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ON 4-ISOMORPHISMS OF GRAPHS

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We show that an edge-bijection between 4-connected graphs preserving homeomorphs of K_4 in both directions is induced by an isomorphism.

In studying duality H. Whitney proved in [7] the following result. A map ϕ between the edge sets $E(G)$ and $E(G')$ of the graphs G and G' respectively is induced by an isomorphism, if G is 3-connected and ϕ is circuit-preserving in both directions. Halin and Jung [1] generalized Whitney's result to n-skein preserving bijections, where an n-skein is a graph spanned by n openly disjoint paths between two vertices. In general, it is natural to ask whether a given edge bijection which preserves homeomorphs of a certain graph H is necessarily induced by an isomorphism.

In this note we study bijections $\phi : E(G) \to E(G')$, ϕ and ϕ^{-1} preserve homeomorphs of K_4, called UK_4. We call such a map a *4-isomorphism*. Our main result is the following

Theorem. *Each* 4-*isomorphism between* 4-*connected graphs is induced by an isomorphism.*

Recently Hemminger, Jung and Kelmans [3] showed that each 3-skein isomorphism between 3-connected graphs with at least 5 vertices is induced by an isomorphism settling a question posed in [2]. As a first step of their proof they show that each 3-skein isomorphism is a 4-isomorphism.

Note that our result does not hold for 3-connected graphs. For example every permutation of the edges of a $K_{3,3}$ is a 4-isomorphism.

Following [2] we call a non-trivial path P in a subgraph H of G a *constituent path* of H if only the inner vertices of P have valency 2 in H. A *special vertex* of H has at least valency 3 in H. If Q is a path connecting two vertices of H, but having no inner vertices in common with H, we call it an H-*jumper*. We denote $\phi(H)$ by H'.

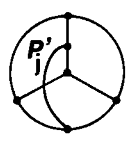
The following lemma, which is due to H.A. Jung (oral communication), simplifies an earlier version of the proof of the theorem.

Lemma. *Let ϕ be a 4-isomorphism from the 4-connected graph G onto G'. If C is a circuit such that $3<|V(C)|<|V(G)|$, then C' is a circuit in G'.*

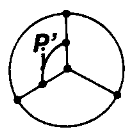
Proof. Pick a vertex x in $V(G)-V(C)$. Since G is 4-connected there exist four paths P_1, P_2, P_3 and P_4 with terminal vertex x, such that $H=C\cup P_1\cup P_2\cup P_3\cup P_4$ is a 4-wheel. For $1\leq j\leq 4$, let H_j denote the unique connected graph with $E(H_j)=E(H)-E(P_j)$. Clearly H contains exactly 4 distinct UK_4, namely H_1, H_2, H_3 and H_4.

First we show that every P_j' is an H_j'-jumper. Without loss of generality suppose that $j=1$. Note that H' is 2-connected since H_1' and H_2' are 2-connected and $H'=H_1'\cup H_2'\supset P_3'$. Hence the graph P_1' contains an H_1'-jumper Q. Now $Q\cup H_1'$ contains at least two UK_4 and so does its preimage. Since each proper subgraph of H – and hence also of H' – contains at most one UK_4, we infer that $Q=P_1'$. However P_1' cannot connect two vertices of the same constituent path of H_1'. For otherwise H' would contain only two distinct UK_4. Therefore H' is a homeomorph of $K_{3,3}$, a prism or a 4-wheel. (See Fig. 1.) In the first two cases H' contains nine or two distinct UK_4 respectively. In the last case P_1' connects a special vertex of H_1' to a nonspecial vertex of H_1'. It follows that H' is a 4-wheel and moreover that each P_i' is a spoke of the 4-wheel. Hence C' is a circuit. □

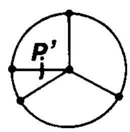
Proof of the Theorem. Let $f=xy$ and $g=xz$ be adjacent edges of G such that the images f', g' are nonadjacent in G'. (See Fig. 2.) Since $G-\{x\}$ is 3-connected, there exist at least three openly disjoint paths from z to y in $G-\{x\}$. At least two of those paths, say A and B, have at least 3 vertices. Consider the circuits C_1, C_2 and C_3 defined by $E(C_1)=E(A)\cup\{f,g\}$, $E(C_2)=E(B)\cup\{f,g\}$ and $E(C_3)=E(A)\cup E(B)$. Clearly $3<|V(C_i)|<|V(G)|$ and therefore the graphs C_i' are circuits in G'. Since f' and g' are not adjacent in G', in C_1' the graph A' decomposes into two nonempty disjoint paths A_1' and A_2'. Similarly in C_2' the graph B' is composed of two nonempty disjoint paths B_1' and B_2'.



$H_j'\cup P_j'$ is a homeomorph of $K_{3,3}$ and contains $9UK_4$



$H_j'\cup P_j'$ is a prism graph and contains $2UK_4$



$H_j'\cup P_j'$ is a 4-wheel and contains $4UK_4$

Fig. 1. The main cases of the Lemma.

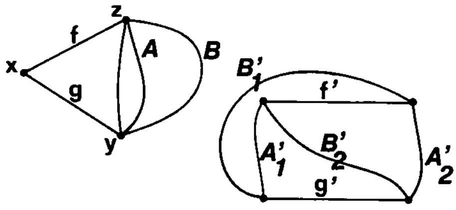


Fig. 2.

For $1 \leqslant i, j \leqslant 2$ the graph $B_i'A_j'$ is a proper subgraph of C_3' and hence not a circuit. Therefore B_i' and A_j' have exactly one vertex common, which is a vertex of f' or g'. It follows that $C_1' \cup C_2'$ is a UK_4. But this is impossible, because $C_1 \cup C_2$ is not a UK_4. We have shown that ϕ is adjacency-preserving. By symmetry, also ϕ^{-1} is adjacency-preserving. Hence by Whitney's Theorem [6], [4] the map ϕ is induced by an isomorphism. □

Note that the result holds for infinite graphs and for graphs with multiple edges.

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