Schur complements and its applications to symmetric nonnegative and Z-matrices

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Abstract

In [Linear Algebra Appl. 177 (1992) 137] Smith proved that if \( H \) is a Hermitian semi-definite matrix and \( A \) is a nonsingular principal submatrix, then the eigenvalues of the Schur complement \( H/A \) interlace those of \( H \). In this paper, we refine the latter result and use it to derive eigenvalues interlacing results on an irreducible symmetric nonnegative matrix that involve Perron complements. For an irreducible symmetric nonnegative matrix, we give lower and upper bounds for its spectral radius and also a lower bound for the maximal spectral radius of its principal submatrices of a fixed order. We apply our results to an irreducible symmetric Z-matrix and to the adjacency matrix or the general Laplacian matrix of a connected weighted graph. The equality cases for the bounds for spectral radii or least eigenvalues are also examined.

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1. Introduction

Schur complement is a very useful tool in matrix analysis. In the literature on nonnegative and Z-matrices, there are many results involving Schur complements;
Another useful but less well-known concept is that of the Perron complement of an irreducible nonnegative matrix. The concept is derived from Schur complement and was introduced by Meyer [11,12] in his construction of an algorithm for computing the stationary distribution vector for Markov chains. For recent works on Perron complement, we refer the reader to the paper by Neumann [15] and the references therein.

In [17, Theorem 5], Smith showed that if $H$ is semidefinite, i.e., $H$ is Hermitian and is either positive semidefinite or negative semidefinite, and if $A$ is a nonsingular principal submatrix of $H$, then the eigenvalues of the Schur complement $H/A$ interlace those of $H$. In Section 3 of this paper, we refine Smith’s result. For completeness, we provide a self-contained proof via a continuity argument. Thereby we obtain eigenvalues interlacing results involving Perron complements of an irreducible symmetric nonnegative matrix.

In the literature, there are many results on bounds for the spectral radius of a nonnegative matrix; see, for instance, [13, Chapter 2]. In Section 4, using Perron complements, we give new lower and upper bounds for the spectral radius of an irreducible symmetric nonnegative matrix, which are expressed in terms of the spectral radii of its complementary principal submatrices and the corresponding eigenvectors. The equality cases for the bounds are also examined. As applications, we obtain the bounds for the minimal eigenvalue of an irreducible symmetric $Z$-matrix, for the spectral radius of the adjacency matrix of a connected simple graph, and also for the minimal eigenvalue of the general Laplacian matrix of a connected weighted graph.

For an irreducible nonnegative matrix $A$, an upper bound for the maximal spectral radius of principal submatrices of a fixed order is given by Friedland and Nabben [7, Theorem 3.1]. In the second half of Section 4, we obtain a lower bound when $A$ is, in addition, symmetric. As a consequence, an upper bound for the minimal eigenvalue of principal submatrices of a fixed order of an irreducible symmetric $Z$-matrix is also established.

2. Preliminaries

Let $G = (V, E)$ be an undirected graph of order $n$ with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E \subseteq V \times V$. By assigning a weight $w[i, j] > 0$ to each edge $[i, j] \in E$, we turn $G$ into a weighted graph; if $[i, j] \notin E$, set $w[i, j] = 0$. By the degree $d_i$ of a vertex $i$ we mean the sum of weight of all edges incident with $i$. The adjacency matrix of $G$ is the matrix of order $n$ given by $A(G) = (w[i, j])$. Clearly, the association $G \mapsto A(G)$ gives a one-to-one correspondence between the set of weighted graphs on $V$ and the set of symmetric nonnegative matrices of order $n$. The general Laplacian matrix of $G$ is given by $L(G) = D(G) - A(G')$ [6], where $D(G)$ equals $\text{diag}(d_1, d_2, \ldots, d_n)$ and $G'$ is the weighted graph obtained from $G$ by deleting all its loops. If $G$ is simple, i.e., if $G$ is loopless and the weight of each edge is 1, then $L(G)$ is the standard Laplacian matrix of $G$. It is known that a weighted graph $G$
A Z-matrix is a real square matrix whose off-diagonal entries are nonpositive. If $A$ is a Z-matrix of order $n$, then it has the form

$$A = tI - B,$$

where $B \geq 0$. \hfill (2.1)

Following Fiedler and Markham [5], for $s = 0, \ldots, n$, we denote by $L_s$ the set of all Z-matrices $A$ of order $n$ of the form (2.1) that satisfy $\rho_s(B) \leq t < \rho_{s+1}(B)$, where $\rho_s(B)$ denotes the maximal spectral radius of all principal submatrices of $B$ of order $s$, $\rho_0(B) = -\infty$, $\rho_{n+1}(B) = +\infty$. In terms of this notation, $L_s$ and $L_{n-1}$ are, respectively, the well-known classes of $M$-matrices and $N_0$-matrices. Notice that the general Laplacian matrix $L(G)$ of a weighted graph $G$ is always an $M$-matrices, because it is a Z-matrix and by the Geršgorin disc theorem it is positive semidefinite (see [1, Theorem 6.4.6] for characterizations of $M$-matrices among Z-matrices).

Let $A$ be a real matrix of order $n$, and let $\alpha, \beta$ be nonempty subset of $\{1, 2, \ldots, n\}$. Denote the submatrix that lies in the rows of $A$ indexed by $\alpha$ and the columns indexed by $\beta$ as $A[\alpha, \beta]$. Abbreviate $A[\alpha, (n) \setminus \beta]$, $A[(n) \setminus \alpha, \beta]$ to $A[\alpha, \beta]$, $A[\alpha, \beta]$, and $A[\alpha, \alpha], A(\alpha, \alpha)$ to $A[\alpha], A(\alpha)$, respectively. In particular, for a vector $x$, treated as a column vector, we use $x[\alpha]$ to denote the subvector of $x$ with entries indexed by $\alpha$. We also abbreviate $x[(n) \setminus \alpha]$ to $x(\alpha)$. Note that in this paper the eigenvalues of a Hermitian matrix $A$ of order $n$ are arranged in nondecreasing order: $\lambda_1(A) < \lambda_2(A) < \cdots < \lambda_n(A)$.

Let $\emptyset \neq \alpha \subset \{n\}$ be such that $A[\alpha]$ is nonsingular. Then the Schur complement of $A[\alpha]$ in $A$ is given by

$$A/A[\alpha] = A(\alpha) - A(\alpha, \alpha)A[\alpha]^{-1}A(\alpha, \alpha).$$

If $A$ is irreducible, nonnegative, then the Perron complement of $A[\alpha]$ in $A$ is given by

$$\mathcal{P}(A/A[\alpha]) = A(\alpha) + A(\alpha, \alpha)(\rho(A)I - A[\alpha])^{-1}A(\alpha, \alpha),$$

where $\rho(B)$ denotes the spectral radius of the matrix $B$. Recall that for an irreducible nonnegative matrix $A$, we have $\rho(A[\alpha]) < \rho(A)$; so the expression on the right-hand side of (2.3) is well defined.

We will need the following results.

**Theorem A** [18]. Let $A \in L_k$, where $k \in \{1, 2, \ldots, n-1\}$, and let $A_{11}$ be a principal submatrix of $A$ of order $s$ which is a nonsingular M-matrix. Then $A/A_{11}$ belongs to $L_{s(i(A_{11}))}$, where, $i(A_{11})$, the index of $A_{11}$ in $A$, is defined to be the smallest positive integer $i$ such that there is an $(i+1) \times (i+1)$ principal submatrix of $A$ which is an $N_0$-matrix and has $A_{11}$ as a principal submatrix.

**Theorem B** [11]. Let $A$ be an irreducible nonnegative matrix of order $n$, and let $\emptyset \neq \alpha \subset \{n\}$. Then $\mathcal{P}(A/A[\alpha])$ is also an irreducible nonnegative matrix with
spectral radius $\rho(A)$. Moreover, if $x$ is a positive eigenvector of $A$ corresponding to $\rho(A)$, then $x(\alpha)$ is a positive eigenvector of $P(A/A[\alpha])$ corresponding to $\rho(A)$.

As noted by a referee, in Theorem A one may replace the assumption “$A \in L_k$, where $k \in \{1, \ldots, n-1\}$” by “$A$ is a $Z$-matrix with nonnegative diagonal entries which is not an $M$-matrix”. Also, it is known (see [16]) that if $A$ is an $M$-matrix and $A_{11}$ is a nonsingular principal submatrix, then $A/A_{11}$ is an $M$-matrix. So Theorem A also covers the case $k = n$ if we adopt the convention $i(A_{11}) = n$ when $A$ is an $M$-matrix. Later in this paper, we will apply Theorem A to $A \in L_k$ for $k \in \langle n \rangle$ with such understanding.

3. Eigenvalues interlacing properties

In [17, Theorem 5] Smith proved that if $A$ is an Hermitian semidefinite matrix of order $n$ and $\alpha$ is a nonempty subset of $\langle n \rangle$ such that $A[\alpha]$ is nonsingular, then
\[
\lambda_r(A) \leq \lambda_r(A/A[\alpha]) \leq \lambda_r(A(\alpha)) \leq \lambda_r(A[\alpha]) \quad \text{for} \quad r = 1, 2, \ldots, n - |\alpha|.
\]
If $A$ is positive semidefinite, then
\[
\lambda_n + 1 - s(A) \leq \lambda_n + 1 - s(A/A[\alpha]) \leq \lambda_n + 1 - s(A(\alpha)), \quad s = 1, 2, \ldots, n - k.
\]

Proof. Let $A_{\epsilon} = A + \epsilon I$, $\epsilon > 0$, and let $|\alpha| = k$. First, suppose $A$ is positive semidefinite. Thus $A_{\epsilon}$ and $A_{\epsilon}[\alpha]$ are both positive definite. By the known fact that $A^{-1}(\alpha) = (A/A[\alpha])^{-1}$ [2] and the Cauchy interlacing theorem (see [8, Theorem 4.3.15]), we have that for any $\epsilon > 0$ and $s = 1, 2, \ldots, n - k$,
\[
0 < \lambda_s(A_{\epsilon}^{-1}) \leq \lambda_s((A_{\epsilon}/A_{\epsilon}[\alpha])^{-1}) \leq \lambda_{s+k}(A_{\epsilon}^{-1}),
\]
\[
\lambda_{n+1-s}(A_{\epsilon}) \geq \lambda_{n-k+1-s}(A_{\epsilon}/A_{\epsilon}[\alpha]) \geq \lambda_{n+1-s-k}(A_{\epsilon}) > 0.
\]

Since the eigenvalues of a matrix depend continuously on its entries, and $A_{\epsilon} \rightarrow A$, $A_{\epsilon}[\alpha] \rightarrow A/A[\alpha]$ when $\epsilon \rightarrow 0$, we obtain
\[
\lambda_{n+1-s}(A) \geq \lambda_{n-k+1-s}(A/A[\alpha]) \geq \lambda_{n+1-s-k}(A), \quad s = 1, 2, \ldots, n - k,
\]
and so
\[
\lambda_{r+k}(A) \geq \lambda_r(A/A[\alpha]) \geq \lambda_r(A), \quad r = 1, 2, \ldots, n - k.
\]
Since \( A(\alpha) = A/A[\alpha] + A(\alpha, \alpha)A[\alpha]^{-1}A(\alpha, \alpha) \) and \( A(\alpha, \alpha)A[\alpha]^{-1}A(\alpha, \alpha) \) is positive semidefinite, by Weyl’s inequality (see [8, Theorem 4.3.1]), \( \lambda_r(A/A[\alpha]) \leq \lambda_r(A(\alpha)) \). So the first inequality follows.

If \( A \) is negative semidefinite, then \(-A\) is positive semidefinite. In addition, \((-A)/(-A[\alpha]) = -(A/A[\alpha])\). By inequality (3.1), we have

\[
\lambda_{n+1-s}(-A) \geq \lambda_{n-k+1-s}(-(A/A[\alpha]))
\]

\[
\geq \lambda_{n+1-s-k}(-A), \quad s = 1, 2, \ldots, n-k,
\]

\[-\lambda_{s}(A) \geq -\lambda_{s}(A/A[\alpha]) \geq -\lambda_{s+k}(A), \quad s = 1, 2, \ldots, n-k.
\]

By Weyl’s inequality again, we also obtain \( \lambda_r(A/A[\alpha]) \leq \lambda_r(A(\alpha)) \). So the theorem follows. □

**Theorem 3.2.** Let \( A \) be a Hermitian semidefinite matrix of order \( n \) and let \( \alpha \) be a nonempty subset of \( \langle n \rangle \) such that \( A[\alpha] \) is nonsingular. Then for any \( \emptyset \neq \alpha' \subset \alpha \subset \langle n \rangle \) and \( r = 1, 2, \ldots, n - |\alpha| \), we have

\[
\lambda_r(A/A[\alpha']) \leq \lambda_r(A/A[\alpha])
\]

\[
\leq \lambda_r(A[\alpha'] \cup (\langle n \rangle \setminus \alpha])/A[\alpha'])
\]

\[
\leq \lambda_{r+|\alpha'|-|\alpha'|}(A/A[\alpha']) \quad \text{if} \ A \text{ is positive semidefinite},
\]

and

\[
\lambda_r(A/A[\alpha']) \leq \lambda_r(A[\alpha'] \cup (\langle n \rangle \setminus \alpha])/A[\alpha'])
\]

\[
\leq \lambda_r(A/A[\alpha])
\]

\[
\leq \lambda_{r+|\alpha'|-|\alpha'|}(A/A[\alpha']) \quad \text{if} \ A \text{ is negative semidefinite}.
\]

**Proof.** Since \( A[\alpha'] \) is a principal submatrix of the definite matrix \( A[\alpha] \), clearly it is indefinite. So the matrices \( A[\alpha] \) and \( A[\alpha'] \) are nonsingular and the Schur complements \( A/A[\alpha] \) and \( A/A[\alpha'] \) are semidefinite. By the quotient formula for Schur complements [4], we have

\[
A/A[\alpha] = (A/A[\alpha'])/(A[\alpha]/A[\alpha']).
\]

Applying Theorem 3.1 (with \( A/A[\alpha'] \) and \( A[\alpha]/A[\alpha'] \) in place of \( A \) and \( A[\alpha] \), respectively) and noting that \( (A/A[\alpha'])(\langle n \rangle \setminus \alpha]) \) is equal to \( (A[\alpha'] \cup (\langle n \rangle \setminus \alpha])/A[\alpha']) \), we readily obtain our assertions. □

Let \( A \) be an irreducible symmetric nonnegative matrix of order \( n \), and let \( B = \rho(A)I - A \). It is well known that \( B \) is a singular \( M \)-matrix and every proper principal submatrix of \( B \) is a nonsingular \( M \)-matrix (see [1, p. 156]). Moreover, \( B \) is symmetric positive semidefinite. Applying Theorems 3.1 and 3.2 to \( B \) and noting that \( B/B[\alpha] = \rho(A)I - \Phi(A/A[\alpha]) \), we obtain the following:
Corollary 3.3. Let $A$ be an irreducible symmetric nonnegative matrix of order $n$. Then for any $\emptyset \neq \alpha \subset \langle n \rangle$ and $r = 1, 2, \ldots, n - |\alpha|$, 
$$
\lambda_r(A) \leq \lambda_r(A(\alpha)) \leq \lambda_r(\mathcal{P}(A/A[\alpha])) \leq \lambda_{r+|\alpha|}(A).
$$

Corollary 3.4. Let $A$ be an irreducible symmetric nonnegative matrix of order $n$. Then for any $\emptyset \neq \alpha' \subset \alpha \subset \langle n \rangle$ and $r = 1, 2, \ldots, n - |\alpha|$, 
$$
\lambda_r(\mathcal{P}(A/A[\alpha'])) \leq \lambda_r(\mathcal{P}(A/A[\alpha])) \leq \lambda_{r+|\alpha|-|\alpha'|}(\mathcal{P}(A/A[\alpha'])).
$$

4. The bounds for eigenvalues of nonnegative and $Z$-matrices

In this section we consider real matrices and real vectors only. We call a vector $x$ a unit vector if $\|x\| = 1$, where $\|x\| = \sqrt{x^Tx}$.

Lemma 4.1. Let $A$ be a real symmetric positive semidefinite of order $n$, and let $x$ be a unit eigenvector of $A$ corresponding to $\rho(A)$. Then for any unit vector $y$,
$$
y^TAy \geq (y^tx)^2\rho(A).
$$
If $A$ is, in addition, positive definite, then the above inequality holds as equality if and only if $y = x$ or $y = -x$.

Proof. Since $A$ is positive semidefinite, there exists an orthonormal basis $x_1, x_2, \ldots, x_n$ of $\mathbb{R}^n$, such that $Ax_i = \lambda_i(A)x_i$, where $x_n = x$, $\lambda_n(A) = \rho(A)$. Let
$$
y = \sum_{i=1}^nc_ix_i.
$$
Then $c_i = y^Tx_i$ for $i = 1, 2, \ldots, n$. Hence
$$
y^TAy = c_1^2\lambda_1(A) + c_2^2\lambda_2(A) + \cdots + c_n^2\lambda_n(A) \geq c_n^2\lambda_n(A) = (y^tx)^2\rho(A).
$$
If $A$ is positive definite, then $y^TAy = (y^tx)^2\rho(A)$ if and only if $c_1 = \cdots = c_{n-1} = 0$, and the result follows. \qed

By the Perron vector of an irreducible nonnegative matrix we mean its unique unit positive eigenvector (necessarily corresponding to its spectral radius).

Theorem 4.2. Let $A$ be an irreducible symmetric nonnegative matrix of order $n$, and let $x$ be the Perron vector of $A$. Then for any $\emptyset \neq \alpha \subset \langle n \rangle$,
$$
\max_{\gamma \in \mathcal{A}(\alpha)} \frac{1}{2} \left[ \rho(A[\alpha]) + \rho(A(\alpha)) + \sqrt{\left(\rho(A[\alpha]) - \rho(A(\alpha))\right)^2 + 4(x^TA[\alpha, \alpha]x_{(\alpha)})^2} \right]
$$
\[ \rho(A) \leq \rho(A[I]) + \rho(A(\alpha)) \]
\[ + \sqrt{\rho(A[I]) - \rho(A(\alpha))}^2 + 4\rho(A[I])A[I]A(I) \]
\[ \leq \rho((\rho(A[I]) - A[I])^{-1}) \|A[I]A[I]A(I)\|_2. \] (4.1)

where the maximum is taken over all unit eigenvectors \(x[I]_\alpha\) and \(x(\alpha)\) of \(A[I]\) and \(A(\alpha)\) corresponding to \(\rho(A[I])\) and \(\rho(A(\alpha))\), respectively. Moreover, the following are equivalent:

1. The first inequality in (4.1) holds as equality.
2. The second inequality in (4.1) holds as equality.
3. The subvectors \(x[I]_\alpha\) and \(x(\alpha)\), respectively, eigenvectors of \(A[I]\) and \(A(\alpha)\) corresponding to their spectral radii.
4. There exist unit (positive) eigenvectors \(x[I]_\alpha\) of \(A[I]\) corresponding to \(\rho(A[I])\) and \(x(\alpha)\) of \(A(\alpha)\) corresponding to \(\rho(A(\alpha))\) such that
\[ \rho(A[I])A[I]A(I) = (x[I]_\alpha A[I]x(\alpha))^2. \]

**Proof.** Consider any \(\emptyset \neq \alpha \subset (n)\). Since \(A\) is irreducible, symmetric and nonnegative, \(\rho(A[I] - A[I])\) is a nonsingular symmetric \(M\)-matrix. So the positive number \(\rho(A) - \rho(A[I])\) is the eigenvalue of \(\rho(A[I] - A[I])\) with minimum modulus; hence
\[ \rho((\rho(A[I]) - A[I])^{-1}) = \frac{1}{\rho(A) - \rho(A[I])}. \]

Since \(\mathcal{P}(A[I])\) is the sum of the symmetric nonnegative matrices \(A[I]\) and \(A[I]\rho(A[I])^{-1}A[I]\), by Theorem B, Weyl’s inequality [8, Theorem 4.3.1] and the fact that \(\lambda_n(C) = \rho(C)\) for any symmetric nonnegative matrix \(C\), we have
\[ \rho(A[I]) = \rho(\mathcal{P}(A[I]) A[I]) \]
\[ = \lambda_n(A[I]) + A[I]\rho(A[I])^{-1}A[I] \]
\[ \leq \lambda_n(A[I]) + \lambda_n(A[I]) A[I]\rho(A[I])^{-1}A[I] \]
\[ = \rho(A[I]) + \rho(A[I]) A[I]\rho(A[I])^{-1}A[I]. \] (4.2)

Choose a unit vector \(u \in [n]\) such that
\[ \rho(A[I]) A[I] = u(A[I]) A[I] u. \]

Then we have
\[ \rho(A[I]) A[I] \leq \rho((\rho(A[I])^{-1}) A[I] u u^T .

Finally, we have
\[ \rho(A[I]) A[I] \leq \rho((\rho(A[I])^{-1}) A[I] u u^T .

Therefore, we can conclude that
\[ \rho(A[I]) A[I] \leq \rho((\rho(A[I])^{-1}) A[I] u u^T .

This completes the proof.
\[
\begin{align*}
\frac{1}{\rho(A) - \rho(A[\alpha])} & = u^t A[\alpha, \alpha] A[\alpha, \alpha] u \\
& \leq \frac{1}{\rho(A) - \rho(A[\alpha])} \rho(A[\alpha, \alpha] A[\alpha, \alpha]).
\end{align*}
\]

By inequality (4.2), therefore
\[
\rho(A) \leq \rho(A[\alpha]) + \frac{1}{\rho(A) - \rho(A[\alpha])} \rho(A[\alpha, \alpha] A[\alpha, \alpha]).
\]

Hence, we obtain the following quadratic inequality in \( \rho(A) \):
\[
\rho(A)^2 - (\rho(A) + \rho(A[\alpha])) \rho(A) + (\rho(A[\alpha]) \rho(A[\alpha])) \leq 0.
\]

Solving inequality (4.4), we obtain the upper bound for \( \rho(A) \) as given by (4.1).

Let \( x_{\alpha[\alpha]}, x_{\alpha(\alpha)} \) be unit eigenvectors of \( A[\alpha], A(\alpha) \) corresponding to \( \rho(A[\alpha]) \), \( \rho(A(\alpha)) \), respectively. Then by Theorem B,
\[
\rho(A) = \rho(\mathcal{P}(A/A[\alpha])) = \lambda_{\alpha}(A(\alpha) + A[\alpha, \alpha] A(\alpha))^{-1} A[\alpha, \alpha] x_{\alpha(\alpha)}
\]
\[
\geq x_{\alpha(\alpha)}^t (A(\alpha) + A[\alpha, \alpha] A(\alpha))^{-1} A[\alpha, \alpha] x_{\alpha(\alpha)}
\]
\[
= \rho(A(\alpha)) + \frac{1}{\rho(A) - \rho(A[\alpha])} (x_{\alpha[\alpha]} A[\alpha, \alpha] x_{\alpha(\alpha)})^2.
\]

where the last inequality follows from Lemma 4.1 by replacing \( A, x \) and \( y \), respectively by \( \rho(A) I - A[\alpha]^{-1}, x_{\alpha} \) and \( z/\|z\| \), where \( z = A[\alpha, \alpha] x(\alpha) \neq 0 \) (noting that the inequality also holds when \( z = 0 \)). We obtain a quadratic inequality in \( \rho(A) \), which is almost the same as inequality (4.4) except that \( "\rho(A[\alpha, \alpha])" \) and \( "\leq" \) are to be replaced by \( "(x_{\alpha[\alpha]} A[\alpha, \alpha] x(\alpha))^2" \) and \( "\geq" \), respectively. So we must have either \( \rho(A) \leq \beta_1 \) or \( \rho(A) \geq \beta_2 \), where \( \beta_1, \beta_2 \), with \( \beta_1 \leq \beta_2 \), denote the roots of the corresponding quadratic equation. But it is readily seen that \( \beta_1 \leq (\rho(A[\alpha]) + \rho(A(\alpha)))/2 < \rho(A) \), so we must have \( \rho(A) \geq \beta_2 \), which gives a lower bound for \( \rho(A) \). Taking the maximum of all these \( \beta_2 \)'s as \( x_{\alpha[\alpha]} \) (respectively, \( x_{\alpha(\alpha)} \)) varies over all possible unit eigenvectors of \( A[\alpha] \) (respectively, \( A(\alpha) \)) corresponding to \( \rho(A[\alpha]) \) (respectively, \( \rho(A(\alpha)) \)), we obtain the first inequality of (4.1).

Below we prove the equivalence of conditions (1)–(4).

(2) \( \Rightarrow \) (3). Suppose that the second inequality of (4.1) holds as equality. Retracing our above proof, we readily see that the inequality in (4.2) and also the inequality (4.4) both hold as equality. By Theorem B, the subvector \( x(\alpha) \) of the Perron vector \( x \) of \( A \) is a positive eigenvector of \( \mathcal{P}(A/A[\alpha]) \) corresponding to \( \rho(A) \). Writing \( x(\alpha) \) for the unit vector \( x(\alpha)/\|x(\alpha)\| \), we have
\[
\lambda_{\alpha}(\mathcal{P}(A/A[\alpha])) = x_{\alpha(\alpha)}^t (A(\alpha) + A[\alpha, \alpha] A(\alpha) - A[\alpha])^{-1} A[\alpha, \alpha] x_{\alpha(\alpha)}.
\]
In order that the inequality in (4.2) becomes equality, we must have $x^T(\alpha)A(\alpha)x(\alpha) = \rho(A(\alpha))$ (and a corresponding equality for $A[\alpha, \alpha]^T(\rho(A)I - A[\alpha, \alpha])^{-1}A[\alpha, \alpha]$). So $x(\alpha)$, and hence $x(\alpha)$, is an eigenvector of $A(\alpha)$ corresponding to $\rho(A(\alpha))$. In view of

$$
\rho(A[\alpha, \alpha])A[\alpha, \alpha]^T = \rho(A[\alpha, \alpha])A[\alpha, \alpha],
$$

where the second equality holds as $A$ is symmetric, we readily see that the second inequality of (4.1) also holds as equality if we replace $\alpha$ by $\langle n \rangle \setminus \alpha$. By what we have done above, it follows that $x[\alpha]$ is also an eigenvector of $A[\alpha, \alpha]$ corresponding to $\rho(A[\alpha, \alpha])$. This establishes (3).

(3) $\Rightarrow$ (4). Since $x$ is an eigenvector of $A$ corresponding to $\rho(A)$ and $x[\alpha]$ is an eigenvector of $A[\alpha, \alpha]$ corresponding to $\rho(A[\alpha, \alpha])$, we have

$$
\rho(A[\alpha])x[\alpha] + A[\alpha, \alpha]x(\alpha) = A[\alpha]x[\alpha] + A[\alpha, \alpha]x(\alpha)
= (Ax)[\alpha] = \rho(A)x[\alpha],
$$

and hence

$$
A[\alpha, \alpha]x(\alpha) = (\rho(A) - \rho(A[\alpha]))x[\alpha]. \quad (4.6)
$$

Similarly, we also have

$$
A[\alpha, \alpha]^T x[\alpha] = (\rho(A) - \rho(A[\alpha]))x(\alpha). \quad (4.7)
$$

From these, we obtain

$$
A[\alpha, \alpha]A[\alpha, \alpha]^T x[\alpha] = (\rho(A) - \rho(A[\alpha]))(\rho(A) - \rho(A[\alpha]))x[\alpha],
$$

and hence $\rho(A[\alpha, \alpha])A[\alpha, \alpha]^T = (\rho(A) - \rho(A[\alpha]))(\rho(A) - \rho(A[\alpha]))$, as $x[\alpha]$ is a positive eigenvector of the nonnegative matrix $A[\alpha, \alpha]A[\alpha, \alpha]^T$.

Let $x_{\langle n \rangle}$, $x_{(\alpha)}$ denote, respectively, the unit vectors $x[\alpha]/\|x[\alpha]\|$ and $x(\alpha)/\|x(\alpha)\|$.

From (4.6), we readily obtain

$$
x_{(\alpha)}^TA[\alpha, \alpha]x(\alpha) = (\rho(A) - \rho(A[\alpha]))\|x[\alpha]\|/\|x(\alpha)\|.
$$

Similarly, from (4.7) we also obtain

$$
x_{\langle n \rangle}^TA[\alpha, \alpha]^Tx[\alpha] = (\rho(A) - \rho(A(\alpha)))\|x(\alpha)\|/\|x[\alpha]\|.
$$

Note that the left-hand sides of the above two equations are equal. Multiplying them up, we obtain

$$
(x_{\langle n \rangle}^TA[\alpha, \alpha]x(\alpha))^2 = (\rho(A) - \rho(A[\alpha]))(\rho(A) - \rho(A(\alpha)))
= \rho(A[\alpha, \alpha])A[\alpha, \alpha]^T.
$$

So condition (4) holds.

It is not difficult to see that when (4) holds (where $x_{\langle n \rangle}$ and $x_{(\alpha)}$ are unit eigenvectors of $A[\alpha]$ and $A(\alpha)$ corresponding to $\rho(A[\alpha])$ and $\rho(A(\alpha))$, respectively, but they need not be positive), the two inequalities in (4.1) both become equalities. So we have (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2).
From the above, we see that conditions (2)–(4) are equivalent, and they imply (1). To complete the proof, we are going to show (1) $\Rightarrow$ (3).

(1) $\Rightarrow$ (3): Suppose that the first inequality in (4.1) holds as equality. Retracing our proof for the first inequality, we see that in this case there exist unit eigenvectors $x_{[\alpha]}$ of $A[\alpha]$ corresponding to $\rho(A[\alpha])$ and $x(\alpha)$ of $A(\alpha)$ corresponding to $\rho(A(\alpha))$ such that $\rho(A)$ equals the larger root of the quadratic equation

$$l^2 - \left(\rho(A[\alpha]) + \rho(A(\alpha))\right)l + \rho(A[\alpha])\rho(A(\alpha)) = 0,$$

and moreover $x(\alpha)$ is an eigenvector of $P(A/A[\alpha])$ corresponding to its spectral radius (as the first inequality of (4.5) holds as equality). By Theorem B, $x(\alpha)$ is (up to multiples) the unique eigenvector of the irreducible nonnegative matrix $P(A/A[\alpha])$ corresponding to $\rho(A)$. This shows that $x(\alpha)$ is an eigenvector of $A(\alpha)$ corresponding to $\rho(A(\alpha))$. Note that if in Eq. (4.8) we replace $\alpha$ by $\langle n \rangle \setminus \alpha$, then we end up with the same quadratic equation (as $A[\alpha, \alpha] = A(\alpha, \alpha)$). It follows that $x_{[\alpha]}$ is also an eigenvector of $A[\alpha]$ corresponding to $\rho(A[\alpha])$. This establishes condition (3). $\square$

Let $G = (V, E)$ be a graph and let $i$ be a vertex of $G$. We will use $N(i)$ and $t_i$ to denote, respectively, the neighbour set and the 2-degree of $i$; that is,

$$N(i) = \{j \in V : \{j, i\} \in E\} \quad \text{and} \quad t_i = \sum_{j \in N(i)} d_j.$$

For any $\emptyset \neq U \subset V$, we denote by $G[U]$ and $G(U)$ the subgraphs of $G$ induced by $U$ and $V \setminus U$, respectively. We also use $G_U$ to denote the graph obtained from $G$ by deleting all edges of $G[U]$ and $G(U)$. In the following we write $\rho(A(G))$ simply as $\rho(G)$.

**Corollary 4.3.** Let $G = (V, E)$ be a connected simple graph of order $n$ and let $\emptyset \neq U \subset V$ such that $G[U]$ is a complete graph. Suppose $U = \{1, \ldots, m\}$. Then

$$\max_y \frac{1}{2} \left[ m - 1 + \rho(G(U)) + \sqrt{(m - 1 - \rho(G(U)))^2 + \frac{4}{m} \left( \sum_{j=m+1}^{n} \tilde{d}_{j y_j} \right)^2} \right] \leq \rho(G) \leq \frac{1}{2} \left[ m - 1 + \rho(G(U)) + \sqrt{(m - 1 - \rho(G(U)))^2 + 4 \rho(G_U)^2} \right].$$

(4.9)
where the maximum is taken over all unit eigenvectors $y = (y_{m+1}, y_{m+2}, \ldots, y_n)^t$ of $A(G(U))$ corresponding to $\rho(G(U))$, and $\tilde{d}_j$ ($j \in [n]$) is the degree of the vertex $j$ of $G_U$. Moreover, when one of the two inequalities in (4.9) holds as equality, both hold as equality, and this happens if and only if for the graph $G_U$, the 2-degrees of the vertices $1, \ldots, m$ are the same, and $(\tilde{d}_{m+1}, \ldots, \tilde{d}_n)^t$ is a positive eigenvector of $A(G(U))$ corresponding to $\rho(G(U))$.

**Proof.** Let $A$ be the adjacency matrix of graph $G$, and let $\alpha = U$. Then $A[\alpha] = A(G[U]) = J - 1$ and $A(\alpha) = A(G(U))$, where $J$ denotes the square matrix of order $m$ with entries all equal to 1. Clearly, $x[\alpha] = (1/\sqrt{m}) \ell$ (and its negative) is the only unit eigenvector of $A[\alpha]$ corresponding to $\rho(A[\alpha]) = m - 1$, where $\ell$ is a column vector consisting of all 1’s. Note that for any unit eigenvector $x(\alpha) = y = (y_{m+1}, \ldots, y_n)^t$ of $A(\alpha)$ corresponding to $\rho(A(\alpha))$, we have

$$x[\alpha]^t A[\alpha] y = \frac{1}{\sqrt{m}} \sum_{j=m+1}^n \tilde{d}_j y_j,$$

and $\rho(A[\alpha]A[\alpha]^t) = \rho(G_U)^2$. By Theorem 4.2 the inequalities in (4.9) follows.

By Theorem 4.2, if one of the inequalities in (4.9) holds as equality, then both inequalities hold as equality. In this case, condition (3) of Theorem 4.2 holds and by (4.6) and (4.7) in the proof of Theorem 4.2, (3) $\Rightarrow$ (4), we have

$$A[\alpha, \alpha] x(\alpha) = (\rho(G) - (m - 1)) x[\alpha]$$

and

$$A[\alpha, \alpha]^t x[\alpha] = (\rho(G) - \rho(G(U))) x(\alpha),$$

where $x = (x_1, \ldots, x_n)^t$ is the Perron vector of the irreducible nonnegative matrix $A$. Note that the vector $x[\alpha]$, being an eigenvector of $A[\alpha]$ corresponding to $\rho(A[\alpha])$, must be a multiple of $\ell$. Also $x(\alpha)$ is an eigenvector of $A(\alpha)$ corresponding to $\rho(A(\alpha))$. Since $x[\alpha]$ and $x(\alpha)$ are positive vectors and $\rho(G) - \rho(G(U)) > 0$, (4.11) implies that $(\tilde{d}_{m+1}, \ldots, \tilde{d}_n)^t (= A[\alpha, \alpha] \ell)$ is a positive multiple of $x(\alpha)$ and hence is a positive eigenvector of $A(G(U))$ corresponding to $\rho(G(U))$. Also, by (4.10) we readily see that the 2-degrees of the vertices $1, \ldots, m$ of graph $G_U$ are same. To prove the converse, take

$$x(\alpha) = \frac{1}{\sqrt{d}} (\tilde{d}_{m+1}, \ldots, \tilde{d}_n)^t, \quad \text{where } d = \tilde{d}_{m+1}^2 + \cdots + \tilde{d}_n^2.$$

Then, by our assumption, $x(\alpha)$ is a unit eigenvector of $A(\alpha)$ corresponding to $\rho(A(\alpha))$. Denote by $t$ the same 2-degrees of vertices $1, 2, \ldots, m$ of $G_U$. Then

$$A[\alpha, \alpha] A[\alpha, \alpha]^t \ell = A[\alpha, \alpha] (\tilde{d}_{m+1}, \ldots, \tilde{d}_n)^t = t \ell,$$

which implies that

$$\rho(G_U)^2 = \rho(A[\alpha, \alpha] A[\alpha, \alpha]^t) = t.$$
Take \( x_{[\alpha]} = (1/\sqrt{m})\ell \) and note that
\[
d = \| A(\alpha, \alpha) \ell \|_2^2 = \ell^t (A(\alpha, \alpha) A(\alpha, \alpha)) \ell = \ell^t (t\ell) = mt.\]
Then a little calculation shows that
\[
(x^t_{[\alpha]} A(\alpha, \alpha) x_{\alpha})^2 = t.\]
Hence, condition (4) of Theorem 4.2 is satisfied. □

**Corollary 4.4.** Let \( G = (V, E) \) be a connected simple graph of order \( n \), and let \( U = \{1, 2, \ldots, m\} \) such that \( G[U] \) is a null graph. Then
\[
\max \frac{1}{2} \left[ \rho(G(U)) + \sqrt{\rho(G(U))^2 + 4 \sum_{k \in (m)} \sum_{j \in (n)} (k,j) \in E} \right] ^2 \leq \rho(G) \leq \frac{1}{2} \left[ \rho(G(U)) + \sqrt{\rho(G(U))^2 + 4 \rho(G[U])^2} \right],
\]
where the maximum is taken over all unit eigenvectors \( y = (y_{m+1}, y_{m+2}, \ldots, y_n)^t \) of \( A(G(U)) \) corresponding to \( \rho(G(U)) \). Moreover, when one of the two inequalities in (4.12) holds as equality, both hold as equality, and this happens if and only if the subvector \( x(U) \) of the Perron vector \( x \) of \( A(G) \) is an eigenvector of \( A(G(U)) \) corresponding to \( \rho(G(U)) \).

**Proof.** To obtain (4.12), apply Theorem 4.2 to the adjacency matrix \( A(G) \) of \( G \) and with \( \alpha = U \), noting that in this case, for any unit eigenvector \( x_{\alpha} \) of \( A(G[U]) \), we have
\[
\max_{x_{[\alpha]}} x_{[\alpha]}^t A[\alpha, \alpha] x_{(\alpha)} = \| A[\alpha, \alpha] x_{(\alpha)} \|,
\]
where \( x_{[\alpha]} \) has the same meaning as given in Theorem 4.2, as any unit vector of \( \mathbb{R}^m \) is an eigenvector of the zero matrix \( A(G[U]) \). The condition for equalities to hold in (4.12) follows from condition (3) of Theorem 4.2. □

**Example 4.5.** Let \( G = (V, E) \) be a bipartite graph. Then the vertex set \( V \) can be partitioned into two (disjoint) subsets \( V_1 \) and \( V_2 \) such that each edge in \( E \) joins a vertex in \( V_1 \) to a vertex in \( V_2 \). Take \( A = A(G) \) and \( \alpha = V_1 \). Then \( A[\alpha] \) and \( A(\alpha) \) are both zero matrices and condition (3) of Theorem 4.2 is satisfied trivially. So in this case the inequalities in (4.1) both hold as equality.

**Example 4.6.** Let \( R_m \) be an \( r_1 \)-regular simple graph of order \( m \) and \( R_{n-m} \) be an \( r_2 \)-regular simple graph of order \( n - m \), and suppose that \( R_m \) and \( R_{n-m} \) are vertex-disjoint. Denote by \( R_m \lor R_{n-m} \) the simple graph obtained from the union of \( R_m \) and \( R_{n-m} \) by adding new edges from each vertex of \( R_m \) to every vertex of \( R_{n-m} \). Considering the adjacency matrix \( A(R_m \lor R_{n-m}) \), and substituting the vertex set of \( R_m \) for \( \alpha \) in Theorem 4.2, we obtain the following:
\[ \frac{1}{2} \left[ r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4m(n-m)} \right] \leq \rho(R_m \vee R_{n-m}) \]
\[ \leq \frac{1}{2} \left[ r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4m(n-m)} \right]. \]

So \( \rho(R_m \vee R_{n-m}) \) is obtained exactly.

For a connected simple graph \( G = (V, E) \), let \( T(G) = \max_{i \in V} t_i \), where \( t_i \) is the 2-degree of the vertex \( i \). In [3, Theorem 1] Cao has obtained the following upper bound for \( \rho(G) \):
\[ \rho(G) \leq \sqrt{T(G)} \]
with equality if and only if \( G \) is either a regular graph or a semiregular bipartite graph (i.e., \( G \) is bipartite and all vertices in the same part of the bipartition of \( G \) have the same degrees). If \( G \) is the graph \( R_m \vee R_{n-m} (r_1 > 0, r_2 > 0) \), by Cao’s bound,
\[ \rho(G) < \sqrt{T(G)}. \]
So in some sense our bound is stronger than Cao’s. However, Cao’s bound is easier to determine than ours.

Example 4.7. Denote by \( C_n \) the cycle of length \( n \) with edges \{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}. Note that \( C_n \) is a 2-regular graph and \((1, \ldots, 1)^T/\sqrt{n} \in \mathbb{R}^n \) is the Perron vector of \( A(C_n) \). If \( n \) is even, then \( C_n \) is bipartite and, by Example 4.5, the inequalities in (4.1) both hold as equality for \( A = A(C_n) \) and \( \alpha = \{ j \in \langle n \rangle : j \text{ is odd} \} \). If \( n \) is a multiple of 3, then, by Corollary 4.4, the inequalities in (4.1) also hold as equality for \( A = A(C_n) \) and \( \alpha = \{ j \in \langle n \rangle : j \text{ is a multiple of 3} \} \). We contend that if \( n \) is not a multiple of 2 or 3, then the inequalities in (4.1) cannot hold as equality for \( A = A(C_n) \) and any choice of \( \emptyset \neq \alpha \subset \langle n \rangle \). Consider a positive integer \( n \) for which there exists \( \emptyset \neq \alpha \subset \langle n \rangle \) such that the inequalities in (4.1) hold as equality. By condition (3) of Theorem 4.2 it follows that for \( U = \alpha \) the subgraphs \( C_n[U] \) and \( C_n(U) \) are both regular. Since \( C_n \) is 2-regular, these subgraphs must be either 0-regular or 1-regular. If they are both 1-regular, then for each vertex \( r \) of \( U \), there is precisely one edge \( \{r, s\} \) with \( s \in \langle n \rangle \backslash U \). So the number of edges between \( U \) and \( \langle n \rangle \backslash U \) is \( |U| \). Interchanging \( U \) and \( \langle n \rangle \backslash U \) in the preceding argument, we find that the number of edges between \( U \) and \( \langle n \rangle \backslash U \) is also equal to \( n - |U| \). Hence, \( n \) is even. By a similar argument, we can also show that if \( C_n[U] \) and \( C_n(U) \) are both 0-regular, then \( n \) is even; if one of them is 0-regular and the other is 1-regular, then \( n \) is a multiple of 3. This establishes our claim.

Remark 4.8.
(i) In Theorem 4.2 the set \( \alpha \) is given. It is clear that we can obtain better lower and upper bounds for \( \rho(A) \) if \( \alpha \) is allowed to vary over all nonempty proper subsets of \( \langle n \rangle \). For some matrices \( A \) and for certain \( \alpha \), the inequalities in (4.1) can hold as equality (see Examples 4.5 and 4.6). But there are also matrices \( A \) for which the inequalities (4.1) cannot hold as equality for any choice of \( \alpha \) (see Example 4.7).
(ii) The following two points are clear from the proof of Theorem 4.2:

First, if the first inequality in (4.1) holds as equality, then the maximum value of the left side must be attained at $x_{(α)} = \pm x[α]/∥x[α]∥$ and $x_{(α)} = \pm x(α)/∥x(α)∥$. Second, the unit eigenvectors $x_{(α)}$, $x_{(α)}$ of condition (4), if they exist, are, respectively, $\pm x[α]/∥x[α]∥$ and $\pm x(α)/∥x(α)∥$.

(iii) In our proof of Theorem 4.2, (2) $⇒$ (3) or (1) $⇒$ (3), by considering the equality case for the inequality in (4.2) or the first inequality in (4.5), we obtain that $x(α)$ is an eigenvector of $A(α)$. To prove the condition that $x[α]$ is an eigenvector of $A[α]$, we replace $α$ by $|n\backslash α|$ and repeat the argument for $x(α)$. Actually, the latter condition can also be obtained by considering the equality case of the first inequality in (4.3) (and the equality case of the inequality in (4.2) and taking $u = x(α)/∥x(α)∥$) or the equality case of the two inequalities in (4.5) (noting that in this case $x_{(α)}$ must be $\pm x(α)/∥x(α)∥$ and making use of the last part of Lemma 4.1).

(iv) Condition (4) in Theorem 4.2 can be reformulated in one of the following ways:


(6) There exist a unit eigenvector of $A[α]$ corresponding to $ρ(A[α])$ and a unit eigenvector of $A(α)$ corresponding to $ρ(A(α))$ which are, respectively, a left singular vector and a right singular vector of $A[α, α]$ corresponding to its largest singular value, i.e., $\sqrt{ρ(A[α, α]A[α, α]^t)}$.

(7) The vectors $x[α]/∥x[α]∥$ and $x(α)/∥x(α)∥$ are, respectively, a left singular vector and a right singular vector of $A[α, α]$ corresponding to its largest singular value.

In view of part (ii) of this remark, it is clear that (5) follows from (4). Suppose that (5) holds. Note that the condition amounts to saying that $(x[α]/∥x[α]∥)^tA[α, α](x(α)/∥x(α)∥)$ is equal to the largest singular value of $A[α, α]$. By the standard theory of singular values, it follows that $A[α, α]$ (respectively, $A[α, α]^t$) takes $x(α)$ (respectively, $x[α]$) to a multiple of $x[α]$ (respectively, $x(α)$). But $A[α]x[α] = ρ(A)x[α] − A[α, α]x(α)$, so $x[α]$ is an eigenvector of $A[α]$, which necessarily corresponds to $ρ(A[α])$, as $x[α]$ is a positive vector. Similarly, $x(α)$ is also an eigenvector of $A(α)$ corresponding to $ρ(A(α))$. So condition (3) of Theorem 4.2 is satisfied.

By (4.6) and (4.7) (in the proof of Theorem 4.2), $A[α, α]x(α)$ and $A[α, α]x[α]$ are, respectively, positive multiples of $x[α]$ and $x(α)$. So condition (3) of Theorem 4.2 can also be reformulated, if desired.

**Theorem 4.9.** Let $A$ be an irreducible symmetric Z-matrix of order $n$. Then for any $∅ \neq α \subset \{n\}$,

$$
\frac{1}{2} \left[ λ_1(A[α]) + λ_1(A(α)) - \sqrt{(λ_1(A[α]) - λ_1(A(α)))^2 + 4ρ(A[α, α]A[α, α]^t)} \right] ≤ λ_1(A)
$$
\[
\leq \min_{x_{\alpha}, x_{(\alpha)}} \left\{ \frac{1}{2} \left[ \lambda_1(A[\alpha]) + \lambda_1(A(\alpha)) \right.ight.
\left. - \sqrt{\lambda_1(A[\alpha]) - \lambda_1(A(\alpha))}^2 + 4(x_{\alpha}^t A[\alpha, \alpha] x_{(\alpha)})^2 \right] \right\},
\]
where the minimum is taken over all unit eigenvectors \(x_{\alpha}, x_{(\alpha)}\) of \(A[\alpha], A(\alpha)\) corresponding to \(\lambda_1(A[\alpha]), \lambda_1(A(\alpha))\), respectively.

**Proof.** Write \(A\) as \(tI - B\) with \(B \geq 0\). Then \(B\) is irreducible, symmetric, \(\rho(B[\alpha]) = t - \lambda_1(A[\alpha])\), and the eigenvectors of \(B[\alpha]\) (respectively, \(B(\alpha)\)) corresponding to \(\rho(B[\alpha])\) (respectively \(\rho(B(\alpha))\)) are the same as those of \(A[\alpha]\) (respectively, \(A(\alpha)\)) corresponding to \(\lambda_1(A[\alpha])\) (respectively, \(\lambda_1(A(\alpha))\)). Applying Theorem 4.2 to \(B\), the result follows. □

**Corollary 4.10.** Let \(A\) be an irreducible symmetric singular \(M\)-matrix of order \(n\). Then for any \(\emptyset \neq \alpha \subset \langle n \rangle\),
\[
\max_{x_{\alpha}, x_{(\alpha)}} \left( x_{\alpha}^t A[\alpha, \alpha] x_{(\alpha)} \right)^2 \leq \lambda_1(A[\alpha]) \lambda_1(A(\alpha)) \leq \rho(A[\alpha, \alpha]),
\]
where the maximum is taken over all unit eigenvectors \(x_{\alpha}, x_{(\alpha)}\) of \(A[\alpha], A(\alpha)\) corresponding to \(\lambda_1(A[\alpha]), \lambda_1(A(\alpha))\), respectively.

**Proof.** The result follows from Theorem 4.9 since \(\lambda_1(A) = 0\). □

**Corollary 4.11.** Let \(G = (V, E)\) be a connected weighted graph of order \(n\), and let \(L\) be its general Laplacian matrix. If \(G\) has loops, then
\[
\lambda_1(L) \leq \sum_{i \in \langle n \rangle} \frac{w[i, i]^2}{\sum_{i \in \langle n \rangle} w[i, i]},
\]

**Proof.** Let \(G' = (V \cup \{n + 1\}, E \cup E_1)\) be the weighted graph obtained from \(G\) by adding a new vertex \(n + 1\) and some edges between \(n + 1\) and vertices of \(G\) such that \(e = (k, n + 1) \in E_1\) if and only if there exists a loop of \(G\) incident with the vertex \(k\). Give \(e\) the weight as that of the loop at \(k\). Let \(\bar{G}\) be the weighted graph obtained from \(G'\) by deleting all its loops. It is clear that \(\bar{G}\) is connected, loopless. Let \(\alpha = \langle n \rangle\). Then \(L(\bar{G})[\alpha] = L\), and \(L(\bar{G})[\alpha]\) is a matrix of order 1, which is \(\sum_{i \in \langle n \rangle} w[i, i]\). Applying Corollary 4.10 to the singular \(M\)-matrix \(L(\bar{G})\), we have
\[
\lambda_1(L) \sum_{i \in \langle n \rangle} w[i, i] \leq \sum_{i \in \langle n \rangle} w[i, i]^2.
\]
The result follows. □

As in Theorem 4.2, we can also give the equivalent conditions for the inequalities in Theorem 4.9 (Corollaries 4.10 and 4.11) to hold as equality. We omit the details.
Note that a zero matrix of order 1 is considered to be irreducible in the following.

**Lemma 4.12.** Let $A$ be an irreducible symmetric nonnegative matrix of order $n$. Then $\rho_1(A) < \rho_2(A) < \cdots < \rho_n(A) = \rho(A)$.

**Proof.** It suffices to prove that for any $k = 1, 2, \ldots, n - 1$, if $B$ is a $k \times k$ principal submatrix of $A$, then there always exists a $(k + 1) \times (k + 1)$ principal submatrix $C$ of $A$ such that $\rho(C) > \rho(B)$. Consider the matrix $A$ to be the adjacency matrix of some weighted connected graph $G$. Clearly there exists an irreducible principal submatrix $B_1$ of $B$ such that $\rho(B) = \rho(B_1)$. So $B_1$ is the adjacency matrix of a connected weighted subgraph $G_1$ of $G$. Since $G$ is connected, there exists a connected weighted subgraph $G_2$ of $G$ with order $k + 1$ which contains $G_1$ as a weighted subgraph. Let $C$ be the adjacency matrix of $G_2$. Then $C$ is a $(k + 1) \times (k + 1)$ principal submatrix of $A$, is irreducible and contains $B_1$ as a proper principal submatrix; so $\rho(C) > \rho(B_1) = \rho(B)$. □

It may be of interest to note that Lemma 4.12 no longer holds if $A$ is not symmetric. For instance, consider

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$  

**Lemma 4.13.** Let $A$ be an irreducible symmetric nonnegative matrix of order $n$, and let $s$ be an integer with $2 \leq s \leq n$. Then for any $\emptyset \neq \alpha \subset (n)$ with $|\alpha| < s$,

$$\rho_i(B[\alpha]) - |\alpha| (\tilde{A}_s(\alpha)) \leq \rho_i(A) < \rho_i(B[\alpha]) - |\alpha| + 1 (\tilde{A}_s(\alpha)),$$

where $B = \rho_i(A)I - A$, $\tilde{A}_s(\alpha) = A(\alpha) + A(\alpha, \alpha)(\rho_i(A)I - A[\alpha])^{-1}A[\alpha, \alpha]$, and for any principal submatrix $B[\alpha]$ of $B$, $i(B[\alpha])$ has the same meaning as that given in Theorem A.

**Proof.** By definition, $B \in L_s$. Since $|\alpha| < s$, we have $\rho(A[\alpha]) < \rho_i(A)$ by Lemma 4.12. So $B[\alpha]$ is a nonsingular $M$-matrix, and $B/B[\alpha] \in L_{i(B[\alpha]) - |\alpha|}$ by Theorem A. Also, we have

$$B/B[\alpha] = B(\alpha) - B(\alpha, \alpha)B[\alpha]^{-1}B[\alpha, \alpha] = \rho_i(A)I - A(\alpha) - A(\alpha, \alpha)(\rho_i(A)I - A[\alpha])^{-1}A[\alpha, \alpha] = \rho_i(A)I - \tilde{A}_s(\alpha),$$

where $\tilde{A}_s(\alpha) \geq 0$ as $B[\alpha]^{-1} \geq 0$. The result now follows from the definition of $L_k$. □
Theorem 4.14. Let $A$ be an irreducible symmetric nonnegative matrix of order $n$, let $s$ be an integer with $2 \leq s \leq n$, and let $\alpha$ be a nonempty proper subset of $\langle n \rangle$ with $|\alpha| < s$. Then
\[
\rho_s(A) \geq \frac{1}{2} \left[ \rho_{s-|\alpha|}(A(\alpha)) + \rho(A[\alpha]) \right]
\]
\[
+ \sqrt{\left( \rho_{s-|\alpha|}(A(\alpha)) - \rho(A[\alpha]) \right)^2 + 4b_\alpha},
\]
where $b_\alpha = \min_{\beta \subseteq \langle n \rangle \setminus \alpha, |\beta| = s - |\alpha|} \max\{ (x_{[\alpha]}^1 A[\alpha, \beta] x_{[\beta]})^2 : x_{[\alpha]} \text{ and } x_{[\beta]} \text{ are, respectively, unit eigenvectors of } A[\alpha] \text{ and } A[\beta] \text{ corresponding to } \rho(A[\alpha]) \text{ and } \rho(A[\beta]) \}$. 

Proof. For any $\alpha \subseteq \langle n \rangle$ with $1 \leq |\alpha| < s$, by definition, $i((\rho_s(A)I-A)[\alpha]) \geq s$, and by Lemma 4.13 we have
\[
\rho_s(A) \geq \rho((\rho_s(A)I-A)[\alpha]) = A(\alpha) + A[\alpha, \alpha] (\rho_s(A)I - A[\alpha])^{-1} A[\alpha, \alpha]
\]
\[
= \max_{\beta \subseteq \langle n \rangle \setminus \alpha} \rho(A[\beta] + A[\alpha, \beta]) (\rho_s(A)I - A[\alpha])^{-1} A[\alpha, \beta]
\]
\[
\geq \max_{\beta \subseteq \langle n \rangle \setminus \alpha} \rho(A[\beta]) + \max_{x_{[\alpha]} \cdots x_{[\beta]}} (x_{[\alpha]}^1 A[\alpha, \beta] x_{[\beta]})^2 \frac{1}{\rho_s(A) - \rho(A[\alpha])}
\]
\[
= \rho_{s-|\alpha|}(A(\alpha)) + \frac{1}{\rho_s(A) - \rho(A[\alpha])} b_\alpha,
\]
where the third inequality above can be obtained by the argument used in establishing (4.5) in the proof of Theorem 4.2, and the vectors $x_{[\alpha]}$, $x_{[\beta]}$ that qualify the max in the inequality are unit eigenvectors of $A[\alpha]$, $A[\beta]$ corresponding to $\rho(A[\alpha])$ and $\rho(A[\beta])$, respectively. Solving the resulting quadratic inequality for $\rho_s(A)$, we obtain the desired lower bound for $\rho_s(A)$. \(\Box\)

For $s = 1, 2, \ldots, n$, denote by $\tau_s(A)$ the minimal eigenvalue of all $s \times s$ principal submatrices of a real symmetric matrix of $A$ of order $n$.

Theorem 4.15. Let $A$ be an irreducible symmetric $Z$-matrix of order $n$, let $s$ be an integer with $2 \leq s \leq n$, and let $\alpha$ be a nonempty proper subset of $\langle n \rangle$ with $|\alpha| < s$. Then
\[ \tau_s(A) \leq \frac{1}{2} \left[ \tau_{s-|\alpha|}(A(\alpha)) + \lambda_1(A[\alpha]) \right] - \sqrt{\left( \tau_{s-|\alpha|}(A(\alpha)) - \lambda_1(A[\alpha]) \right)^2 + 4c_\alpha}, \]

where \( c_\alpha = \min_{\beta \subseteq \langle n \rangle \setminus \alpha, |\beta| = s - |\alpha|} \max \{ (x_\alpha^1 A[\alpha, \beta] x_\beta^1)^2 : x_\alpha, x_\beta \text{ are, respectively, unit eigenvectors of } A[\alpha] \text{ and } A[\beta] \text{ corresponding to } \lambda_1(A[\alpha]) \text{ and } \lambda_1(A[\beta]) \}. \]

**Proof.** Write \( A \) in the form \( tI - B \) with \( B \geq 0 \). For any \( \emptyset \neq \alpha \subseteq \langle n \rangle \), \( \rho_s(B[\alpha]) = t - \tau_s(A[\alpha]) \). The result follows from Theorem 4.14. \( \square \)

Finally, we note that the bounds for \( \rho(A) (= \rho_n(A)) \) and \( \lambda_1(A) (= \tau_n(A)) \) as given by Theorems 4.14 and 4.15 agree with those as given, respectively, by Theorems 4.2 and 4.9. Also, the inequality in Theorem 4.14 (respectively, Theorem 4.15) holds as equality for any choice of \( s \) and \( \alpha \) if we take \( A \) to be \( J \) (respectively, \( tI - J \)).

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References