

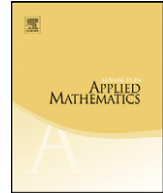


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# Stability of the reverse Blaschke–Santaló inequality for zonoids and applications

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## ABSTRACT

An important  $GL(n)$  invariant functional of centred (origin symmetric) convex bodies that has received particular attention is the volume product. For a centred convex body  $A \subset \mathbb{R}^n$  it is defined by  $P(A) := |A| \cdot |A^*|$ , where  $|\cdot|$  denotes volume and  $A^*$  is the polar body of  $A$ . If  $A$  is a centred zonoid, then it is known that  $P(A) \geq P(C^n)$ , where  $C^n$  is a centred affine cube, i.e. a Minkowski sum of  $n$  linearly independent centred segments. Equality holds in the class of centred zonoids if and only if  $A$  is a centred affine cube. Here we sharpen this uniqueness statement in terms of a stability result by showing in a quantitative form that the Banach–Mazur distance of a centred zonoid  $A$  from a centred affine cube is small if  $P(A)$  is close to  $P(C^n)$ . This result is then applied to strengthen a uniqueness result in stochastic geometry.

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## 1. Introduction

In the preceding two decades, affine or  $GL(n)$  invariant functionals of convex bodies have been studied intensively. A major reason for interest in such functionals is due to the observation that they often lead to strong geometric and analytic inequalities (cf. [2,11,12,32–34]). Another incentive comes from tantalising open problems in this field. The Petty projection inequality and Mahler's inequality

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are just two examples of conjectured sharp inequalities which remain to be established; see the survey articles [19,31,55] and the introduction of [33] for further information and relations to important analytic inequalities [5–7].

A fundamental  $GL(n)$  invariant functional is the volume product. For a centred convex body  $A \subset \mathbb{R}^n$ , i.e. an  $n$ -dimensional compact convex set which is symmetric with respect to the origin, it is defined by

$$P(A) := |A| \cdot |A^*|,$$

where  $|\cdot|$  denotes volume and  $A^*$  is the polar body of  $A$ . (We refer to the beginning of Section 2 for some definitions and further background information.) Since  $P$  is continuous and  $GL(n)$  invariant, it attains its maximum and minimum. The maximum of  $P$  is provided by the famous Blaschke–Santaló inequality. For centred convex bodies  $A \subset \mathbb{R}^n$  it states that

$$P(A) \leq P(B_2^n), \quad (1)$$

where  $B_2^n$  denotes the Euclidean unit ball centred at the origin  $o \in \mathbb{R}^n$ . Equality holds in (1) if and only if  $A$  is a centred ellipsoid; see e.g. [26,31,35,41,44,48,49].

A major open problem, which became known as Mahler's conjecture or as the (conjectured) reverse Blaschke–Santaló inequality, asks for a sharp lower bound for  $P$ . This problem originates from a paper by K. Mahler [37]. The Mahler conjecture for centred convex bodies states that

$$P(A) \geq P(C^n), \quad (2)$$

for all centred convex bodies  $A \subset \mathbb{R}^n$ , where  $C^n$  is a centred affine cube. This inequality has been established in full generality for  $n = 2$  by Mahler [38]; see [55] for a recent exposition of Mahler's proof. Equality holds for  $n = 2$  if and only if  $A$  is a parallelogram. The latter fact has been proved by Reisner [45,46], who established the conjecture for the class of centred zonoids in general dimensions. It turned out that the centred affine cubes are the only centred zonoids for which equality holds in (2). Reisner's approach is based on previous work of Schneider [50] on the average number of vertices of random polyhedral sets, and also on a purely geometric estimate for the number of vertices of centrally symmetric polytopes, due to Bárány and Lovász [4]. Later Gordon, Meyer and Reisner [22] found a very elegant new proof in the case of zonoids.

Mahler's conjecture is settled for specific classes of convex bodies, such as unconditional convex bodies (cf. [39,47,48]), or centred polytopes having at most  $2n + 2$  vertices or at most  $2n + 2$  facets if  $3 \leq n \leq 8$ ; see [36] (cf. also [3]). The discussion of the equality cases of (2) for these specific classes of sets already indicates that it will be difficult to determine all cases in which equality holds in (2). Strong functional versions of the Blaschke–Santaló inequality and its reverse form have been studied recently [1,15–18]. In [10], Bourgain and Milman proved that  $P(A) \geq c^n \cdot P(B_2^n)$ , where  $c$  is a universal constant (cf. [30]). Very recently, Nazarov et al. (see [43]) showed that centred affine cubes are local minimisers for the volume product, but the general conjecture is still open. This is also true for the nonsymmetric version of Mahler's conjecture (see [40,42] for particular cases).

The main purpose of the present work is to establish a strengthening of the estimate (2) in terms of a stability result for centred zonoids, and to apply this to a problem in stochastic geometry. In this context, a stability result provides an improved lower bound for  $P(A)$  if some information on the distance between  $A$  and the extremal bodies of (2) (yielding equality in (2)) is available (cf. [23] for an introduction to geometric stability results). Equivalently, it implies that if  $P(A)$  is close to the lower bound  $P(C^n)$ , then  $A$  must be close to some centred affine cube. In order to measure the distance between centred convex bodies, a suitable  $GL(n)$  invariant notion of distance for centred convex bodies  $K, M \subset \mathbb{R}^n$  is

$$\delta_{\text{BM}}(K, M) = \min\{\lambda \geq 1 : K \subset T(M) \subset \lambda K \text{ for some } T \in GL(n)\},$$

the (symmetric) Banach–Mazur distance of  $K$  and  $M$ . Note that  $\delta_{\text{BM}}(K, M) \geq 1$  with equality if and only if  $K$  and  $M$  differ only by a linear transformation.

**Theorem 1.1.** *There is a constant  $\gamma_n$ , depending only on the dimension  $n$ , such that the following is true. If  $A \subset \mathbb{R}^n$  is a centred zonoid satisfying*

$$P(A) \leq (1 + \varepsilon) \cdot P(C^n)$$

for some  $\varepsilon \in [0, 1]$ , then

$$\delta_{\text{BM}}(A, C^n) \leq 1 + \gamma_n \cdot \varepsilon^{\frac{1}{n}}.$$

The order of the stability estimate in Theorem 1.1, which is  $\frac{1}{n}$ , can be improved to the optimal order 1 by combining our global result with the local result in [43]. The main result of [43] states the existence of a constant  $\delta_0(n) > 0$ , depending only on  $n$ , such that the following is true. If  $A \subset \mathbb{R}^n$  is a centred convex body with  $\delta_{\text{BM}}(A, C^n) \leq 1 + \delta_0(n)$ , then  $P(A) \geq P(C^n)$  with equality if and only if  $A$  is a centred affine cube. An inspection of the arguments in [43] shows that the paper implicitly contains a proof of the following stronger local statement.

There exist constants  $\delta_0(n) > 0$  and  $\beta_n$ , depending only on  $n$ , such that the following is true. If  $A \subset \mathbb{R}^n$  is a centred convex body with  $\delta_{\text{BM}}(A, C^n) \leq 1 + \delta_0(n)$ , then

$$P(A) \geq (1 + \beta_n(\delta_{\text{BM}}(A, C^n) - 1))P(C^n).$$

Now we put  $\varepsilon_0(n) := \min\{(\delta_0(n)/\gamma_n)^n, 1\}$  and choose  $\varepsilon \in [0, \varepsilon_0(n)]$ . If  $A \subset \mathbb{R}^n$  is a centred zonoid such that

$$P(A) \leq (1 + \varepsilon) \cdot P(C^n), \quad (3)$$

then by Theorem 1.1, we have

$$\delta_{\text{BM}}(A, C^n) \leq 1 + \gamma_n \cdot \varepsilon^{\frac{1}{n}} \leq 1 + \delta_0(n).$$

Hence the result in [43] shows that indeed

$$\delta_{\text{BM}}(A, C^n) \leq 1 + \beta_n \cdot \varepsilon. \quad (4)$$

We put  $\alpha_n := \max\{\beta_n, \sqrt{n}/\varepsilon_0(n)\}$ . Then (4) holds if (3) is satisfied for some  $\varepsilon \in [0, \varepsilon_0(n)]$ . On the other hand, if  $\varepsilon \geq \varepsilon_0(n)$ , then

$$\delta_{\text{BM}}(A, C^n) \leq \sqrt{n} \leq 1 + \alpha_n \cdot \varepsilon_0(n) \leq 1 + \alpha_n \cdot \varepsilon.$$

Any explicit estimate for  $\delta_0(n)$  and  $\beta_n$  thus will immediately imply an explicit estimate for the constant  $\alpha_n$ . At the end of Section 3, we comment on the size of  $\gamma_n$ , for which a rough estimate follows from the proof of Theorem 1.1.

**Corollary 1.2.** *There is a constant  $\alpha_n$ , depending only on the dimension  $n$ , such that the following is true. If  $A \subset \mathbb{R}^n$  is a centred zonoid satisfying*

$$P(A) \leq (1 + \varepsilon) \cdot P(C^n)$$

for some  $\varepsilon \geq 0$ , then

$$\delta_{\text{BM}}(A, C^n) \leq 1 + \alpha_n \cdot \varepsilon.$$

Introducing  $\tilde{\delta}_{\text{BM}}(A, C^n) := \ln \delta_{\text{BM}}(A, C^n)$ , we can restate the assertion of Corollary 1.2 in the following form: There exists a constant  $\tilde{\alpha}_n$ , depending only on the dimension  $n$ , such that, for any centred zonoid  $A \subset \mathbb{R}^n$ ,

$$P(A) \geq (1 + \tilde{\alpha}_n \cdot \tilde{\delta}_{\text{BM}}(A, C^n)) \cdot P(C^n).$$

A general stability result for the Blaschke–Santaló inequality has recently been found in [8], for symmetric and for not necessarily symmetric convex bodies.

Stability results for various geometric inequalities have been applied to the solution of generalised versions and variants of Kendall’s problem in stochastic geometry (see e.g. [27,29]). In the present paper, we apply the stability results for the volume product of zonoids to generalise known uniqueness results in stochastic geometry. More specifically, we consider the zero cell  $Z_0$  of a stationary Poisson hyperplane mosaic  $\widehat{X}$  in  $\mathbb{R}^n$ . The expected number of vertices of  $Z_0$  is known to be minimal if  $\widehat{X}$  is a parallel mosaic, and maximal if  $\widehat{X}$  is isotropic. In Section 5, we show in a quantitative form that  $\widehat{X}$  is close to a parallel mosaic if the expected number of vertices of  $Z_0$  is almost minimal. A similar result is obtained for the upper bound.

Our proof of Theorem 1.1 is based on and refines the inductive argument in [22]. Section 2 contains a geometric stability result for [22, Lemma 3], in Section 3 the induction step is carried out. The geometric structure of centred zonoids which are close to a centred affine cube is described in Section 4.

## 2. An auxiliary stability estimate

The approach of this and the next section is based on the arguments provided by Gordon, Meyer and Reisner in [22]. Therefore, we partly use the notation of that paper. Although the statement of our main result is  $\text{GL}(n)$  invariant, we introduce an auxiliary Euclidean structure. Thus the general setting is the Euclidean space  $\mathbb{R}^n$  with scalar product  $\langle \cdot, \cdot \rangle$  and induced norm  $\| \cdot \|$ . The Euclidean unit ball is denoted by  $B_2^n$ , its boundary  $\mathbb{S}^{n-1}$  is the unit sphere. Let  $e_1, \dots, e_n$  be the Euclidean standard basis of  $\mathbb{R}^n$ . The convex hull of points  $x_1, \dots, x_k \in \mathbb{R}^n$  is denoted by  $[x_1, \dots, x_k]$ .

We write  $B_\infty^n := [-e_1, e_1] + \dots + [-e_n, e_n]$  for the centred Euclidean cube having edge length 2. As usual, a set  $A \subset \mathbb{R}^n$  is called centred if  $A = -A$ . A (centred) zonotope is a Minkowski sum of (centred) segments. The class of (centred) zonoids is obtained as the closure of the class of (centred) zonotopes in the space of convex bodies with the Hausdorff metric. Hence each (centred) zonoid is a limit of (centred) zonotopes. For an introduction to zonoids, we refer to [51, Section 3.5]; see also the surveys by Schneider and Weil [53] and by Goodey and Weil [21].

For a centred convex body  $A \subset \mathbb{R}^n$ , the polar body  $A^*$  is defined by

$$A^* := \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \text{ for all } a \in A\},$$

which is again a centred convex body. Clearly, the polar body of  $B_\infty^n$  is the regular crosspolytope  $B_1^n := [\pm e_1, \dots, \pm e_n]$ .

To simplify notation, we write  $|A|$  for the relative volume of a compact, convex set  $A \subset \mathbb{R}^n$  relative to its affine hull, which is zero for the empty set. The support function of a nonempty compact, convex set  $A \subset \mathbb{R}^n$  is  $h(A, u) := \max\{\langle x, u \rangle : x \in A\}$ , for  $u \in \mathbb{R}^n$ .

First, we establish an improvement of [22, Lemma 3] in terms of a stability result. Given a centred convex body  $B$  in  $\mathbb{R}^n$  and  $x \in \mathbb{S}^{n-1}$ , we write  $B(x)$  for the central section  $B \cap x^\perp$  of  $B$  with the orthogonal complement  $x^\perp$  of  $x$ .

For a hyperplane  $H$  in  $\mathbb{R}^n$  orthogonal to  $x \in \mathbb{S}^{n-1}$  with  $B \cap H \neq \emptyset$ , let  $B_2(H)$  denote the  $(n - 1)$ -dimensional Euclidean ball which is contained in  $H$ , centred at  $H \cap \mathbb{R}x$ , and has the same  $(n - 1)$ -dimensional volume as  $B \cap H$ . The union of all such balls  $B_2(H)$  is again a convex body which is called the Schwarz rounding of  $B$  with respect to the line  $\mathbb{R}x$  (cf. [25, p. 178]). Let  $C_x$  be the centred double cone with  $C_x \cap x^\perp = B_x \cap x^\perp$ ,  $|C_x| = |B| = |B_x|$  and rotational symmetry around the line  $\mathbb{R}x$ .

In particular, if  $B_x$  is not a double cone, then there exists some  $t_0 > 0$  such that if  $y \in B_x \setminus C_x$ , then  $|\langle x, y \rangle| < t_0$ , and if  $y \in C_x \setminus B_x$ , then  $|\langle x, y \rangle| > t_0$ . The importance of  $C_x$  is explained by the formula

$$\int_{C_x} |\langle x, y \rangle| dy = \frac{n}{2(n+1)} \cdot \frac{|B|^2}{|B(x)|}.$$

It follows that

$$\begin{aligned} \frac{n}{2(n+1)} \cdot \frac{|B|^2}{|B(x)|} - \int_B |\langle x, y \rangle| dy &= \int_{C_x} |\langle x, y \rangle| dy - \int_{B_x} |\langle x, y \rangle| dy \\ &\geq t_0 V(C_x \setminus B_x) - t_0 V(B_x \setminus C_x) = 0, \end{aligned}$$

where the inequality is strict, if  $B$  is not a double cone. This is just a restatement of [22, Lemma 3] by Gordon, Meyer and Reisner.

In order to obtain a stability version of [22, Lemma 3], we introduce

$$\Delta_x(B) := \frac{|C_x \setminus B_x| + |B_x \setminus C_x|}{|B|}$$

as a very rough measure for quantifying the distance  $\Delta_x(B)$  of the associated symmetral  $B_x$  of  $B$  from the double cone  $C_x$  with common base  $B(x)$ . Clearly, the numerator is the symmetric difference metric of  $B_x$  and  $C_x$ , and the ratio is scaling invariant. Moreover, in the present situation we have  $|B_x \setminus C_x| = |C_x \setminus B_x|$ , and thus  $\Delta_x(B) = 2|C_x \setminus B_x|/|B|$ .

Observe that in the statement of the following lemma, the integral on the left-hand side and the ratio  $|B|^2/|B(x)|$  on the right-hand side remain unchanged if  $B$  is replaced by its Schwarz rounding  $B_x$ .

**Lemma 2.1.** *If  $B$  is a centred convex body in  $\mathbb{R}^n$  and  $x \in \mathbb{S}^{n-1}$ , then*

$$\int_B |\langle x, y \rangle| dy \leq \left[ 1 - \frac{1}{8} \Delta_x(B) \right] \cdot \frac{n}{2(n+1)} \cdot \frac{|B|^2}{|B(x)|}.$$

**Proof.** If  $B_x$  is a double cone (equivalently,  $B$  is a double cone), then  $C_x = B_x$ ,  $\Delta_x(B) = 0$  and the asserted inequality holds with equality. Hence, we may assume that  $B_x$  (i.e.  $B$ ) is not a double cone. Let

$$h := \frac{n|B|}{2|B(x)|}$$

denote the height of the “upper half” (with respect to  $x$ ) of  $C_x$ . Thus we have  $h > h(B, x)$ . We define

$$\begin{aligned} \varrho(t) &:= |(tx + x^\perp) \cap B|^{\frac{1}{n-1}}, \\ \tilde{\varrho}(t) &:= |(tx + x^\perp) \cap C_x|^{\frac{1}{n-1}} = \left( 1 - \frac{t}{h} \right) |B(x)|^{\frac{1}{n-1}}, \end{aligned}$$

for  $t \in [0, h]$ . Note that by definition  $\varrho(t) = 0$  for  $t \in (h(B, x), h)$ . The definitions immediately imply that

$$\Delta_x(B) = \frac{2}{|B|} \int_0^h |\varrho(t)^{n-1} - \tilde{\varrho}(t)^{n-1}| dt.$$

The Brunn–Minkowski inequality yields that  $\varrho(t)$  is concave on  $[0, h(B, x)]$ , and hence the same is true for  $\varrho(t) - \tilde{\varrho}(t)$  on  $[0, h(B, x)]$ . We observe that  $\varrho(0) = \tilde{\varrho}(0)$  and

$$\int_0^h \varrho(t)^{n-1} dt = \frac{1}{2}|B| = \frac{1}{2}|C_x| = \int_0^h \tilde{\varrho}(t)^{n-1} dt.$$

Since  $B_x$  is not a double cone and the line  $(\tilde{\varrho}(t), t)$ ,  $t \in \mathbb{R}$ , meets the interior of  $B_x$ , there exists a unique  $t_0 \in (0, h)$  such that  $\varrho(t) > \tilde{\varrho}(t)$  if  $t \in (0, t_0)$ , and  $\varrho(t) < \tilde{\varrho}(t)$  if  $t \in (t_0, h)$ . It also follows that

$$\int_0^{t_0} (\varrho(t)^{n-1} - \tilde{\varrho}(t)^{n-1}) dt = \int_{t_0}^h (\tilde{\varrho}(t)^{n-1} - \varrho(t)^{n-1}) dt = \frac{1}{4} \Delta_x(B) |B|.$$

We put  $\varphi(s) := \varrho(s) - \tilde{\varrho}(s)$ ,  $s \in [0, t_0]$ , and recall that  $\varphi$  is concave on  $[0, t_0]$ ,  $\varphi(0) = \varphi(t_0) = 0$  and  $\varphi(s) > 0$  on  $(0, t_0)$ . For  $\lambda \in (0, 1)$ ,  $s \in [0, t_0]$  and  $t = \lambda s$ , we get

$$\begin{aligned} \varrho(t) - \tilde{\varrho}(t) &= \varphi(\lambda s) = \varphi((1 - \lambda)0 + \lambda s) \\ &\geq (1 - \lambda)\varphi(0) + \lambda\varphi(s) = \lambda\varphi(s) \\ &= \lambda(\varrho(s) - \tilde{\varrho}(s)). \end{aligned}$$

Moreover, since  $\varrho$  and  $\tilde{\varrho}$  are decreasing, we conclude that

$$\begin{aligned} \varrho(t)^{n-1} - \tilde{\varrho}(t)^{n-1} &= (\varrho(t) - \tilde{\varrho}(t)) \left( \sum_{i=0}^{n-2} \varrho(t)^i \tilde{\varrho}(t)^{n-2-i} \right) \\ &\geq \lambda(\varrho(s) - \tilde{\varrho}(s)) \left( \sum_{i=0}^{n-2} \varrho(s)^i \tilde{\varrho}(s)^{n-2-i} \right) \\ &= \lambda(\varrho(s)^{n-1} - \tilde{\varrho}(s)^{n-1}). \end{aligned}$$

Therefore

$$\int_0^{\lambda t_0} (\varrho(t)^{n-1} - \tilde{\varrho}(t)^{n-1}) dt \geq \lambda^2 \int_0^{t_0} (\varrho(t)^{n-1} - \tilde{\varrho}(t)^{n-1}) dt = \lambda^2 \frac{1}{4} \Delta_x(B) |B|. \tag{5}$$

On the other hand, the component of  $C_x \setminus B_x$  containing the apex  $hx$  is starshaped with respect to  $hx$ , and hence

$$\int_{(1-\lambda)h+\lambda t_0}^h (\tilde{\varrho}(t)^{n-1} - \varrho(t)^{n-1}) dt \geq \lambda^n \int_{t_0}^h (\tilde{\varrho}(t)^{n-1} - \varrho(t)^{n-1}) dt = \lambda^n \frac{1}{4} \Delta_x(B) |B|. \tag{6}$$

We define  $t_1 \in (0, t_0)$  and  $t_2 \in (t_0, h)$  such that

$$\int_0^{t_1} (\varrho(t)^{n-1} - \tilde{\varrho}(t)^{n-1}) dt = \int_{t_2}^h (\tilde{\varrho}(t)^{n-1} - \varrho(t)^{n-1}) dt = \frac{1}{4} \cdot \frac{\Delta_x(B)}{4} |B|.$$

From (5) and (6) it follows that

$$t_1 \leq \frac{1}{2}t_0 \leq \left(1 - \frac{1}{n}\right)t_0 \quad \text{and} \quad t_2 \geq \left(1 - \frac{1}{n}\right)t_0 + \frac{1}{n}h,$$

and hence

$$t_2 - t_1 \geq \frac{h}{n}.$$

We put

$$\tilde{\varphi}(t) := \tilde{Q}(t)^{n-1} - Q(t)^{n-1}.$$

Then

$$\begin{aligned} \int_{t_1}^{t_0} |\tilde{\varphi}(t)| dt &= \frac{3}{4} \cdot \frac{\Delta_x(B)}{4} |B| = \int_{t_0}^{t_2} \tilde{\varphi}(t) dt, \\ \int_0^{t_1} |\tilde{\varphi}(t)| dt &= \int_{t_2}^h \tilde{\varphi}(t) dt, \end{aligned}$$

and hence

$$\frac{n}{2(n+1)} \frac{|B|^2}{|B(x)|} - \int_B |\langle x, y \rangle| dy = \int_{C_x} |\langle x, y \rangle| dy - \int_{B_x} |\langle x, y \rangle| dy \tag{7}$$

$$= 2 \int_0^h t \tilde{\varphi}(t) dt \tag{8}$$

$$\begin{aligned} &\geq 2t_2 \int_{t_2}^h \tilde{\varphi}(t) dt + 2t_0 \int_{t_0}^{t_2} \tilde{\varphi}(t) dt \\ &\quad - 2t_0 \int_{t_1}^{t_0} |\tilde{\varphi}(t)| dt - 2t_1 \int_0^{t_1} |\tilde{\varphi}(t)| dt \end{aligned}$$

$$\geq \frac{2h}{n} \int_{t_2}^h \tilde{\varphi}(t) dt \geq \frac{h \cdot \Delta_x(B) |B|}{8n}$$

$$= \frac{\Delta_x(B) |B|^2}{16 |B(x)|},$$

from which we obtain the required estimate.  $\square$

**Remark.** The order of the error term is optimal, as (8) in the argument above implies that

$$\frac{n}{2(n+1)} \frac{|B|^2}{|B(x)|} - \int_B |\langle x, y \rangle| dy \leq 2h \int_{t_0}^h \tilde{\varphi}(t) dt = h|C_x \setminus B_x| = \frac{n\Delta_x(B)|B|^2}{4|B(x)|},$$

which in turn yields

$$\int_B |\langle x, y \rangle| dy \geq [1 - n\Delta_x(B)] \cdot \frac{n}{2(n+1)} \cdot \frac{|B|^2}{|B(x)|}.$$

**3. The induction step**

In order to prove the theorem, we use induction on  $n$ , thus following the approach in [22].

The statement to be proved is the following: There is a constant  $\gamma_n$ , depending only on the dimension  $n$ , such that

$$P(A) \leq (1 + \varepsilon) \cdot P(C^n),$$

for a centred zonoid  $A \subset \mathbb{R}^n$  and some  $\varepsilon \in [0, 1]$ , implies that

$$\delta_{BM}(A, C^n) \leq 1 + \gamma_n \cdot \varepsilon^{\frac{1}{n}}.$$

If  $n = 1$  this trivially holds with  $\gamma_1 = 0$ .

Assume that the assertion of the theorem is true in any  $(n - 1)$ -dimensional Euclidean space. Let  $A$  be a centred zonoid for which

$$P(A) \leq (1 + \varepsilon) \cdot P(C^n), \tag{9}$$

where  $\varepsilon \in [0, 1]$  and  $C^n$  is a centred affine cube.

In the following, we denote by  $\text{pr}(A, x^\perp)$  the orthogonal projection of  $A$  to the subspace  $x^\perp$ . Lemma 1 by Gordon, Meyer and Reisner [22] yields a vector  $x \in \mathbb{S}^{n-1}$  such that

$$2|A^*| \cdot |\text{pr}(A, x^\perp)| \leq (n + 1)|A| \int_{A^*} |\langle x, y \rangle| dy.$$

We define  $A^*(x) := A^* \cap x^\perp$ . It follows from Lemma 2.1 that

$$2|A^*| \cdot |\text{pr}(A, x^\perp)| \leq (n + 1)|A| \cdot \left(1 - \frac{1}{8}\Delta_x(A^*)\right) \cdot \frac{n}{2(n+1)} \cdot \frac{|A^*|^2}{|A^*(x)|}.$$

Using that  $(\text{pr}(A, x^\perp))^* = A^*(x)$ , where the polar body on the left-hand side is taken with respect to the Euclidean structure induced on  $x^\perp$ , we get

$$\left(1 + \frac{1}{8}\Delta_x(A^*)\right) \cdot \frac{4}{n} \cdot P(\text{pr}(A, x^\perp)) \leq P(A). \tag{10}$$

From this we deduce that



$$\begin{aligned} \left(1 + \frac{1}{8} \Delta_x(A^*)\right) \cdot \frac{4}{n} \cdot P(B_\infty^{n-1}) &\leq \left(1 + \frac{1}{8} \Delta_x(A^*)\right) \cdot \frac{4}{n} \cdot P(\text{pr}(A, x^\perp)) \\ &\leq P(A) \leq (1 + \varepsilon) \cdot P(B_\infty^n) \\ &= (1 + \varepsilon) \cdot \frac{4}{n} \cdot P(B_\infty^{n-1}), \end{aligned}$$

and therefore

$$\Delta_x(A^*) \leq 8 \cdot \varepsilon. \tag{11}$$

On the other hand, since (10) implies that

$$\frac{4}{n} \cdot P(\text{pr}(A, x^\perp)) \leq P(A),$$

we deduce from (9) that

$$\frac{4}{n} \cdot P(\text{pr}(A, x^\perp)) \leq P(A) \leq (1 + \varepsilon) \cdot P(B_\infty^n) = (1 + \varepsilon) \cdot \frac{4}{n} \cdot P(B_\infty^{n-1}).$$

Thus we have

$$P(\text{pr}(A, x^\perp)) \leq (1 + \varepsilon) \cdot P(C^{n-1}),$$

where  $C^{n-1}$  denotes a centred affine cube in  $x^\perp$ . The projection  $\text{pr}(A, x^\perp)$  of the zonoid  $A$  is again a zonoid. Therefore, the induction hypothesis yields that

$$\delta_{\text{BM}}(\text{pr}(A, x^\perp), C^{n-1}) \leq 1 + \gamma_{n-1} \cdot \varepsilon^{\frac{1}{n-1}} \tag{12}$$

with  $\gamma_{n-1} \geq 0$  depending on  $n$ .

For centred convex bodies  $K, L \subset \mathbb{R}^n$  the Banach–Mazur distance satisfies  $\delta_{\text{BM}}(K, L) = \delta_{\text{BM}}(K^*, L^*)$ . Using again the relation  $(\text{pr}(A, x^\perp))^* = A^*(x)$ , we conclude that (12) is equivalent to

$$\delta_{\text{BM}}(A^*(x), B_1^{n-1}) \leq 1 + \gamma_{n-1} \cdot \varepsilon^{\frac{1}{n-1}}. \tag{13}$$

Here  $B_1^{n-1}$  is a centred regular crosspolytope in  $x^\perp$ .

Next we introduce another way to measure the distance of a centred convex body  $B$  in  $\mathbb{R}^n$  from double cones. For  $x \in \mathbb{S}^{n-1}$ , let  $M$  be the maximal volume of double cones contained in  $B$  with base  $B(x)$ , and let

$$\tilde{\Delta}_x(B) := \frac{|B| - M}{|B|}.$$

As before, let  $C_x$  be the centred double cone with  $C_x \cap x^\perp = B_x \cap x^\perp$ ,  $|C_x| = |B|$  and rotational symmetry around the line  $\mathbb{R}x$ . Note that with  $\tilde{h} := h(B, x)$  and  $h := h(C_x, x)$ , we have

$$\tilde{\Delta}_x(B) = 1 - \frac{M}{|C_x|} = 1 - \frac{\tilde{h}}{h}.$$

Now if  $\Pi$  is the parallel strip bounded by the two supporting hyperplanes of  $B$  parallel to  $x^\perp$ , then

$$|C_x \setminus B_x| \geq |C_x \setminus \Pi| = \left(\frac{h - \tilde{h}}{h}\right)^n |C_x| = \tilde{\Delta}_x(B)^n |B|,$$

and hence

$$\Delta_x(B) \geq 2\tilde{\Delta}_x(B)^n. \tag{14}$$

Note that  $\Delta_x(B) = 2\tilde{\Delta}_x(B)^n$  if  $B$  is the intersection of  $\Pi$  and a double cone with base  $B(x)$ .

To proceed further, we use that (13) implies that there is a linear transformation  $\tilde{B}_1^{n-1}$  of the regular crosspolytope  $B_1^{n-1}$  such that

$$\tilde{B}_1^{n-1} \subset A^*(x) \subset (1 + \gamma_{n-1}\varepsilon^{\frac{1}{n-1}})\tilde{B}_1^{n-1}. \tag{15}$$

We choose a point  $z$  from the support set of  $A^*$  with exterior unit normal vector  $x$ . Then  $[A^*(x), \pm z]$  is a double cone in  $A^*$  having base  $A^*(x)$  and maximal volume. Hence, combining (11) and (14), we get

$$\frac{|A^*| - |[A^*(x), \pm z]|}{|A^*|} = \tilde{\Delta}_x(A^*) \leq (4 \cdot \varepsilon)^{\frac{1}{n}}.$$

Since, by (15),

$$\begin{aligned} |[\tilde{B}_1^{n-1}, \pm z]| &\leq |[A^*(x), \pm z]| \\ &\leq |[ (1 + \gamma_{n-1}\varepsilon^{\frac{1}{n-1}})\tilde{B}_1^{n-1}, \pm z]| \\ &= (1 + \gamma_{n-1}\varepsilon^{\frac{1}{n-1}})^{n-1} |[\tilde{B}_1^{n-1}, \pm z]|, \end{aligned}$$

we deduce

$$\begin{aligned} \frac{|A^*| - |[\tilde{B}_1^{n-1}, \pm z]|}{|A^*|} &\leq \tilde{\Delta}_x(A^*) + \frac{((1 + \gamma_{n-1}\varepsilon^{\frac{1}{n-1}})^{n-1} - 1)|[\tilde{B}_1^{n-1}, \pm z]|}{|A^*|} \\ &\leq 4^{\frac{1}{n}}\varepsilon^{\frac{1}{n}} + ((1 + \gamma_{n-1}\varepsilon^{\frac{1}{n-1}})^{n-1} - 1), \end{aligned}$$

and hence

$$\frac{|A^*| - |[\tilde{B}_1^{n-1}, \pm z]|}{|A^*|} \leq \tilde{\gamma}_n \cdot \varepsilon^{\frac{1}{n}}, \tag{16}$$

where

$$\tilde{\gamma}_n := 4^{\frac{1}{n}} + n(1 + \gamma_{n-1})^{n-2}\gamma_{n-1}.$$

From the last estimate, we finally deduce the required estimate for the Banach–Mazur distance.

For this, we first apply to  $A^*$  a linear transformation  $T \in GL(n)$  which satisfies

$$T(\tilde{B}_1^{n-1}) = B_1^{n-1} \subset x^\perp \quad \text{and} \quad T([\tilde{B}_1^{n-1}, \pm z]) = B_1^n := B_1^{n-1} + [-x, x],$$

where  $T(\pm x) = \pm x$ . We put  $D := T(A^*)$ . From (15), we deduce that

$$B_1^{n-1} \subset D(x) := D \cap x^\perp \subset (1 + \gamma_{n-1} \varepsilon^{\frac{1}{n-1}}) B_1^{n-1} \quad \text{and} \quad B_1^n \subset D,$$

and from (16), we get

$$\frac{|D| - |B_1^n|}{|D|} \leq \tilde{\gamma}_n \cdot \varepsilon^{\frac{1}{n}}.$$

Now if  $F$  is any facet of  $B_1^n$  with exterior unit normal  $u \in \mathbb{S}^{n-1}$ , then

$$\frac{2}{n} \cdot (h(D, u) - h(B_1^n, u)) \cdot |F| \leq |D| - |B_1^n|.$$

Moreover, since  $D$  is centrally symmetric,  $h(D, \pm x) = 1$ , and by the above inclusions for  $D(x)$ , we have

$$|D| \leq 2(1 + \gamma_{n-1} \varepsilon^{\frac{1}{n-1}})^{n-1} |B_1^{n-1}| \leq (1 + \gamma_{n-1} \varepsilon^{\frac{1}{n-1}})^{n-1} \frac{2^n}{(n-1)!}.$$

Since  $|F| = \sqrt{n}/(n-1)!$ , we obtain

$$\begin{aligned} h(D, u) - h(B_1^n, u) &\leq |D| \cdot \frac{|D| - |B_1^n|}{|D|} \cdot \frac{n}{|F|} \cdot \frac{1}{2} \\ &\leq \tilde{\gamma}_n (1 + \gamma_{n-1} \varepsilon^{\frac{1}{n-1}})^{n-1} \sqrt{n} 2^{n-1} \cdot \varepsilon^{\frac{1}{n}}. \end{aligned}$$

Since  $h(B_1^n, u) \geq 1/\sqrt{n}$ , we finally get

$$h(D, u) \leq (1 + \gamma_n \varepsilon^{\frac{1}{n}}) \cdot h(B_1^n, u),$$

where

$$\gamma_n := n 2^{n-1} (4^{\frac{1}{n}} + n(1 + \gamma_{n-1})^{n-2} \gamma_{n-1}) (1 + \gamma_{n-1})^{n-1}. \tag{17}$$

This yields

$$B_1^n \subset D \subset (1 + \gamma_n \varepsilon^{\frac{1}{n}}) B_1^n,$$

and therefore

$$\delta_{\text{BM}}(A^*, B_1^n) \leq 1 + \gamma_n \cdot \varepsilon^{\frac{1}{n}},$$

that is

$$\delta_{\text{BM}}(A, B_\infty^n) \leq 1 + \gamma_n \cdot \varepsilon^{\frac{1}{n}} \tag{18}$$

which completes the induction.

The preceding argument also implies bounds for the constant  $\gamma_n$ ; cf. the recursion (17). Since  $\gamma_1 = 0$ , it follows for instance that  $\gamma_2 = 8$  is a suitable choice. Moreover, we have  $\gamma_3 \leq \frac{1}{4} \cdot 10^6$ . A general estimate is  $\gamma_n \leq 4^{n!2^n}$ , which follows from (17) by induction.

#### 4. Zonoids close to affine cubes

The conclusion of Theorem 1.1 yields a centred zonoid  $Z$  which is close with respect to the Banach–Mazur distance to a centred affine cube. The following proposition provides some further information about the structure of a centred zonoid close to a centred affine cube. Essentially, such a zonoid can be written as the Minkowski sum of  $n$  basic zonoids each of which is close to a centred segment.

**Proposition 4.1.** *If  $Z$  is a zonoid in  $\mathbb{R}^n$  with  $\delta_{\text{BM}}(Z, C^n) \leq \varepsilon$  for some  $\varepsilon \in (0, \frac{1}{4n^2})$ , then there exist zonoids  $Z_1, \dots, Z_n$  and centred independent segments  $s_1, \dots, s_n$  such that*

$$Z = Z_1 + \dots + Z_n,$$

$$s_i \subset Z_i \subset s_i + 4n\varepsilon(s_1 + \dots + s_n).$$

**Proof.** Once the result has been proved for zonotopes, the general case follows by approximation. Hence, let  $Z$  be a zonotope. Then there exist vectors  $x_1, \dots, x_k \in \mathbb{R}^n \setminus \{0\}$ , any  $n$  of them linearly independent, such that

$$Z = [-x_1, x_1] + \dots + [-x_k, x_k].$$

Since the statement of the proposition is  $\text{GL}(n)$  invariant, we can assume that there exists an orthonormal base  $v_1, \dots, v_n$  of  $\mathbb{R}^n$  such that, for  $Z_\infty = \sum_{i=1}^n [-v_i, v_i]$ , we have

$$Z_\infty \subset Z \subset (1 + \varepsilon)Z_\infty,$$

the absolute values of the coordinates of any  $x_j$  are different with respect to this base, and  $x_j$  or  $x_j - x_m$ ,  $m \neq j$ , are not parallel to any of the coordinate hyperplanes. We partition the vectors  $x_1, \dots, x_k$  into  $n$  groups  $X_1, \dots, X_n$ , where  $x_j \in X_i$  if the  $i$ th coordinate of  $x_j$  has maximal absolute value. Now let us define

$$Z_i := \sum_{x_j \in X_i} [-x_j, x_j],$$

and hence  $Z = Z_1 + \dots + Z_n$ . In addition, for  $i \in \{1, \dots, n\}$ , we choose a boundary point  $y_i$  of  $Z_i$  having  $v_i$  as an exterior unit normal vector, then we put  $s_i := [-y_i, y_i]$ .

First, we show that

$$Z_i \subset [-v_i, v_i] + 2\varepsilon Z_\infty. \tag{19}$$

We have  $|\langle x, v_i \rangle| \leq 1 + \varepsilon$  for  $x \in Z_i$  because  $Z_i \subset Z$ . If  $m \neq i$ ,  $1 \leq m \leq n$ , then we write  $\pi_{m,i}$  to denote the orthogonal projection into the coordinate plane spanned by  $v_i$  and  $v_m$ . Now we further partition  $X_i$ . For  $x_j \in X_i$ , let  $x_j \in X_i^+$  if  $\langle x_j, v_i \rangle \langle x_j, v_m \rangle > 0$ , and let  $x_j \in X_i^-$  if  $\langle x_j, v_i \rangle \langle x_j, v_m \rangle < 0$ .

If  $x_j \in X_i^+$  then there exists a side  $e_j$  of  $\pi_{m,i}Z$  with exterior unit vector  $w_j$  such that  $e_j$  is a translate of  $[-\pi_{m,i}x_j, \pi_{m,i}x_j]$ , and for  $y \in e_j$ , we have  $\langle w_j, y \rangle \geq \langle w_j, v_m - v_i \rangle$ . Since  $\langle v_i, x_j \rangle > \langle v_m, x_j \rangle$  and  $e_j \subset (1 + \varepsilon)\pi_{m,i}Z_\infty$ , for any  $y \in e_j$  we have

$$1 - \varepsilon \leq \langle v_m, y \rangle \leq 1 + \varepsilon.$$

It follows that

$$\sum_{x_j \in X_i^+} |\langle v_m, x_j \rangle| \leq \varepsilon.$$

Together with the analogous result for  $X_i^-$ , we conclude that  $|\langle x, v_m \rangle| \leq 2\varepsilon$  for  $x \in Z_i$ , which in turn yields (19).

Now combining (19) and  $Z_\infty \subset Z_1 + \dots + Z_n$ , we obtain

$$1 \leq \langle v_i, y_i \rangle + (n - 1)2\varepsilon,$$

and hence  $Z_i \subset s_i + 2n\varepsilon Z_\infty$  follows by another application of (19). Since

$$Z_\infty \subset Z \subset \sum_{i=1}^n (s_i + 2n\varepsilon Z_\infty) = (s_1 + \dots + s_n) + 2n^2\varepsilon Z_\infty \subset (s_1 + \dots + s_n) + \frac{1}{2}Z_\infty,$$

we conclude  $Z_i \subset s_i + 4n\varepsilon(s_1 + \dots + s_n)$ .  $\square$

### 5. Applications to stochastic geometry

In this section, we show how two known uniqueness results in stochastic geometry can be strengthened by applying the present stability result for the volume product of zonoids and the stability improvement of the Blaschke–Santaló inequality (see [8]). For notation and details from stochastic geometry, we refer to [54]. Throughout we assume that  $d \geq 3$ . All results remain true for  $d = 2$ , but then improved stability estimates are available.

Let  $\widehat{X}$  be a nondegenerate stationary Poisson hyperplane process in  $\mathbb{R}^n$  of intensity  $\widehat{\gamma}$  and with direction distribution  $\widehat{\varphi}$ . For each realisation of  $\widehat{X}$ , the hyperplanes dissect  $\mathbb{R}^n$  into nonoverlapping cells, which are almost surely convex polytopes. With probability one, the origin  $o$  is contained in the interior of a unique cell  $Z_0$ , which is called the zero cell. Thus  $Z_0$  is a random polytope generated by  $\widehat{X}$ . The number of vertices  $f_0(Z_0)$  of  $Z_0$  is a random variable and  $\mathbb{E}f_0(Z_0)$  denotes its mathematical expectation. It is known that

$$\mathbb{E}f_0(Z_0) \geq 2^n \tag{20}$$

with equality if and only if the hyperplanes of  $\widehat{X}$  are almost surely parallel to  $n$  fixed hyperplanes with linearly independent normal vectors, i.e. if and only if the associated mosaic is a parallel mosaic with fixed directions of the bounding hyperplanes (cf. [54, Theorem 10.4.9]). It is clear that such a mosaic yields the value  $2^n$ , but it is not clear at all that this is the minimal value and that this minimal value is only attained for parallel mosaics. For a proof of (20), it is first shown by integral geometric methods that (cf. [54, p. 505])

$$\mathbb{E}f_0(Z_0) = 2^{-n}n! \cdot |\Pi_{\widehat{X}}| \cdot |\Pi_{\widehat{X}}^*| = 2^{-n}n! \cdot P(\Pi_{\widehat{X}}), \tag{21}$$

where  $\Pi_{\widehat{X}}$  is the associated (centred) zonoid of  $\widehat{X}$  (cf. [54, p. 156, (4.59)]) and  $\Pi_{\widehat{X}}^* := (\Pi_{\widehat{X}})^*$  is its polar body. The zonoid  $\Pi_{\widehat{X}}$  is defined by

$$h(\Pi_{\widehat{X}}, u) = \frac{\widehat{\gamma}}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \widehat{\varphi}(dv).$$

From (21) and the lower bound  $4^n/n!$  for the volume product of zonoids (cf. (2)), the estimate (20) follows.

We now aim to show that

$$\mathbb{E}f_0(Z_0) \leq (1 + \varepsilon) \cdot 2^n, \tag{22}$$

for some  $\varepsilon \in [0, 1]$ , implies that the associated mosaic is almost (in a suitable sense) a parallel mosaic. This statement will be made precise, by providing a stability result for the underlying direction distribution  $\widehat{\varphi}$  of  $\widehat{\mathcal{X}}$ .

Assume that (22) is satisfied for some  $\varepsilon \in [0, 1]$ . Then (21) implies that

$$P(\Pi_{\widehat{\mathcal{X}}}) = |\Pi_{\widehat{\mathcal{X}}}| \cdot |\Pi_{\widehat{\mathcal{X}}}^*| \leq (1 + \varepsilon) \cdot \frac{4^n}{n!}.$$

An application of Corollary 1.2 yields that

$$\delta_{\text{BM}}(\Pi_{\widehat{\mathcal{X}}}, C^n) \leq 1 + \alpha_n \cdot \varepsilon.$$

Hence there exist positive numbers  $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n > 0$  and unit vectors  $v_1, \dots, v_n$  such that the parallelo-  
tope (centred affine cube)

$$P^n := \sum_{i=1}^n \frac{1}{2} \tilde{\lambda}_i [-v_i, v_i]$$

satisfies

$$P^n \subset \Pi_{\widehat{\mathcal{X}}} \subset (1 + \alpha_n \varepsilon) P^n. \tag{23}$$

Putting

$$\tilde{\rho} := \sum_{i=1}^n \tilde{\lambda}_i \delta_{\{-v_i, v_i\}},$$

we have

$$h(P^n, u) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \tilde{\rho}(dv), \quad u \in \mathbb{S}^{n-1}.$$

Thus (23) yields that, for  $u \in \mathbb{S}^{n-1}$ ,

$$\int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \rho_0(dv) \leq \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \widehat{\varphi}(dv) \leq (1 + \alpha_n \varepsilon) \cdot \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \rho_0(dv), \tag{24}$$

where  $\rho_0 := (1/\widehat{\mathcal{V}}) \cdot \tilde{\rho}$ . Note that since  $\widehat{\varphi}$  is a probability measure,

$$0 < \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \rho_0(dv) \leq \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \widehat{\varphi}(dv) \leq 1.$$

Let  $e_1, \dots, e_n$  denote an orthonormal basis of  $\mathbb{R}^n$ . Then

$$\sum_{i=1}^n |\langle e_i, v \rangle| \geq 1 \tag{25}$$

for all  $v \in \mathbb{S}^{n-1}$ , and thus

$$\rho_0(\mathbb{S}^{n-1}) \leq \sum_{i=1}^n \int_{\mathbb{S}^{n-1}} |\langle e_i, v \rangle| \widehat{\varphi}(dv) \leq n.$$

This yields the rough estimate

$$\|\widehat{\varphi} - \rho_0\|_{TV} \leq n + 1$$

for the total variation norm  $\|\widehat{\varphi} - \rho_0\|_{TV}$  of  $\widehat{\varphi} - \rho_0 =: \mu$ . After these preparations, we apply Theorem 5.1 and the subsequent discussion on page 44 in [28] with  $\Phi(t) := |t|$  (cf. [9,24]). This yields

$$\left| \int_{\mathbb{S}^{n-1}} F d\mu \right| \leq c_1(n, \tau) \|F\|_{BL} \|\mu\|_{TV}^{1-\tau} \|T_\Phi \mu\|^\tau,$$

where  $c_1(n, \tau)$  is a constant depending only on  $n$  and  $\tau$  (for  $c_1(n, \tau)$  an explicit estimate can be provided),  $F : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  is an arbitrary bounded Lipschitz function with bounded Lipschitz norm

$$\|F\|_{BL} := \|F\|_\infty + \|F\|_L, \quad \|F\|_\infty := \sup_x |F(x)|, \quad \|F\|_L := \sup_{x \neq y} \frac{F(x) - F(y)}{\|x - y\|},$$

$\tau \in (0, 2/(n + 4))$ ,  $\|\cdot\|$  denotes the  $L_2$ -norm on  $\mathbb{S}^{n-1}$ , and

$$(T_\Phi \mu)(u) := \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \mu(dv), \quad u \in \mathbb{S}^{n-1}.$$

From (24), we get

$$\|T_\Phi \mu\| \leq n\kappa_n \|T_\Phi \mu\|_\infty \leq c_2(n) \cdot \varepsilon.$$

Thus, choosing  $\tau = 1/(n + 4)$ , we conclude

$$\left| \int_{\mathbb{S}^{n-1}} F d\mu \right| \leq c_3(n) \|F\|_{BL} \cdot \varepsilon^{\frac{1}{n+4}}.$$

Let  $d_D$  denote the Dudley metric (also called Fortet–Mourier metric) on the space of finite Borel measures. This metric is defined by

$$d_D(\nu_1, \nu_2) := \sup \left\{ \left| \int_{\mathbb{S}^{n-1}} F d(\nu_1 - \nu_2) \right| : \|F\|_{BL} \leq 1 \right\},$$

for finite Borel measures  $\nu_1, \nu_2$  on  $\mathbb{S}^{n-1}$ ; see [14,20,28], [56, Chapter 6] for some additional information on this metric. Then we obtain

$$d_D(\widehat{\varphi}, \rho_0) \leq c_3(n) \cdot \varepsilon^{\frac{1}{n+4}}.$$

Choosing  $F \equiv 1$  and putting  $\rho := (1/\rho_0(\mathbb{S}^{n-1})) \cdot \rho_0$ , we get

$$d_D(\rho, \rho_0) \leq |1 - \rho_0(\mathbb{S}^{n-1})| \leq c_3(n) \cdot \varepsilon^{\frac{1}{n+4}},$$

and therefore

$$d_D(\widehat{\varphi}, \rho) \leq 2c_3(n) \cdot \varepsilon^{\frac{1}{n+4}}.$$

Hence, by the proof of [14, Theorem 11.3.3; p. 311, l. 10] (cf. also [13] and [20, Lemma 9.5]), we obtain a similar estimate for the Prokhorov metric  $d_P$ , that is

$$d_P(\widehat{\varphi}, \rho) \leq 2d_D(\widehat{\varphi}, \rho)^{\frac{1}{2}} \leq 4c_3(n) \cdot \varepsilon^{\frac{1}{2(n+4)}}.$$

Better known than the Dudley metric is the Wasserstein distance  $W_1$  (which is also called the Kantorovich–Rubinstein distance) for probability measures. If  $\nu_1, \nu_2$  are two probability measures, then it can be defined by

$$d_W(\nu_1, \nu_2) := \sup \left\{ \left| \int_{\mathbb{S}^{n-1}} F d(\nu_1 - \nu_2) \right| : \|F\|_L \leq 1 \right\},$$

according to a well-known duality formula for the Kantorovich–Rubinstein distance. Obviously, we have  $d_D \leq d_W$ . Let  $e \in \mathbb{S}^{n-1}$  be arbitrary, but fixed. If  $F : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  satisfies  $\|F\|_L \leq 1$  and  $F(e) = 0$ , then  $\|F(x)\| \leq 2$  for all  $x \in \mathbb{S}^{n-1}$ , and therefore  $\|\frac{1}{3}F\|_{BL} \leq 1$ . Using that  $\nu_1, \nu_2$  are probability measures, we get

$$\begin{aligned} d_W(\nu_1, \nu_2) &= \sup \left\{ \left| \int_{\mathbb{S}^{n-1}} F d(\nu_1 - \nu_2) \right| : \|F\|_L \leq 1, F(e) = 0 \right\} \\ &\leq \sup \left\{ 3 \cdot \left| \int_{\mathbb{S}^{n-1}} \frac{1}{3} F d(\nu_1 - \nu_2) \right| : \left\| \frac{1}{3} F \right\|_{BL} \leq 1 \right\} \\ &= 3 \cdot d_D(\nu_1, \nu_2), \end{aligned}$$

and hence  $d_W \leq 3 \cdot d_D$ .

This shows in a quantitative way that the direction distribution of  $\widehat{X}$  is close to the direction distribution of a parallel mosaic.

**Theorem 5.1.** *Let  $\widehat{X}$  be a nondegenerate stationary Poisson hyperplane process in  $\mathbb{R}^n$  with intensity  $\widehat{\gamma}$  and direction distribution  $\widehat{\varphi}$ . Then there is a constant  $c(n)$  such that the following is true. If*

$$\mathbb{E}f_0(Z_0) \leq (1 + \varepsilon) \cdot 2^n,$$

for some  $\varepsilon \in [0, 1]$ , then there exist positive numbers  $\lambda_1, \dots, \lambda_n > 0$  and linearly independent unit vectors  $\nu_1, \dots, \nu_n$  such that

$$\rho := \sum_{i=1}^n \lambda_i \delta_{\{-\nu_i, \nu_i\}}$$



is a probability measure and

$$d_W(\widehat{\varphi}, \rho) \leq c(n) \cdot \varepsilon^{\frac{1}{n+4}}, \quad d_P(\widehat{\varphi}, \rho) \leq c(n) \cdot \varepsilon^{\frac{1}{2(n+4)}}.$$

The derivation of Theorem 5.1 is based on Corollary 1.2. Therefore the constant  $c(n)$  is not explicitly given. Using instead Theorem 1.1, we obtain an explicit bound for  $c(n)$ , but then we have to replace  $\varepsilon$  by  $\varepsilon^{1/n}$  in the conclusion of Theorem 5.1.

We turn to the upper bound

$$\mathbb{E}f_0(Z_0) \leq 2^{-n} n! \kappa_n^2, \tag{26}$$

where  $\kappa_n$  is the volume of the Euclidean unit ball  $B_2^n$ . Equality holds in (26) if and only if there is some  $\alpha \in GL(n)$  such that  $\alpha \widehat{X}$  is isotropic, i.e. the direction distribution  $\widehat{\varphi}_\alpha$  of  $\alpha \widehat{X}$  is normalised spherical Lebesgue measure. These assertions follow from (21) and from the Blaschke–Santaló inequality (1) for centred convex bodies, that is

$$P(\Pi_{\widehat{X}}) = |\Pi_{\widehat{X}}| \cdot |\Pi_{\widehat{X}}^*| \leq \kappa_n^2$$

with equality if and only if  $\Pi_{\widehat{X}}$  is a centred ellipsoid. The latter is equivalent to saying that there is some  $\alpha \in GL(n)$  such that  $\alpha^{-t} \Pi_{\widehat{X}} = \Pi_{\alpha \widehat{X}}$  is a ball, i.e.  $\alpha \widehat{X}$  is isotropic (cf. [54, pp. 505–506]). Here  $\alpha^t$  is the adjoint linear map of  $\alpha$  and  $\alpha^{-t}$  denotes the inverse of the adjoint linear map.

Now assume that

$$\mathbb{E}f_0(Z_0) \geq (1 - \varepsilon) \cdot 2^{-n} n! \kappa_n^2, \tag{27}$$

for some  $\varepsilon \in [0, 1/2)$ . Then (21) and (27) imply that

$$P(\Pi_{\widehat{X}}) = |\Pi_{\widehat{X}}| \cdot |\Pi_{\widehat{X}}^*| \geq (1 - \varepsilon) \cdot \kappa_n^2.$$

By Theorem 1.1 and the subsequent remark in [8], we deduce that there is some  $\alpha \in GL(n)$  such that

$$B_2^n \subset \alpha^{-t} \Pi_{\widehat{X}} \subset (1 + c_5(n) \varepsilon^{\frac{1}{3n}} |\log \varepsilon|^{\frac{1}{3}}) \cdot B_2^n.$$

This implies that

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \sigma_0(dv) &\leq \frac{\widehat{\gamma}_\alpha}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \widehat{\varphi}_\alpha(dv) \\ &\leq (1 + c_5(n) \varepsilon^{\frac{1}{3n}} |\log \varepsilon|^{\frac{1}{3}}) \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \sigma_0(dv), \end{aligned} \tag{28}$$

where

$$\sigma_0 := \frac{n}{2} \frac{\kappa_n}{\kappa_{n-1}} \sigma,$$

$\sigma$  is the normalised spherical Lebesgue measure, and  $\widehat{\gamma}_\alpha, \widehat{\varphi}_\alpha$  are the intensity and the direction distribution of  $\alpha \widehat{X}$ . Specifically, we have

$$h(\alpha^{-t} \Pi_{\widehat{X}}, u) = \frac{\widehat{\gamma}_\alpha}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \widehat{\varphi}_\alpha(dv),$$

where

$$\widehat{\gamma}_\alpha := \widehat{\gamma} \cdot \int_{\mathbb{S}^{n-1}} \|\alpha^{-t}v\| \widehat{\varphi}(dv)$$

and

$$\widehat{\varphi}_\alpha := \frac{\widehat{\gamma}}{\widehat{\gamma}_\alpha} \cdot \int_{\mathbb{S}^{n-1}} \mathbf{1}\left\{\frac{\alpha^{-t}v}{\|\alpha^{-t}v\|} \in \cdot\right\} \|\alpha^{-t}v\| \widehat{\varphi}(dv).$$

Thus (28) leads to

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \sigma_1(dv) &\leq \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \widehat{\varphi}_\alpha(dv) \\ &\leq (1 + c_5(n)\varepsilon^{\frac{1}{3n}} |\log \varepsilon|^{\frac{1}{3}}) \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| \sigma_1(dv), \end{aligned} \tag{29}$$

where  $\sigma_1 := (\widehat{\gamma}_\alpha)^{-1} \frac{n\kappa_n}{\kappa_{n-1}} \cdot \sigma$ .

Using (28) and choosing  $\tau = 3/(2(n+4))$ , we can argue as before and finally get, for any bounded Lipschitz function  $F$  on  $\mathbb{S}^{n-1}$ ,

$$\left| \int_{\mathbb{S}^{n-1}} F d(\widehat{\varphi}_\alpha - \sigma_1) \right| \leq c_6(n) \|F\|_{BL} \cdot \varepsilon^{\frac{1}{6n^2}}.$$

The preceding estimate implies that  $|1 - \sigma_1(\mathbb{S}^{n-1})| \leq c_6(n) \cdot \varepsilon^{\frac{1}{6n^2}}$ , and hence

$$d_D(\widehat{\varphi}_\alpha, \sigma) \leq 2c_6(n) \cdot \varepsilon^{\frac{1}{6n^2}}.$$

As before, this yields estimates for the Wasserstein and the Prokhorov distance.

**Theorem 5.2.** *Let  $\widehat{X}$  be a nondegenerate stationary Poisson hyperplane process in  $\mathbb{R}^n$  with intensity  $\widehat{\gamma}$  and direction distribution  $\widehat{\varphi}$ . Then there is a constant  $c'(n)$  such that the following is true. If*

$$\mathbb{E}f_0(Z_0) \geq (1 - \varepsilon) \cdot 2^{-n} n! \kappa_n^2,$$

for some  $\varepsilon \in [0, 1/2)$ , then there is some  $\alpha \in GL(n)$  such that the direction distribution  $\widehat{\varphi}_\alpha$  of  $\alpha\widehat{X}$  satisfies

$$d_W(\widehat{\varphi}_\alpha, \sigma) \leq c'(n) \cdot \varepsilon^{\frac{1}{6n^2}}, \quad d_P(\widehat{\varphi}_\alpha, \sigma) \leq c'(n) \cdot \varepsilon^{\frac{1}{12n^2}}.$$

This shows that the direction distribution of  $\alpha\widehat{X}$  is almost isotropic.

Similar results follow for  $\mathbb{E}f_1(Z_0)$ , since the zero cell is almost surely a simple polytope, and hence

$$\mathbb{E}f_1(Z_0) = \frac{n}{2} \mathbb{E}f_0(Z_0).$$

In three dimensions, one can use Euler's relation to obtain a result for

$$\mathbb{E}f_2(Z_0) = 2 + \frac{1}{2}\mathbb{E}f_0(Z_0),$$

cf. a remark in [52, Section 2].

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