Journal of Computational and Applied Mathematics 234 (2010) 518–525

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Existence of positive, negative and sign-changing periodic solutions for a class of integral equations[☆]

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ARTICLE INFO

Article history:
Received 21 June 2008
Received in revised form 5 February 2009

MSC:
34B15

Keywords:
Integral equations
Cone
e-continuous
Sign-changing periodic solution
Fixed point theorem

ABSTRACT

In this paper, the simultaneous existence of positive, negative and sign-changing periodic solutions for a class of integral equations of the form

$$\phi(x) = \int_{[x,x+\omega]\cap\Omega} K(x,y)f(y,\phi(y-\tau(y)))\mathrm{d}y, \quad x \in \Omega,$$

is considered, where Ω is a closed subset of R^N with a periodic structure. Our main result is different from most existing results since they provide three *constant sign* periodic solutions only.

1. Introduction

There are now numerous results on the existence of multiple solutions of functional (differential or difference) equations under additional side conditions. In particular, by means of the Krasnoselski fixed point theorem, the Leggett–Williams fixed point theorem and/or the Avery fixed point theorems, the existence of three *constant sign* solutions for many differential or difference boundary value problems have been proved; see e.g. [1–19].

In [3], three *positive* periodic solutions for a class of integral equations are established by means of the Leggett–Williams fixed point theorem. The question then arises as to whether there are three solutions with different types of sign regularities. This is a relatively difficult question. But in a recent paper by Li et al. [1], the existence of sign-changing solutions for nonlinear operator equations is discussed using the topological degree and fixed point index theory. The corresponding results are shown to be useful in deriving sign-changing solutions of integral equations. In this paper, we will also be interested in deriving the simultaneous existence of three solutions for an integral equation which are positive, negative and 'sign-changing'. Our integral equation, however, involves spaces with periodic structure and hence is different from the Hammerstein type equation in [1].

To be more precise, let R^N be the N-dimensional Euclidean space endowed with componentwise ordering $\leq$. For any $u, v \in R^N$, the 'interval' $[u, v]$ is the set $\{x \in R^N \mid u \leq x \leq v\}$. Let $\omega = (\omega_1, \ldots, \omega_N) \in R^N$ with positive components and let $e^{(1)} = (1, 0, \ldots, 0), \ldots, e^{(N)} = (0, \ldots, 0, 1)$ be the standard orthonormal vectors in R^N. Let Ω be a closed subset of R^N with the positive Lebesgue measure $\mu(\Omega)$ and which has the following 'periodic' structure: for each $x \in \Omega$,

$$x + \omega_i e^{(i)} \in \Omega,$$

☆ Project supported by the Natural Science Foundation of Shanxi Province (2008011002-1) and Shanxi Datong University and by the Development Foundation of Higher Education Department of Shanxi Province and by the Science and Technology Bureau of Datong City.

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doi:10.1016/j.cam.2009.12.043

and for each pair $y, z \in \Omega$,

$$\mu\left([y, y+\omega] \cap \Omega\right) = \mu\left([z, z+\omega] \cap \Omega\right) > 0.$$

For the sake of convenience, we will set

$$\Omega(x) = [x, x+\omega] \cap \Omega.$$

We will be concerned with integral equations of the form where the functions K, f and τ satisfy the following conditions:

$$\phi(x) = \int_{[x,x+\omega]\cap\Omega} K(x, y) f(y, \phi(y - \tau(y)))\mathrm{d}y, \quad x \in \Omega, \tag{1}$$

- $K \in C(\Omega \times \Omega, (0, \infty))$ and $K(x+\omega_i e^{(i)}, y+\omega_i e^{(i)}) = K(x, y)$ for any $(x, y) \in \Omega \times \Omega$ and $i \in \{1, 2, \dots, N\}$, K is uniformly continuous[1] on $\Omega \times \Omega$,
- $f \in C(\Omega \times R, R)$ and $f(x + \omega_i e^{(i)}, u) = f(x, u)$ for $i \in \{1, 2, \dots, N\}$ and $x \in \Omega$,
- $\tau : \Omega \to \Omega$ is a function such that $\tau(x + \omega_i e^{(i)}) = \tau(x)$ for any $x \in \Omega$ and $i \in \{1, 2, \dots, N\}$, and there is a partition $\{\Omega_1, \Omega_2, \dots, \Omega_m\}$ of $\Omega(x_0)$ such that $\tau(x) = (n_1^{(i)}\omega_1, n_2^{(i)}\omega_2, \dots, n_N^{(i)}\omega_N)$ for $x \in \Omega_i$, where $n_1^{(i)}, n_2^{(i)}, \dots, n_N^{(i)}$ are integers and $x_0 \in \Omega$.

To see an example of the function τ. Let $(\omega_1, \omega_2) = (4\pi, 4\pi)$ and

$$\Omega = \left\{(x, y) \in R^2 | 4n\pi \le x \le 4n\pi + 2\pi, 4m\pi \le y \le 4m\pi + 2\pi \;\; m, n = 0, \pm 1, \pm 2, \dots\right\}.$$

Let

$$\begin{aligned}
l_1 &= \{(x, y) | x = 0, 0 \le y < \pi\} \cup \{(x, y) | 0 < x < \pi, y = 0\} \cup \{(x, y) | x = \pi, 0 \le y < \pi\},\\
l_2 &= \{(x, y) | \pi < x \le 2\pi, y = 0\} \cup \{(x, y) | x = 2\pi, 0 < y \le \pi\} \cup \{(x, y) | \pi < x < 2\pi, y = \pi\},\\
l_3 &= \{(x, y) | 0 \le x \le \pi, y = \pi\} \cup \{(x, y) | x = 0, \pi < y \le 2\pi\} \cup \{(x, y) | 0 < x < \pi, y = 2\pi\},\\
l_4 &= \{(x, y) | x = 2\pi, \pi < y \le 2\pi\} \cup \{(x, y) | \pi \le x < 2\pi, y = 2\pi\} \cup \{(x, y) | x = \pi, \pi < y < 2\pi\},
\end{aligned}$$

and

$$\begin{aligned}
\Theta_1 &= \{(x, y) | 0 < x < \pi, 0 < y < \pi\},\\
\Theta_2 &= \{(x, y) | \pi < x < 2\pi, 0 < y < \pi\},\\
\Theta_3 &= \{(x, y) | 0 < x < \pi, \pi < y < 2\pi\},\\
\Theta_4 &= \{(x, y) | \pi < x < 2\pi, \pi < y < 2\pi\}.
\end{aligned}$$

Then $\{l_1 \cup \Theta_1, l_2 \cup \Theta_2, l_3 \cup \Theta_3, l_4 \cup \Theta_4\}$ is a partition of $\Omega(0) = [0, 2\pi] \times [0, 2\pi]$, and τ defined by

$$\tau(x) = \begin{cases} (\omega_1, \omega_2) & x \in l_1 \cup \Theta_1 \\ (\omega_1, 2\omega_2) & x \in l_2 \cup \Theta_2 \\ (2\omega_1, 3\omega_2) & x \in l_3 \cup \Theta_3 \\ (4\omega_1, 2\omega_2) & x \in l_4 \cup \Theta_4 \end{cases}$$

is an example.

A concrete example of (1) is the integral equation

$$\phi(x) = \int_x^{x+2\pi} K(x, y) f(\phi(y))\mathrm{d}y, \quad x \in R, \tag{2}$$

where

$$K(x, y) = \frac{\exp \int_x^y a(t)\mathrm{d}t}{\exp \int_0^{2\pi} a(t)\mathrm{d}t - 1}, \quad x, y \in R,$$

which arises when periodic solutions are sought for the differential equation

$$\phi'(x) = -a(x)\phi(x) + f(\phi(x)), \quad x \in R, \tag{3}$$

where $a = a(x)$ is a positive continuous 2π-periodic function defined on R (see e.g. [5,6]).

We will look for solutions in the set of all real continuous functions of the form $\phi : \Omega \to R$ such that

$$\phi(x + \omega_i e^{(i)}) = \phi(x), \quad x \in \Omega.$$

[1] This assumption can be relaxed. Indeed, we may assume in the sequel that for any $\varepsilon > 0$, there exists positive δ, which does not depend on y, such that $|K(x_1, y) - K(x_2, y)| < \varepsilon$ for all $x_1, x_2 \in \Omega$ that satisfy $|x_1 - x_2| < \delta$.

This set will be denoted by $C_\omega(\Omega)$ in the sequel, when endowed with the usual linear and ordering structure as well as the norm $\|\phi\| = \max_{z\in\Omega(x),x\in\Omega} |\phi(z)|$, $C_\omega(\Omega)$ is a real Banach space with a normal and total cone $P = \{\phi \in C_\omega(\Omega) : \phi(x) \geq 0, x \in \Omega\}$.

A function ϕ in $C_\omega(\Omega)$ is said to be an ω-periodic solution of (1) if substitution of it into (1) yields an identity for all $x \in \Omega$. A periodic solution ϕ is said to be positive if $\phi \in P$, negative if $\phi \in -P$ and sign-changing if $\phi \notin P \cup (-P)$.

The simultaneous existence of such solutions will be based on the recent existence theorem in [1]. For this reason, we first quote several results to be used in the sequel. Let E be a real Banach space with cone P_1 and null vector θ. The semi-order induced by the cone P_1 is denoted by "$\leq$". Let $D \subset E$ and $D \neq \emptyset$. An operator $A : D \to E$ is said to be increasing on D if $Ax \leq Ay$ for any $x, y \in D$ and $x \leq y$. A fixed point u of operator A is said to be positive if $u \in P_1$, negative if $u \in -P_1$ and sign-changing if $u \notin P_1 \cup (-P_1)$.

Definition 1 ([7]). Let $A : D \to E$ be an operator, $e \in P_1 \setminus \{\theta\}$ and $x_0 \in D$. If for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $-\varepsilon e \leq Ax - Ax_0 \leq \varepsilon e$ for all $x \in D$ with $\|x - x_0\| < \delta$, then A is called e-continuous at x_0. If A is e-continuous at each point $x \in D$, then A is called e-continuous on D.

It is easy to see that if $A : D \to E$ is a linear operator, then A is e-continuous on D iff A is e-continuous at θ.

Theorem A (*Leray–Schauder [8]*). *Let $A : E \to E$ be completely continuous, $A\theta = \theta$, and Fréchet differentiable at θ. Assume that 1 is not an eigenvalue of the Fréchet derivative $A'(\theta)$. Let $F = \{x \in E \setminus \{\theta\} : Ax = x\}$. Then there exists $\tau > 0$ such that $F \cap B_\tau = \phi$, where $B_\tau = \{x \in E : \|x\| < \tau\}$. That is, θ is an isolated zero point of the completely continuous vector field $I - A$ and*

$$i(I - A, \theta) = i(I - A'(\theta), \theta) = (-1)^k,$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(\theta)$ in $(1, +\infty)$, and i is the index of isolated zero point (see e.g. [4]).

Theorem B (*Krein–Rutman [8]*). *Let E be a Banach space, $P_1 \subset E$ a total cone and K a linear compact positive operator with $r(K) > 0$, where $r(K)$ denotes the spectral radius of K. Then $r(K)$ is an eigenvalue of K with a positive eigenvector. Furthermore, $r(K)$ is an eigenvalue of K^*, the dual operator of K, with positive eigenvector in P_1^*, where P_1^* is the dual cone of P_1.*

Theorem C (*Li et al. [1]*). *Let P_1 be a normal cone in E, $A = KF$, where $F : E \to E$ is a continuous and bounded increasing operator, $K : E \to E$ is a positive linear completely continuous operator which is also e-continuous on E. Suppose that*

L1 *$F\theta = \theta$, F is Fréchet differentiable at θ, and $KF'(\theta)$ has an eigenvalue $\lambda_0 \in (1, \infty)$ with eigenvector u satisfying $\nu e \leq u \leq \lambda e$ for some positive ν and λ;*
L2 *1 is not an eigenvalue of the operator $KF'(\theta)$, and $i(I - KF'(\theta), \theta) = 1$;*
L3 *there exist $u_0 \in (-P_1) \setminus \{\theta\}$ and $v_0 \in P_1 \setminus \{\theta\}$ such that $u_0 \leq Au_0$ and $Av_0 \leq v_0$, and there also exists $\beta > 0$ such that $u_0 \leq -\beta e$ and $\beta e \leq v_0$;*
L4 *there exists $h \geq \gamma e$ with $\gamma > 0$ such that $\|x\|h \leq x$ for all $x \in P_1$ with $Ax = x$, and $x \leq -\|x\|h$ for all $x \in (-P_1)$ with $Ax = x$.*

Then A has at least one sign-changing fixed point, one positive fixed point and one negative fixed point.

2. Main results

We will assume that
(C_0) there are m and M such that

$$0 < m \leq K(x, y) \leq M < +\infty \quad \text{for } x, y \in \Omega(t) \text{ and } t \in \Omega. \tag{4}$$

Then $\hat{c} = m/M > 0$. Let

$$e(x) = \int_{\Omega(x)} K(x, s)\mathrm{d}s, \quad x \in \Omega,$$

we have $e(x) > 0$ for $x \in \Omega$ (since $e(x) = \int_{\Omega(x)} K(x, s)\mathrm{d}s \geq m\mu(\Omega(x)) > 0$). And we may also verify that if

$$0 < \nu \leq m/(M^2\mu(\Omega(t))) \Rightarrow \nu e \leq \hat{c} \quad \text{and} \quad K(x, y) \geq \hat{c}K(z, y) \tag{5}$$

for $x, y, z \in \Omega(t)$ and $t \in \Omega$.

Now we define operators $F, G, A : C_\omega(\Omega) \to C_\omega(\Omega)$ respectively by

$$(Fu)(x) = f(x, u(x - \tau(x))), \quad x \in \Omega,\ u \in C_\omega(\Omega), \tag{6}$$

$$(Gu)(x) = \int_{\Omega(x)} K(x, s)\, u(s)\, \mathrm{d}s, \quad x \in \Omega,\ u \in C_\omega(\Omega), \tag{7}$$

and

$$A = GF.$$

Then $F : C_\omega(\Omega) \to C_\omega(\Omega)$ is a continuous and bounded operator, $G : C_\omega(\Omega) \to C_\omega(\Omega)$ is a linear positive continuous operator and $G(P) \subset P$.

Furthermore, by standard arguments (see e.g. [3]), we may also show that G is completely continuous. So $A : C_\omega(\Omega) \to C_\omega(\Omega)$ is also completely continuous on $C_\omega(\Omega)$. By the Riesz–Schauder theorem, we may suppose that the sequence $\{\lambda_n\}$ of all positive eigenvalues of G satisfies

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots > 0.$$

Lemma 1. *Suppose* (C_0) *holds. Then the operators* $G, A : C_\omega(\Omega) \to C_\omega(\Omega)$ *are e-continuous on* $C_\omega(\Omega)$.

Proof. For any given $u_0 \in C_\omega(\Omega)$, and any $u \in C_\omega(\Omega)$,

$$\begin{aligned} |Gu(t) - Gu_0(t)| &\le \int_{\Omega(t)} K(t,s)\,|u(s) - u_0(s)|\,\mathrm{d}s \\ &\le \|u - u_0\| \int_{\Omega(t)} K(t,s)\,\mathrm{d}s = \|u - u_0\| e(t), \quad t \in \Omega. \end{aligned}$$

So G is e-continuous at u_0 and it follows from the continuity of $F : C_\omega(\Omega) \to C_\omega(\Omega)$ that $A = GF$ is also e-continuous at u_0. $\square$

Lemma 2. *Suppose* (C_0) *holds. Suppose further that*

(C_1) *$f(\cdot, 0) = 0$ on Ω, and for each $t \in \Omega$, $f(t,u)$ is nondecreasing in u.*

Then

$$\|Gu\|\hat{c} \le Gu \quad \text{for } u \in P \quad \text{and} \quad Gu \le -\|Gu\|\hat{c} \quad \text{for } u \in (-P),$$

and

$$\|Au\|\hat{c} \le Au \quad \text{for } u \in P \quad \text{and} \quad Au \le -\|Au\|\hat{c} \quad \text{for } u \in (-P).$$

Proof. For any $u \in P$, from the definition of G.

$$\begin{aligned} (Gu)(t) &= \int_{\Omega(t)} K(t,s)\,u(s)\,\mathrm{d}s \ge m \int_{\Omega(t)} u(s)\,\mathrm{d}s = \hat{c}M \int_{\Omega(z)} u(s)\,\mathrm{d}s \\ &\ge \hat{c} \int_{\Omega(z)} K(z,s)\,u(s)\,\mathrm{d}s = \hat{c}(Gu)(z), \end{aligned}$$

for $t, z \in \Omega$. Then $Gu \ge \|Gu\|\hat{c}$. Similarly, we can obtain that $Gu \le -\|Gu\|\hat{c}$ for $u \in (-P)$. It follows from condition (C_1) that $F(P) \subset P$ and $F(-P) \subset -P$. So $Au = GFu \ge \|GFu\|\hat{c} = \|Au\|\hat{c}$ for $u \in P$, and $Au = GFu \le -\|GFu\|\hat{c} = -\|Au\|\hat{c}$ for $u \in (-P)$. $\square$

Lemma 3. *Suppose* (C_0) *holds. Assume that $f(\cdot, 0) = 0$ on Ω, and*

$$\lim_{u \to 0} f(t,u)/u = a \tag{8}$$

uniformly with respect to $t \in \Omega$. Then the operator A is Fréchet differentiable at θ and $A'(\theta) = aG$.

Proof. From (8), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t,u)/u - a| < \varepsilon$ for all $t \in \Omega$ and $|u| \in (0, \delta)$. So we have $\|Fu - au\| \le \varepsilon\|u\|$ for all $u \in C_\omega(\Omega)$ with $\|u\| < \delta$. Consequently,

$$\lim_{\|u\| \to 0} \frac{\|Fu - F\theta - au\|}{\|u\|} = 0.$$

This implies that the operator F is Fréchet differentiable at θ and $F'(\theta) = aI$. It follows from the definition of A and the chain rule for derivatives of composite operators [8] that $A'(\theta) = GF'(\theta) = aG$. $\square$

Theorem 1. *Suppose that conditions* (C_0) *and* (C_1) *hold. Assume further that*

(C_2) $\lim_{u \to 0} f(t,u)/u = a$ *uniformly with respect to $t \in \Omega$, there exists a positive integer n_0 such that*

$$1/\lambda_{2n_0} < a < 1/\lambda_{2n_0+1}$$

and the sum of the algebraic multiplicities of the eigenvalues λ_i for all $1 \le i \le 2n_0$ is even;

(C_3) $\lim_{u \to \infty} f(t,u)/u = f_\infty$ *uniformly with respect to $t \in \Omega$, and $f_\infty < 1/\|e\|$.*

Then the Eq. (1) has at least three nontrivial periodic solutions, one of which is positive, another is negative, and the third solution is sign-changing.

Proof. We only need to verify all the conditions of Theorem C.

(1) It follows from (C_2) and Lemma 3 that the eigenvalues of the operator aG in $(1, +\infty)$ are $a\lambda_1, a\lambda_2, \ldots, a\lambda_{2n_0}$, and 1 is not an eigenvalue of aG. According to condition (C_2) and the Leray–Schauder theorem, we may deduce $i(I - A'(\theta), \theta) = 1$. That is the condition (2) of Theorem C holds.

(2) Since P is a total cone in $C_\omega(\Omega)$, $G : C_\omega(\Omega) \to C_\omega(\Omega)$ is a completely continuous positive linear operator and the spectral radius $r(G) = \lambda_1 > 0$. It follows from the Krein–Rutman theorem that there exists $v \in P \setminus \{\theta\}$ such that $Gv = \lambda_1 v$. Choose ν such that $0 < \nu \le m/(M^2 \mu(\Omega(t)))$, then by (5), $\nu e \le \hat{c}$, and according to Lemma 2, we have

$$\nu\lambda_1 \|v\| e \le \|\lambda_1 v\| \hat{c} = \|Gv\| \hat{c} \le Gv = \lambda_1 v = Gv \le \|v\| e.$$

So $\nu\|v\|e \le v \le \lambda_1^{-1}\|v\|e$. The condition (1) of Theorem C holds.

(3) According to Lemma 2, we have

$$u = Au \ge \|Au\|\hat{c} = \|u\|\hat{c} \quad \text{for all } u \in P \text{ and } Au = u;$$
$$u = Au \le -\|Au\|\hat{c} = -\|u\|\hat{c} \quad \text{for all } u \in (-P) \text{ and } Au = u.$$

It is easy to see that the condition (4) of Theorem C is satisfied.

(4) By condition (C_3), for some large $R > 0$, we have

$$f(t, R)/R < 1/\|e\|, \qquad f(t, -R)/(-R) < 1/\|e\|, \quad t \in \Omega.$$

Let $u_0 = -R,\ v_0 = R$. Then $u_0 = -R \le -R\|e\|^{-1}e, R\|e\|^{-1}e \le R = v_0$. It follows that

$$\begin{aligned}
(Au_0)(t) &= \int_{\Omega(t)} K(t, s) f(s, -R)\, ds \ge -R\|e\|^{-1} \int_{\Omega(t)} K(t, s)\, ds \\
&= -R\|e\|^{-1} e(t) \ge -R = u_0(t), \quad t \in \Omega; \\
(Av_0)(t) &= \int_{\Omega(t)} K(t, s) f(s, R)\, ds \le R\|e\|^{-1} \int_{\Omega(t)} K(t, s)\, ds \\
&= R\|e\|^{-1} e(t) \le R = v_0(t), \quad t \in \Omega.
\end{aligned}$$

So $u_0 \le Au_0,\ Av_0 \le v_0$. This implies that the condition (3) of Theorem C holds. The proof is completed. □

3. An example

It is important to see an example that illustrates the above result. For this purpose, let us first consider the eigenvalue problem

$$\lambda u(x) = \int_0^1 (1 + xy + x^2y^2) u(y) dy. \tag{9}$$

Since

$$\int_0^1 (1 + xy + x^2y^2) u(y) dy = \int_0^1 u(y) dy + x \int_0^1 y u(y) dy + x^2 \int_0^1 y^2 u(y) dy,$$

we see that

$$\lambda u(x) = \int_0^1 u(y) dy + x \int_0^1 y u(y) dy + x^2 \int_0^1 y^2 u(y) dy. \tag{10}$$

We now look for $\lambda \in R$ such that there is a nontrivial function u defined on $[0, 1]$ which satisfies the above equation.

For the special case where $\lambda = 0$, we can pick any u such that

$$\int_0^1 u(y) dy = \int_0^1 y u(y) dy = \int_0^1 y^2 u(y) dy = 0.$$

But if we require $u \in C([0, 1], [0, \infty))$, then $u \equiv 0$, and hence $\lambda = 0$ cannot be an eigenvalue. In view of (10), an eigenfunction u must satisfy

$$u(x) = a + bx + cx^2$$

for some a, b, c. Since

$$\begin{aligned}
&\int_0^1 (a + by + cy^2) dy = a + \frac{1}{3}c + \frac{1}{2}b, \\
&\int_0^1 y\left(a + by + cy^2\right) dy = \frac{1}{4}c + \frac{1}{3}b + \frac{1}{2}a, \\
&\int_0^1 y^2\left(a + by + cy^2\right) dy = \frac{1}{5}c + \frac{1}{4}b + \frac{1}{3}a,
\end{aligned}$$

we see that

$$\lambda a + \lambda bx + \lambda cx^2 = \left(a + \frac{1}{3}c + \frac{1}{2}b\right) + x\left(\frac{1}{4}c + \frac{1}{3}b + \frac{1}{2}a\right) + x^2\left(\frac{1}{5}c + \frac{1}{4}b + \frac{1}{3}a\right),$$

so that (since $1, x, x^2$ are linearly independent in $C[0, 1)$)

$$\lambda a = \left(a + \frac{1}{3}c + \frac{1}{2}b\right)$$
$$\lambda b = \left(\frac{1}{4}c + \frac{1}{3}b + \frac{1}{2}a\right)$$
$$\lambda c = \left(\frac{1}{5}c + \frac{1}{4}b + \frac{1}{3}a\right)$$

or

$$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{60}\begin{pmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{pmatrix}$ are roots 84.499, 0.16124, 7.3396 of the characteristic polynomial

$$X^3 - 92X^2 + 635X - 100,$$

with corresponding eigenvectors

$$\begin{pmatrix} 0.8270 \\ 0.4599 \\ 0.3233 \end{pmatrix}, \begin{pmatrix} -0.1277 \\ 0.7137 \\ -0.6887 \end{pmatrix}, \begin{pmatrix} 0.5474 \\ -0.5283 \\ -0.6490 \end{pmatrix}.$$

Therefore we have found a positive kernel

$$K_1(x, y) = 1 + xy + x^2y^2$$

over $[0, 1]^2$ and three positive and simple eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Next we consider an eigenvalue problem of the form (1). Let $T = 2$,

$$\Omega_1 = \{x \in R : 2n \leq x \leq 2n + 1, n \in Z\},$$
$$\Omega_1(x) = [x, x + 2] \cap \Omega_1,$$

and for any $m, n \in Z$,

$$K(x, y) = 1 + (x - 2n)(y - 2m) + (x - 2n)^2(y - 2m)^2, \quad 2n \leq x \leq 2n + 1,\ 2m \leq y \leq 2m + 1.$$

Then it is easily checked that

$$K(x + 2, y) = K(x, y), \qquad K(x, y + 2) = K(x, y), \quad (x, y) \in \Omega_1 \times \Omega_1,$$
$$+\infty \geq M_1 = \max_{0 \leq s,t \leq 1} K(t, s) \geq K(t, s) \geq \min_{0 \leq s,t \leq 1} K(t, s) = m_1 > 0,$$

and

$$1 \geq \frac{K(t, s)}{\max\limits_{t \in \Omega_1, s \in [t, t+T]} K(t, s)} \geq \frac{\min\limits_{t \in \Omega_1, s \in [t, t+T]} K(t, s)}{\max\limits_{t \in \Omega_1, s \in [t, t+T]} K(t, s)} = \frac{m_1}{M_1}.$$

Let $C_T(\Omega_1)$ be the set of all real T-periodic continuous functions, endowed with the usual linear structure as well as the norm $\|y\| = \sup_{t \in [0,1]} |y(t)|$. Then $C_T(\Omega_1)$ is a Banach space. Define a cone of $C_T(\Omega_1)$ by

$$P = \{y(t) \in C_T(\Omega_1) : y(t) \geq 0,\ t \in \Omega_1\}.$$

Then P is a normal and total cone. Define operators $F, G, A : C_T(\Omega_1) \to C_T(\Omega_1)$ respectively by

$$(Fu)(t) = f(u(t)), \quad t \in \Omega_1,\ u \in C_T(\Omega_1), \tag{11}$$

$$(Gu)(t) = \int_t^{t+T} K(t, s)\, u(s)\, ds, \quad t \in \Omega_1,\ u \in C_T(\Omega_1), \tag{12}$$

and $A = GF$. Then $F : C_T(\Omega_1) \to C_T(\Omega_1)$ is a continuous and bounded operator, $G : C_T(\Omega_1) \to C_T(\Omega_1)$ is a linear completely continuous operator and $G(P) \subset P$. So $A : C_T(\Omega_1) \to C_T(\Omega_1)$ is also completely continuous on $C_T(\Omega_1)$. Consider

$$u(x) = \int_{\Omega_1(x)} K(x, y) f(u(y)) \mathrm{d}y, \quad x \in \Omega_1. \tag{13}$$

By the definition of K, for any $2n \le x \le 2n + 1$,

$$\begin{aligned} u(x) &= \int_{\Omega_1(x)} K(x, y) f(u(y)) \mathrm{d}y \\ &= \int_x^{2n+1} K(x, y) f(u(y)) \mathrm{d}y + \int_{2n+2}^{x+2} K(x, y) f(u(y)) \mathrm{d}y \\ &= \int_{x-2n}^{1} K(x, y) f(u(y)) \mathrm{d}y + \int_0^{x-2n} K(x, y) f(u(y)) \mathrm{d}y \\ &= \int_0^1 K(x, y) f(u(y)) \mathrm{d}y \\ &= \int_0^1 K(x - 2n, y) f(u(y)) \mathrm{d}y. \end{aligned}$$

Then by setting $e_1(t) = \int_{\Omega_1(t)} K(t, s)\mathrm{d}s$ ($\|e_1\|$ can be calculated (at least numerically) and is a positive number), we have

Theorem 2. *Assume that* $\lim_{u\to 0} f(u)/u = f_0$ *and* $60/7.3394 < f_0 < 60/0.16124$, $\lim_{u\to\infty} f(u)/u = f_\infty < 1/\|e_1\|$. *Then the Eq. (13) has at least three nontrivial periodic solutions, one of which is positive, another is negative, and the third solution is sign-changing.*

For instance, if we choose f to be the nondecreasing function $f(u) = \eta \arctan u + \xi u$ where $\xi < \frac{1}{\|e_1\|}$ and $60/7.3396 < \eta + \xi < 60/0.16124$, then $f(0) = 0$, $\lim_{u\to 0} f(u)/u = \eta + \xi = f_0$, and $60/7.3396 < f_0 < 60/0.16124$, and $\lim_{u\to\infty} f(u)/u = \xi = f_\infty < 1/\|e_1\|$. Hence the conditions in Theorem 2 are satisfied, so that the equation

$$u(x) = \int_0^1 (1 + xy + x^2y^2) f(u(y)) \mathrm{d}y, \quad x \in [0, 1],$$

or,

$$u(x) = \int_{\Omega_1(x)} K(x, y) f(u(y)) \mathrm{d}y, \quad x \in \Omega_1,$$

has at least three nontrivial periodic solutions, one of which is positive, another is negative, and the third solution is sign-changing.

In conclusion, we have investigated the simultaneous existence of positive, negative and sign-changing periodic solutions of a class of integral equations with periodic structure. An existence criterion based on the spectral structure of an associated linear eigenvalue problem and the asymptotic behavior of an associated nonlinear function is derived which is different from those in the literature. An example is also constructed to illustrate our result.

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