right] + 4 \cdot \sum_{n=1}^{[\sqrt{d}]\wedge 1} \sqrt{d^2 - 6n^2}$$

$$> 1 + 2d - 2 - \frac{2d}{\sqrt{6}} + \frac{2d^2}{\sqrt{6}} \cdot \frac{\pi}{2} - 2d - 2d - \alpha$$

$$> \frac{\pi}{\sqrt{6}} d^2 - 2 \left(\frac{1 + \sqrt{6}}{\sqrt{6}}\right) d - 2 \quad (0 < \alpha < 1).$$

Write $g(x) = (\pi/\sqrt{6}) x^2 - 2((1 + \sqrt{6})/\sqrt{6}) x - 2 - (3x)^{4/3}$. Then $g'(x) > 0$ when $x = 5$, $g''(x) > 0$ when $x = 0.65$. It follows that $g(x)$ is a strictly increasing function when $x \geq 5$. On the other hand, $f(10) > 0$, so, if $x \geq 10$, then $g(x) > 0$. Hence, if $d > 10$, equivalently, $Nv > 900$, then $\#(D) > (3d)^{4/3}$, in other words, $(\#(D))^2 > (9d^2)^2 = Nv^2$.

**Lemma 3.5.** Let $D = \{x \in O_\ell | |x| \leq d\}$, and let $E$ and $W$ be as in Lemma 2.2. If $d^2 < Nv^2/(4(1 + \sqrt{2Nv})^2)$ then for $e_1, e_2, e_3, e_4 \in E$ and $w \in W$, $N(e_1 e_2 - e_3 e_4) < Nv^2$, $N(e_1 w - e_2) < Nv^2$.

**Proof.** For any $e \in E$, $e = d' - d''$, where $d', d'' \in D$, $|e|^2 \leq (|d'| + |d''|)^2 \leq (2|d|)^2 = 4d^2 < Nv^2/(1 + \sqrt{2Nv})^2$.

$$N(e_1 e_2 - e_3 e_4) = |e_1 e_2 - e_3 e_4|^2 \leq (|e_1 e_2| + |e_3 e_4|)^2$$

$$< \frac{4Nv^4}{(1 + \sqrt{2Nv})^4} < \frac{4Nv^4}{(\sqrt{2Nv})^4} = \frac{4Nv^4}{4Nv^3} = Nv^2.$$
By Lemma 3.1, for any \( w \in W \), \( |w| \leq \sqrt{2N_v} \), hence,

\[
\begin{align*}
N(e_1w - e_2) &= |e_1w - e_2|^2 \\
&\leq (|e_1w| + |e_2|)^2 \\
&\leq (\max(|w| + 1))^2 \\
&< \frac{N_v^2}{(1 + \sqrt{2N_v})^2} (\sqrt{2N_v + 1})^2 = N_v^2. \quad \Box
\end{align*}
\]

Lemma 3.6. When \( 293 < N_v < 1045 \), \( \tau_v \) is bijective.

**Proof.** The smallest \( N_v \) satisfying \( N_v > 293 \) is \( N_v = 313 \). Choose \( d = 6 \), then \( d^2 = 36 < 313^2 / (4(1 + \sqrt{2 \cdot 313})^2) \). In this time, \( \#(D) = 49, 49^{3.2} = 343 > 313 \).

347 is the smallest of all primes which are larger than 343. Choose \( d^2 = 40 \), then \( \#(D) = 53, 53^{3.2} > 385. \)

Choose \( d^2 = 44 \), then \( \#(D) = 57, 57^{3.2} > 430. \)

Choose \( d^2 = 50 \), then \( \#(D) = 63, 63^{3.2} > 500. \)

Choose \( d^2 = 58 \), then \( \#(D) = 77, 77^{3.2} > 675. \)

Choose \( d^2 = 79 \), then \( \#(D) = 103, 103^{3.2} > 1045. \)

By Lemma 2.2, \( \tau_v \) is bijective if \( 293 < N_v < 1045. \) \( \Box \)

4. Case 2: \( 11 \leq N_v \leq 293 \)

**Lemma 4.1.** Assume \( Q^2 = 2O_F \). If we view \( Q \) as a lattice in \( C \), then the maximum distance from \( Q \) to \( C \) is \( \sqrt{10}/2. \)

**Lemma 4.2.** Suppose \( M \) is a non-principal ideal, then every residue class \( (\text{mod } M) \) can be represented by an element \( c \in O_F \) with \( Nc \leq (5/4) NM \).

**Lemma 4.3.** Let \( (b) \) be a principal ideal prime to \( Q \), then every residual class \( (\text{mod } (b)) \) can be represented by an element \( c \in Q \) with \( Nc \leq (5/2)Nb \).

The proofs of the above three lemma are analogous to similar results in [7].

**Lemma 4.4.** Suppose that \( P \) (corresponding to \( v \)) is a non-principal prime ideal with \( 11 \leq N_v \leq 293 \), then \( \tau_v \) is bijective.

**Proof.** Apply Lemma 2.1. We construct \( C, G \) and \( W \) for each \( P \) (corresponding to \( v \) with \( 11 \leq N_v \leq 293 \)).

Let \( C' = \{ c \in O_F | |c|^2 \leq (5/4)Nv \} \), \( W = \{ w \in O_F \cap U | |w|^2 \leq 2Nv \} \). Put \( T = \{ t_1, \ldots, t_r | t_i \in C', \ t_i \notin U, \ 1 \leq i \leq r \} \), \( S = \{ s_1, \ldots, s_r | s_i \equiv t_i \ (\text{mod } P), \ s_i \in U, \ 1 \leq i \leq r \} \). Let \( C = (C' \setminus T) \cup S, \ m = \max_{c \in C} |c| \). By Lemma 2.3 we know that

(1) if \( \sqrt{2N_v} + m < N_v \), then \( W \subset CU_1 \),

(2) if \( G = \{ g \} \) and \( m|g| + m = m(|g| + 1) < N_v \), then \( CG \subset CU_1 \),
Suppose that $PP = p = Nr$, where $P$ is an ideal in $O_r$ and $p$ is a prime in $Z$.

If we construct $C$, $G$ and $W$ for $P$, then we need not to do so for $\tilde{P}$ because $C$, $G$, $W$ will suit $\tilde{P}$.

Now we give $T$, $S$, $G = \{g\}$ and $m$ for every non-principal prime ideal $P$ (corresponding to $r$ with $11 \leq Nr \leq 293$). Direct computations show that in each case conditions (1), (2) and (3) are satisfied.

egin{align*}
Nr = 11, & \quad P = (11, 4 + \sqrt{-6}), \\
T = S = \emptyset, & \quad g = 2, \quad m \leq \frac{1}{3} \sqrt{55}. \\
Nr = 29, & \quad P = (29, 2 + 3\sqrt{-6}), \\
T = \{3 \pm \sqrt{-6}, 5 - \sqrt{-6}\}, & \quad S = \{3 - 2\sqrt{-6}, -4\}, \\
g = 2, & \quad m \leq \frac{1}{3} \sqrt{143}. \\
Nr = 53, & \quad P = (53, 10 + \sqrt{-6}), \\
T = S = \emptyset, & \quad g = 2, \quad m \leq \frac{1}{3} \sqrt{265}. \\
Nr = 59, & \quad P = (59, 8 + 3\sqrt{-6}), \\
T = \{7 + 2\sqrt{-6}, 7 - 2\sqrt{-6}\}, & \quad S = \{1 - \sqrt{-6}, -2 + 2\sqrt{-6}\}, \\
g = 2, & \quad m \leq \frac{1}{3} \sqrt{295}. \\
Nr = 83, & \quad P = (83, 4 + 5\sqrt{-6}), \\
T = \{1 + 4\sqrt{-6}, 1 - 4\sqrt{-6}, 7 + 3\sqrt{-6}, 7 - 3\sqrt{-6}\}, \\
S = \{3 - \sqrt{-6}, 5 - \sqrt{-6}, 7 - 2\sqrt{-6}, 7 + 2\sqrt{-6}\}, \\
g = 2, & \quad m = \sqrt{145}. \\
Nr = 101, & \quad P = (101, 14 + \sqrt{-6}), \\
T = \{7 + 3\sqrt{-6}, 7 - 3\sqrt{-6}\}, & \quad S = \{-7 + 2\sqrt{-6}, -7 - 4\sqrt{-6}\}, \\
g = 2, & \quad m = \sqrt{145}. \\
Nr = 107, & \quad P = (107, 8 + 5\sqrt{-6}), \\
T = \{11 + \sqrt{-6}, 11 - \sqrt{-6}\}, & \quad S = \{3 - 4\sqrt{-6}, 3 - 6\sqrt{-6}\}, \\
g = 2, & \quad m = \sqrt{225}.
\end{align*}
\( Nv = 131, \quad P = (131, 16 + \sqrt{-6}), \)
\( T = \{ 1 + 5\sqrt{-6}, 1 - 5\sqrt{-6} \}, \quad S = \{ -15 + 4\sqrt{-6}, 17 - 4\sqrt{-6} \}, \)
\( g = 2, \quad m = \sqrt{385}. \)

\( Nv = 149, \quad P = (149, 2 + 7\sqrt{-6}), \)
\( T = \{ 1 + 5\sqrt{-6}, 1 - 5\sqrt{-6} \}, \quad S = \{ -1 - 2\sqrt{-6}, 3 + 2\sqrt{-6} \}, \)
\( g = 2, \quad m \leq \frac{1}{2}\sqrt{745}. \)

\( Nv = 173, \quad P = (173, 14 + 5\sqrt{-6}), \)
\( T = \{ 13 + 2\sqrt{-6}, 13 - 2\sqrt{-6}, 7 + 5\sqrt{-6}, 7 - 5\sqrt{-6} \}, \)
\( S = \{ -1 - 3\sqrt{-6}, -2 + 5\sqrt{-6}, -7 - 8 + 2\sqrt{-6} \}, \)
\( g = 2, \quad m \leq \frac{1}{2}\sqrt{865}. \)

\( Nv = 179, \quad P = (179, 8 + 7\sqrt{-6}), \)
\( T = \{ 13 + 2\sqrt{-6}, 13 - 2\sqrt{-6}, 7 + 5\sqrt{-6}, 7 - 5\sqrt{-6} \}, \)
\( S = \{ -1 - 3\sqrt{-6}, -2 + 5\sqrt{-6}, -7 - 8 + 2\sqrt{-6} \}, \)
\( g = 2, \quad m \leq \frac{1}{2}\sqrt{895}. \)

\( Nv = 227, \quad P = (227, 20 + 3\sqrt{-6}), \)
\( T = \{ 5 + 6\sqrt{-6}, 5 - 6\sqrt{-6}, 11 + 5\sqrt{-6}, 11 - 5\sqrt{-6} \}, \)
\( S = \{ -15 + 3\sqrt{-6}, -4 + 4\sqrt{-6}, -9 + 2\sqrt{-6}, -5\sqrt{-6} \}, \)
\( g = 2, \quad m \leq \frac{1}{2}\sqrt{1135}. \)

\( Nv = 251, \quad P = (251, 4 + 9\sqrt{-6}), \)
\( T = \{ 11 + 5\sqrt{-6}, 11 - 5\sqrt{-6}, 17 + 2\sqrt{-6}, 17 - 2\sqrt{-6} \}, \)
\( S = \{ 7 - 4\sqrt{-6}, -16 - 3\sqrt{-6}, -10 + 4\sqrt{-6}, -10 \}, \)
\( g = 6, \quad m \leq \frac{1}{2}\sqrt{1255}. \)

\( Nv = 269, \quad P = (269, 22 + 3\sqrt{-6}), \)
\( T = \{ 11 + 5\sqrt{-6}, 11 - 5\sqrt{-6}, 17 + 2\sqrt{-6}, 17 - 2\sqrt{-6} \}, \)
Now we turn to dealing with principal prime ideals. Write \( C' = \{ z \in \mathbb{Q} \mid Q^3 = 2O \}, \)
\[ |z| \leq (5/2)Nc, \]
\[ T = \{ t_1, \ldots, t_r \in C' \mid t_i \notin U, \quad 1 \leq i \leq r \}, \]
\[ S = \{ s_1, \ldots, s_r \in \mathbb{Q} \subseteq C' \mid s_i \in U, \quad s_i \equiv t_i \pmod{P}, \quad 1 \leq i \leq r \}. \]
Let \( C = \{ 1 \} \cup (C' \setminus T) \cup S, \) \( m = \max_{z \in C} |z|, \)
\( W = \{ w \in O_F \cap U \mid |w|^2 \leq 2Nc \}. \)

Applying Lemma 2.3, we conclude that

1. if \( |g| + 1 < Nc \) then \( WC \subseteq C_U, \)
2. suppose that \( G = \{ g \}, \) if \( (|g| + 1)m < \sqrt{2Nc} \) and \( |g| + m < Nc \) then \( CG \subseteq C_U, \)
3. if \( m + 1 < Nc \) then \( 1 \in C \cap \ker \beta \subseteq C_U. \)

It is easy to see that (2) implies (3). \( \square \)

**Lemma 4.5.** If \( P \) (corresponding to \( v \)) is a principal prime ideal with \( 11 \leq Nc \leq 293, \) then \( \tau_v \) is bijective.

**Proof.** As in the proof of Lemma 4.4, we give \( S, T \) and \( G = \{ g \} \) for each \( P. \) Of course, we only do with one prime ideal for each \( Nc. \) In all cases below conditions (1) and (2) are satisfied.

\[ Nc = 31, \quad P = (5 + \sqrt{-6}). \]
\[ T = S = 0, \]
\[ g = 3, \quad m \leq \sqrt{135}. \]

\[ Nc = 73, \quad P = (7 + 2\sqrt{-6}). \]
\[ T = \{ 4 + 5\sqrt{-6}, 4 - 5\sqrt{-6} \}, \quad S = \{ -10 + \sqrt{-6}, -8 + \sqrt{-6} \}. \]
\[ g = 5, \quad m \leq \sqrt{385}. \]

\[ Nc = 79, \quad P = (5 + 3\sqrt{-6}). \]
\[ T = \{ 4 + 5\sqrt{-6}, 4 - 5\sqrt{-6} \}, \quad S = \{ -6 - \sqrt{-6}, -14 \}. \]
\[ g = 3, \quad m \leq \sqrt{\frac{395}{2}}. \]

\[ N_r = 97, \quad P = (1 + 4\sqrt{-6}). \]
\[ T = \left\{ 14 + \sqrt{-6}, 14 - \sqrt{-6}, 8 + 5\sqrt{-6}, 8 - 5\sqrt{-6} \right\}. \]
\[ S = \left\{ 2 + 2\sqrt{-6}, 2.6 + \sqrt{-6}, 10 - \sqrt{-6} \right\}. \]
\[ g = 5, \quad m \leq \sqrt{\frac{485}{2}}. \]

\[ N_r = 103, \quad P = (7 + 3\sqrt{-6}). \]
\[ T = \left\{ 8 + 5\sqrt{-6}, 8 - 5\sqrt{-6} \right\}. \]
\[ S = \left\{ 6 - \sqrt{-6}, 10 + 2\sqrt{-6} \right\}. \]
\[ g = 5, \quad m \leq \sqrt{\frac{515}{2}}. \]

\[ N_r = 127, \quad P = (11 + \sqrt{-6}). \]
\[ T = \left\{ 16 + \sqrt{-6}, 16 - \sqrt{-6}, 2 - 7\sqrt{-6} \right\}. \]
\[ S = \left\{ 6 - \sqrt{-6}, 8 - 6 - 3\sqrt{-6}, 4\sqrt{-6}, 4 + 4\sqrt{-6} \right\}. \]
\[ g = 3, \quad m \leq \sqrt{\frac{635}{2}}. \]

\[ N_r = 151, \quad P = (1 + 5\sqrt{-6}). \]
\[ T = \left\{ 14 + 5\sqrt{-6}, 14 - 5\sqrt{-6}, 8 + 7\sqrt{-6}, 8 - 7\sqrt{-6} \right\}. \]
\[ S = \left\{ 12 - 5\sqrt{-6}, 16 - 3\sqrt{-6}, 6 - 3\sqrt{-6}, 10 + 3\sqrt{-6} \right\}. \]
\[ g = 6, \quad m \leq \sqrt{\frac{755}{2}}. \]

\[ N_r = 193, \quad P = (13 + 2\sqrt{-6}). \]
\[ T = \left\{ 20 + 3\sqrt{-6}, 20 - 3\sqrt{-6} \right\}. \]
\[ S = \left\{ 6 - \sqrt{-6}, 6 - 7\sqrt{-6} \right\}. \]
\[ g = 5, \quad m \leq \sqrt{\frac{985}{2}}. \]

\[ N_r = 199, \quad P = (7 + 5\sqrt{-6}). \]
\[ T = \left\{ 20 + 3\sqrt{-6}, 20 - 3\sqrt{-6} \right\}. \]
\[ S = \left\{ 6 - 7\sqrt{-6}, 10 + 4\sqrt{-6} \right\}. \]
\[ g = 3, \quad m \leq \sqrt{\frac{995}{2}}. \]

\[ N_r = 223, \quad P = (13 + 3\sqrt{-6}). \]
\[ T = \left\{ 20 + 3\sqrt{-6}, 20 - 3\sqrt{-6}, 6.4 + 9\sqrt{-6}, 4 - 9\sqrt{-6}, 2.22 + 3\sqrt{-6}, 3.62 - 3\sqrt{-6} \right\}. \]
\[ S = \{-6 - 3\sqrt{-6}, -6 - 9\sqrt{-6}, 22 - 4\sqrt{-6}, -14 + 4\sqrt{-6}, -4 - 3\sqrt{-6}, 4 + 10\sqrt{-6}\}, \]
\[ g = 3, \quad m \leq \sqrt{\frac{11145}{2}}. \]
\[ N_r = 241, \quad P = (5 + 6\sqrt{-6}), \]
\[ T = \{4 + 9\sqrt{-6}, 4 - 9\sqrt{-6}, 22 + 3\sqrt{-6}, 22 - 3\sqrt{-6}, 10 + 9\sqrt{-6}, 10 - 9\sqrt{-6}\}, \]
\[ S = \{-6 - 3\sqrt{-6}, 14 + 3\sqrt{-6}, -14 + 8\sqrt{-6}, -14 + 2\sqrt{-6}, -3\sqrt{-6}, 20 + 3\sqrt{-6}\}, \]
\[ g = 7, \quad m \leq \sqrt{\frac{1205}{2}}. \]
\[ N_r = 271, \quad P = (11 + 5\sqrt{-6}), \]
\[ T = \{10 + 9\sqrt{-6}, 10 - 9\sqrt{-6}, 22 + 5\sqrt{-6}, 22 - 5\sqrt{-6}\}, \]
\[ S = \{-12 - \sqrt{-6}, -20 + 2\sqrt{-6}, -5\sqrt{-6}, -8 + 6\sqrt{-6}\}, \]
\[ g = 6, \quad m \leq \sqrt{\frac{1335}{2}}. \]

When \(11 \leq N_r \leq 293\), there exist two inert prime ideals, one is \((13)\) and the other is \((17)\).

For \((13)\), take \(C = \{a + b\sqrt{-6} | -6 \leq a, b \leq 6 \text{ and } a, b \text{ are integers}\} \cup \{\pm 8 \pm 6\sqrt{-6}\}\} \cup \{\pm 5 \pm 6\sqrt{-6}\}. \]
\[ W = \{w \in O_F \cap U \mid |w|^2 < 2 \cdot 169\}, \quad G = \{1 + \sqrt{-6}\}. \]

Apply Lemma 2.1 to show \(\tau_r\) is bijective.

For \((17)\), applying Lemma 2.2, we choose \(d = \sqrt{33}\) and let \(D = \{x \in O_F \mid |x| \leq d\}\) then \(#(D) = 47\) and \(d^2 < N_r^2/(4(1 + \sqrt{2N_r})^2)\). Note that \(47^2 > 289^2 = N_r^2\), hence \(\tau_r\) is bijective.

This completes the proof. \(\square\)

We conclude from Sections 3 and 4 the following.

**Theorem 4.6.** Suppose that \(S\) consists all finite places with \(N_r \leq 7\) and Archimedean places in \(F = \mathbb{Q}(\sqrt{-6})\). Then \(K_2O_F \subset K_2^3 F\).

**5. The determination of \(K_2\mathbb{Z}[^\sqrt{-6}]\)**

**Theorem 5.1.** Let \(O_F = \mathbb{Z}[\sqrt{-6}]\), the ring of integers of \(F = \mathbb{Q}(\sqrt{-6})\). Then \(#(K_2O_F) = 1\).
Proof. Write $S = S \cup \{Q, P_1, P_2, P_3, P_4, P_5\}$, where $Q^2 = O_F$, $P_1^2 = 3O_F$, $P_2 = (5, 2 + \sqrt{-6})$, $P_3 = P_2$, $P_4 = (1 + \sqrt{-6})$, $P_5 = P_4$. By Lemma 4.6, $K_2O_F < K_2F$. We observe below the relations of generators of $K_2F$. By Browkin [2], for any $x \in F$, we have the following identities: $\{x, x + 1\}^2 = 1$, $\{x, x^2 + x + 1\}^3 = 1$, $\{x, x^2 + 1\}^4 = 1$. By Browkin and Schinzel [4], $r_2(K_2O_F) = 0$.

Thus $\{-1, -1\} = 1$, $\{-1, 1\} = 1$.

It is easy to show that $u_0 = -1$, $u_1 = 2$, $u_2 = -6$, $u_3 = 1 + \sqrt{-6}$, $u_4 = 1 - \sqrt{-6}$, $u_5 = 2 + \sqrt{-6}$, $u_6 = 2 - \sqrt{-6}$ are the generators of $U = O_5$. We have that:

$\{u_0, u_i\} = 1$ for $0 \leq i \leq 6$.

$\{u_1, u_2\} = \{2, -6\} = \{2, 3\} = \{2, -2\} = 1$.

$\{u_1, u_5\} = \{2, -2 + 4\sqrt{-6}\} = \{2, 1 + 2\sqrt{-6}\}$,

$\{2, -1 + 2\sqrt{-6}\} = \{1 - \sqrt{-6}, 1 + 2\sqrt{-6}\}$

$= \{2, 2\sqrt{-6} - 1 + 2\sqrt{-6}\} = 1$,

$\{1 - \sqrt{-6}, 1 + 2\sqrt{-6}\}$

$= \{1 - \sqrt{-6}, 2\sqrt{-6}\}^{-1} \cdot \{1 - \sqrt{-6} + 2\sqrt{-6}\}^{-1}$,

because

$(1 - \sqrt{-6})^4 + 1 = -4 - 2\sqrt{-6}$, $\{1 - \sqrt{-6}, -4 - 2\sqrt{-6}\}^4 = 1$,

$\{1 - \sqrt{-6}, 2\sqrt{-6}\}^{-1} = \{1 - \sqrt{-6}, -2\}^{-4} = \{1 - \sqrt{-6}, 2\}^{-4}$.

$\{2, 2 + \sqrt{-6}\} = \{2, 10\}$, $\{2, 10\} = 1$.

$\{u_1, u_3\} = 1$.

$\{u_1, u_4\} = 1$.

$\{u_2, u_5\} = \{\sqrt{-6}, -2 + 4\sqrt{-6}\} = \{\sqrt{-6}, -2\} \cdot \{\sqrt{-6}, 1 - 2\sqrt{-6}\}$

$= \{\sqrt{-6}, -2\} \cdot \{2\sqrt{-6} - 1 + 2\sqrt{-6}\}^{-1}$

$= \{\sqrt{-6}, -2\} \cdot \{2, 1 - 2\sqrt{-6}\}^{-1}$.

$\{u_2, u_6\} = \{\sqrt{-6}, 10\} \cdot \{\sqrt{-6}, 2 + \sqrt{-6}\}^{-1} \cdot \{\sqrt{-6}, 10\}^4 = 1$.

$\{u_3, u_4\} = \{(1 + \sqrt{-6})/2, (1 - \sqrt{-6})/2\} \cdot \{2, 1 - \sqrt{-6}\} \cdot \{2, 1 + \sqrt{-6}\}^{-1}$

$= \{2, 1 - \sqrt{-6}\} \cdot \{2, 1 + \sqrt{-6}\}^{-1}$. 
since \((1 + \sqrt{-6})/2 + (1 - \sqrt{-6})/2 = 1\).

\[\{u_3, u_5\}^2 = 1.\]

\[\{u_3, u_6\} = \{1 + \sqrt{-6}, -1\}^{-1} \{1 + \sqrt{-6}, 2\}^{-1} \{1 + \sqrt{-6}, -4 + 2\sqrt{-6}\}.\]

\[\{1 + \sqrt{-6}, -4 + 2\sqrt{-6}\}^4 = \{1 + \sqrt{-6}, (1 + \sqrt{-6})^2 + 1\}^{-1}.\]

\[\{u_3, u_5\}^2 = \{1 - \sqrt{-6}, -2 + 4\sqrt{-6}\} = \{1 - \sqrt{-6}, 2\}^{-1}.\]

\[\{u_4, u_6\} = \{1 - \sqrt{-6}, -1 + 2\sqrt{-6}\} = \{1 - \sqrt{-6}, 2\}^{-1}.\]

\[\{u_4, u_6\}^2 = 1.\]

\[\{u_5, u_6\} = \{2 + \sqrt{-6}, 4/3\}^{-1} \{4, 4\}^{-1}.\]

\[\{u_5, u_6\} = \{2 + \sqrt{-6}, 4/3\}^{-1} \{4, 4\}^{-1}.\]

\[\{u_5, u_6\} = \{2 + \sqrt{-6}, 4/3\}^{-1} \{4, 4\}^{-1}.\]

It follows from the relations we just obtained that for any generator \(x\) of \(K_2^F\), there exist non-negative integers \(n, n_1\) and \(n_2\) such that \(x^{2n} = \{1 + \sqrt{-6}, 2\}^{n_1} \cdot \{1 - \sqrt{-6}, 2\}^{n_2}.\) On the other hand, \(\{1 + \sqrt{-6}, 8\} = \{1 - \sqrt{-6}, 8\} = \{7, 8\} = 1.\)

Let \(m = (1 + \sqrt{-6})/2.\) Then \(m^2 + m + 1 = -3/4.\) Hence \(\{m + \sqrt{-6}, -3/4\}^3 = 1.\)

Let \(l = -\frac{1}{3}.\) Then \(l^2 + l + 1 = 3/4.\) Hence \(\{-1/2, 3/4\} = 1.\)

Note that

\[\{1 + \sqrt{-6}, 2\}^{n_1} \cdot \{1 - \sqrt{-6}, 2\}^{n_2} = \{1 + \sqrt{-6}, 8\} = \{1 - \sqrt{-6}, 8\} = \{7, 8\} = 1.\]

Therefore \(\{1 - \sqrt{-6}, -3/4\} = \{1 - \sqrt{-6}, 1/8\};\) it follows \(\{1 - \sqrt{-6}, 8\}^3 = 1.\)

Because \(r_2(K_2^O_F) = 0,\) for any non-negative integer \(n\) and \(x \in K_2^O_F,\) the order of \(x^{2n}\) is the same as that of \(x.\) But

\[\tau_r(\{1 + \sqrt{-6}, 2\}^{-n_1} \cdot \{1 - \sqrt{-6}, 2\}^{-n_2}) = \begin{cases} 2^{-n_1} \pmod{P}, & \text{if } P = (1 + \sqrt{-6}), \\ 2^{-n_2} \pmod{P}, & \text{if } P = (1 - \sqrt{-6}), \\ 1 \pmod{P}, & \text{if } P \neq 7. \end{cases}\]

Thus we have \(\{1 + \sqrt{-6}, 2\}^{n_1} \cdot \{1 - \sqrt{-6}, 2\}^{n_2} \in K_2^O_F\) if and only if \(n_1 \equiv n_2 \equiv 0 \pmod{3}.\) Hence for any \(x \in K_2^O_F,\) there exist non-negative integers \(n, n'\) such that \(x^{2n} = \{1 + \sqrt{-6}, 8\}^{n'} .\)

Therefore, \(#(K_2^O_F) = 1\) or \(#(K_2^O_F) = 3.\)

Let \(h = 1 + \sqrt{-6},\) then \(h^2 + h + 1 = -3(1 - \sqrt{-6}).\) For any place \(v\) (corresponding to \(P\)) of \(F,\)

\[\tau_v(\{1 + \sqrt{-6}, -3(1 - \sqrt{-6})\} \{\frac{1}{3}, 1 - \sqrt{-6}\}) = 1 \pmod{P}.\]
Remark. The main point of the method used in [1] may be to get a reasonably low value of \( m \) such that \( K_2O_F \subseteq \mathbb{K}_2F \). In [7], Tate uses Proposition 1 (Lemma 2.1 in the present paper) to construct \( W \), \( C \) and \( G \) for \( d = 1, -2, -3, -7, -11, -15 \). If \( C \) is constructed by Tate's method, then \( C \subseteq U \) in all cases above. But it is not hard to see that in other cases \( C \not\subseteq U \) if \( C \) is done as in [1]. One step of the improved method which we propose in this paper is to construct \( t \) such that \( C \subseteq U \), of course. \( C \) will also satisfy some others. Here, \( C \) may be quite "large". The other step is to use an analytical method to get \( m \), which is different from Tate's method. Although this method is based on a theorem due to Bass and Tate, we can see that our method can be used to deal with other imaginary quadratic fields. Using this method, the author proves that \( K_2O_F \cong \mathbb{Z}/2\mathbb{Z} \) for \( F = \mathbb{Q}(-\sqrt{-35}) \) (see [5]).

Note added in proof. After finishing this paper, the author has found a paper of M. Skalba (see [6]). In [6] Skalba proposes a method which is different from ours and by which he shows that \( K_2O_F = 1 \) for \( F = \mathbb{Q}(\sqrt{-5}) \) and \( F = \mathbb{Q}(\sqrt{-19}) \). This can also be proved by our method. In fact, to do this we need fewer computations as compared with the case of \( F = \mathbb{Q}(\sqrt{-6}) \). Since both discriminants of \( F = \mathbb{Q}(\sqrt{-5}) \) and \( F = \mathbb{Q}(\sqrt{-19}) \) are less than that of \( F = \mathbb{Q}(\sqrt{-6}) \).

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References

[5] Qin Hourong. Computation of \( K_2O_F \cong \mathbb{Z}/142\mathbb{Z} \) for \( F \in K \). to appear.