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The Dotted Straightening Algorithm

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If a homogeneous bracket polynomial is antisymmetric in certain subsets of its points, then it can be represented in an abbreviated form called a dotted bracket expression. These dotted bracket expressions lead to a more compact expression in terms of tableaux than the usual representation. Consequently, we can derive a much more efficient straightening algorithm than the ordinary one for bracket polynomials already given in dotted form. This dotted straightening algorithm gives precisely the same result as the ordinary one, and preserves the dotted property at every step.

Dotted bracket expressions are bracket expressions, or vector invariants, which have additional anti-symmetry indicated on certain sets of points. In White (this volume) we have seen dotted bracket expressions arise as the result of evaluating simple Cayley algebra expressions. They are also very useful in expressing invariants of anti-symmetric tensors or Cayley algebra extensors, as seen in McMillan (1989), an application which we will not explore in the present work. We will achieve some notational convenience by equating dotted bracket expressions with tableaux. We will obtain identities for dotted brackets and then mimic some observations of classical invariant theory, in particular the classical straightening algorithm of Young (1928), that created a standard basis for the space of vector invariants. We will show that in the case of a dotted bracket expression, the classical straightening algorithm can be expressed in a much more compact form, which can also be implemented in a much more efficient fashion. This compact straightening algorithm maintains the dotted form of the expression at every step, and achieves the same outcome as the ordinary straightening algorithm. We will restrict our attention to homogeneous multilinear dotted bracket expressions, that is, those bracket expressions which have precisely the same set of points occurring among the brackets of each monomial, with each such point occuring precisely once in each monomial.

For us, a tableau is an $m \times n$ array, delimited by parentheses. The tableau entries will be lower case Roman letters. We reserve upper case letters to denote blocks of entries in rows of a tableau.

Suppose we have a bracket product of degree m in brackets of length n, involving distinct points, that is dotted in the sets of points $a_1 a_2 \ldots a_{k_1}$, $b_1 b_2 \ldots b_{k_2}, \ldots, d_1 d_2 \ldots d_{k_j}$. We define the tableau described below to be equal to this dotted product.

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1) Rows of the tableau correspond to brackets, the first row to the first bracket etc.

2) The entries of the rows will be the same letters as in the corresponding brackets without subscripts.

3) There is a $\pm$ sign attached, that being sign(σ) where σ is the permutation that takes the bracket entries in the order they are given in the dotted bracket expression and orders them lexicographically, $a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2}, \ldots$.

EXAMPLE.

$$[a_1 a_2 a_3 \overset{\bullet}{c}_1][\overset{\blacktriangle}{b}_1 \overset{\blacktriangle}{b}_2 \overset{\bullet}{c}_2 \overset{\bullet}{c}_3][\overset{\blacktriangle}{b}_3 d_1 d_2 d_3] = - \begin{pmatrix} a & a & a & c \\ b & b & c & c \\ b & d & d & d \end{pmatrix}.$$

Note that the above example is a signed sum of nine ordinary bracket monomials.

We have the negative sign since the odd permutation $\sigma = (b_1 c_1 c_2 c_3 b_3 b_2)$ gives

$$\sigma\{a_1 a_2 a_3 c_1 b_1 b_2 c_2 c_3 b_3 d_1 d_2 d_3\} = \{a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3 d_1 d_2 d_3\}.$$

We will sometimes refer to this type of tableau representation as a **compact tableau** to distinguish it from ordinary tableau representation of bracket polynomials, which uses one tableau for each monomial.

Note that a particular compact tableau corresponds to different appearing yet equal versions of a dotted expression. For example:

$$\begin{aligned} \begin{pmatrix} a & a & a & c \\ b & b & c & c \\ b & d & d & d \end{pmatrix} &= -[a_1 a_2 a_3 \overset{\bullet}{c}_1][\overset{\blacktriangle}{b}_1 \overset{\blacktriangle}{b}_2 \overset{\bullet}{c}_2 \overset{\bullet}{c}_3][\overset{\blacktriangle}{b}_3 d_1 d_3 d_3] \\ &= [a_1 a_2 a_3 \overset{\bullet}{c}_2][\overset{\blacktriangle}{b}_1 \overset{\blacktriangle}{b}_2 \overset{\bullet}{c}_1 \overset{\bullet}{c}_3][\overset{\blacktriangle}{b}_3 d_1 d_2 d_3] \\ &= [a_1 a_2 a_3 \overset{\bullet}{c}_1][\overset{\blacktriangle}{b}_1 \overset{\blacktriangle}{b}_3 \overset{\bullet}{c}_2 \overset{\bullet}{c}_3][\overset{\blacktriangle}{b}_2 d_1 d_2 d_3] \end{aligned}$$

etc.

It will usually be our convention when associating a tableau with dotted brackets to use the dotted bracket expression where the subscripts of each letter are in order across the bracket expression. So among all the choices we would use:

$$\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} = -[a_1 a_2 \overset{\bullet}{b}_1 \overset{\blacktriangle}{c}_1][\overset{\bullet}{b}_2 \overset{\blacktriangle}{c}_2 d_1 d_2].$$

EXAMPLE. An example of a non-trivial dotted bracket expression is the superbracket of six pairs of points in rank four, which is the determinant of the Plücker coordinate vectors of the six lines determined by the pairs. This invariant is very important in structural rigidity, for if two rigid bodies are joined by six rigid bars attached at flexible endpoints, then the whole structure is infinitesimally rigid if and only if the superbracket of the six lines determined by the bars is non-zero. The superbracket yields the following tableaux in McMillan(1989):

$$\begin{pmatrix} a & a & d & e \\ b & b & d & f \\ c & c & e & f \end{pmatrix} - \begin{pmatrix} a & b & d & d \\ a & c & e & e \\ b & c & f & f \end{pmatrix}.$$

Each of these two tableaux represents a bracket monomial dotted in three pairs of points, and each is therefore equal to a signed sum of 8 ordinary bracket monomials.

The dotted brackets impose an algebra on compact tableaux. We will establish identities in compact tableaux which correspond to ordinary syzygies on the corresponding dotted bracket expressions. First we need a preliminary observation.

Suppose T is a compact tableau in letters $a, b, c, d, e, \ldots$ with k occurrences of letter d, with h of them in the first row, i in the second row, ..., and j in the last row. Assume that d corresponds to k letters $d_1, d_2, \ldots, d_k$ in the equivalent dotted bracket expression. We show that T is equal to a sum of tableaux in the modified letters $a, b, c, d_1, d_2, \ldots, d_k, e, \ldots$.

THEOREM 1. *Let T be a compact tableau on letters $a, b, c, d, e, \ldots$ with k occurrences of letter d. If*

$$T = \begin{pmatrix} X & \ldots & d & \ldots & d & \ldots & Y \\ Z & \ldots & d & \ldots & d & \ldots & W \\ & & & \vdots & & & \\ U & \ldots & d & \ldots & d & \ldots & V \end{pmatrix}$$

and σ is a permutation of $1, 2, \ldots, k$, we define T_σ, a compact tableau in the letters $a, b, c, d_1, d_2, \ldots, d_k, e, \ldots$ by

$$T_\sigma = \begin{pmatrix} X & \ldots & d_{\sigma_1} & \ldots & d_{\sigma_h} & \ldots & Y \\ Z & \ldots & d_{\sigma_{h+1}} & \ldots & d_{\sigma_i} & \ldots & W \\ & & & \vdots & & & \\ U & \ldots & d_{\sigma_{k-j+1}} & \ldots & d_{\sigma_k} & \ldots & V \end{pmatrix}$$

Then

$$T = \sum_\sigma T_\sigma,$$

where the sum runs over all split-shuffles σ of the d_i, and is unsigned.

PROOF:

$$T = \begin{pmatrix} X & \dots & d & \dots & d & \dots & Y \\ Z & \dots & d & \dots & d & \dots & W \\ & & & \vdots & & & \\ U & \dots & d & \dots & d & \dots & V \end{pmatrix}$$

$$= sign(T)[\overset{\bullet}{a_1} \ldots \overset{\blacktriangle}{d_1} \overset{\blacktriangle}{d_2} \ldots \overset{\blacktriangle}{d_h} \overset{\blacksquare}{e_1} \ldots][\ldots \overset{\blacktriangle}{d_{h+1}} \ldots \overset{\blacktriangle}{d_i} \ldots] \ldots [\ldots \overset{\blacktriangle}{d_k} \ldots] \ldots.$$

The dottings of the distinct letters are split-shuffles of disjoint sets. We can expand a particular dotting and leave the others intact as dottings. We expand the d_i and get

$$T = sign(T) \sum_{\sigma} sign(\sigma)[\overset{\bullet}{a_1} \ldots d_{\sigma_1} d_{\sigma_2} \ldots d_{\sigma_h} \overset{\blacksquare}{e_1} \ldots][\ldots d_{\sigma_{h+1}} \ldots d_{\sigma_i} \ldots] \ldots [\ldots d_{\sigma_k} \ldots] \ldots.$$

Writing the bracket sum as a sum of tableaux we have:

$$T = sign(T) \sum_{\sigma} sign(\sigma) sign(T_\sigma) T_\sigma$$

Now we show that for all σ the product $sign(T)sign(\sigma)sign(T_\sigma) = 1$. Suppose δ is the permutation that orders the vectors of the dotted bracket expression corresponding to T, so $sign(T) = sign(\delta) = sign(T_I)$, where I is the identity permutation. Let δ_σ be the permutation that orders the vectors of the dotted bracket expression corresponding to T_σ. Then clearly $\delta = \delta_I = \delta_\sigma \sigma$ for all σ, thus $sign(T)sign(\sigma)sign(T_\sigma) = 1$, completing the proof.

We will now establish an identity in compact tableaux, i.e. dotted brackets. First we define a split sum over a multiset. Let our split be a partition of a multiset whose elements are from a linearly ordered set, $(a, b, c, \ldots)$. A shuffle of a particular split of the multiset is a permutation of the elements of the multiset such that each block of the split is ordered by the linear order on the underlying set. If we pick certain entries of our tableau with repeated letters, the rows of the tableau effect a split. We will call the sum over all shuffles that net distinct summands the **multiset split-sum**. For the tableau

$$\begin{pmatrix} a & a & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{pmatrix},$$

the multiset split-sum over the boldface letters is

$$\begin{pmatrix} a & a & c & d \\ b & b & c & d \end{pmatrix} + \begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} + \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}$$

$$+\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

Note that there is some ambiguity over which shuffle caused certain terms to appear in the split sum. For example, the first term of the sum could be a consequence of the identity permutation on the original tableau or a consequence of the transposition of the c in the first row and the c in the second row. Because of a tableau signature convention we will not need to be concerned with which shuffle netted a particular term, only that every possible term appear in the sum. We can however characterize the permutations of a multiset split-sum as the set of all shuffles that do not exchange copies of the same letter between rows.

Note also that if the multiset happens to have no repeated letters, we have a split of m distinct letters into k blocks of size i_k. Then the tableau split-sum will have $\binom{m}{i_1 \ldots i_k}$ terms.

The identity we will prove will have the form of the van der Waerden syzygies on brackets. Split-sums with appropriate coefficients will add to zero. We will use boldface to denote the set of letters to be shuffled in the identity.

THEOREM 2. *If T is a rectangular compact tableau with two rows, n columns, and at least $n + 1$ letters boldface and T has the property that if any letter is boldface, then all occurrences of that letter in the same row are boldface, then we have the following identity:*

$$\sum_{\delta} (c_{\delta_a} c_{\delta_b} c_{\delta_c} \ldots c_{\delta_d}) \quad \delta \begin{pmatrix} \ldots & A & \ldots & \mathbf{a} & \ldots & \mathbf{b} & \ldots \\ \ldots & \mathbf{c} & \ldots & \mathbf{d} & \ldots & E & \ldots \end{pmatrix} = 0,$$

where the sum ranges over all multiset split shuffles of the boldfaced letters and the coefficients, c_{δ_x}, are determined relative to the effect of δ on the boldface letters as follows:

for the letter x, δ moves $p - j$ x's into a row with j x's, $p \geq j \geq 0$.

1) if the j x's are boldface, then $c_{\delta_x} = 1$.

2) if the j x's are not boldface, then $c_{\delta_x} = \binom{p}{j}$.

This theorem, proved independently by McMillan (1989), also follows from the Exchange Lemma, p. 60, of Grosshans et. al. (1987). See also Huang et. al. (1990).

EXAMPLE. The identity on the boldface letters of

$$\begin{pmatrix} a & b & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{pmatrix}$$

gives

$$\begin{pmatrix} a & b & c & d \\ b & b & c & d \end{pmatrix} + 1 \cdot 2 \cdot 1 \cdot 1 \begin{pmatrix} a & b & b & d \\ b & c & c & d \end{pmatrix} + 1 \cdot 2 \cdot 1 \cdot 2 \begin{pmatrix} a & b & b & c \\ b & c & d & d \end{pmatrix}$$

$$+1\cdot 3\cdot 1\cdot 2\begin{pmatrix} a & b & b & b \\ c & c & d & d \end{pmatrix} + 1\cdot 1\cdot 1\cdot 2\begin{pmatrix} a & b & c & c \\ b & b & d & d \end{pmatrix} = 0.$$

Equivalently,

$$\begin{pmatrix} a & b & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{pmatrix}$$

$$= -2\begin{pmatrix} a & b & b & d \\ b & c & c & d \end{pmatrix} - 4\begin{pmatrix} a & b & b & c \\ b & c & d & d \end{pmatrix} - 6\begin{pmatrix} a & b & b & b \\ c & c & d & d \end{pmatrix}$$

$$-2\begin{pmatrix} a & b & c & c \\ b & b & d & d \end{pmatrix}.$$

This is the form of the identity we will use, where we substitute the sum on the right side of the equation for the tableau on the left side. Note that in our notation, a tableau with boldface letters is equal to the same tableau without boldface, since the boldface notation is used only to inform the reader how Theorem 2 is being applied.

Theorem 2 was stated and proven for tableaux with two rows, for convenience. It is evident that the identity is equally valid for tableaux with more than two rows, where the boldface letters are restricted to two of the rows. The rows with no boldface letters are unchanged by the identity.

The entries of our tableaux are from a linearly ordered set. As we have noted the signature convention allows us to order the entries of the rows as we wish. Now we adopt the convention of ordering the row entries in ascending order. We also order the rows lexicographically in ascending order, treating the rows as n-letter words. In the dotted bracket correspondence this is just a matter of commuting brackets in products and we record any sign changes imposed on our compact tableaux by the signature convention. With these ordering conventions, we can now impose an order on the tableaux. If T_1 and T_2 are $m \times n$ tableaux on the same letters and w_1 and w_2 are words obtained by concatenating the rows of the respective T_i, row 1 joined by row 2, etc., then we say $T_1 \leq T_2$ if and only if $w_1 \leq w_2$ (in the lexicographical order on the w_i). We can make this comparison only after adopting the convention of ordering the row entries and the rows of the tableau. From now on when we refer to tableaux we will assume that this convention is adopted unless explicitly stated otherwise.

We define a **standard compact tableau** to be a tableau whose row entries are ascending and whose column entries are strictly ascending.

EXAMPLE.

$\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}$ is a standard compact tableau, while

$\begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix}$ is a compact tableau which is non-standard.

LEMMA 1. *Suppose T is a nonstandard compact tableau with entry y in row i and entry x in row j, with both entries in the same column and $x \leq y, i < j$. If we boldface y and all the entries of row i to the right of y together with all other occurrences of y in row i, and boldface x and all the entries of row j to the left of x together with all other occurrences of x in row j, and apply Theorem 2, then we realize T as a sum of compact tableaux $T = -\sum_\delta c_\delta T_\delta$ where $T_\delta < T$ for every δ.*

PROOF:

$$T = \begin{pmatrix} & \vdots & \\ A & y & B \\ & \vdots & \\ C & x & D \\ & \vdots & \end{pmatrix}.$$

By our row ordering convention if $b \in B$ and $c \in C$, then $c \leq x \leq y \leq b$. A shuffle, δ, exchanges certain boldface letters of T. Since the shuffles do not transpose the same letters between rows, the smallest letter shuffled from row j is strictly less than the smallest letter shuffled from row i. Then the resulting row i of T_δ is necessarily smaller than row i of T. In fact, row i of T_δ may be smaller than some previous row of T_δ, so by our convention we reorder the rows of T_δ, but in any case, $T_\delta < T$.

EXAMPLE.

$$\begin{pmatrix} a & a & \mathbf{e} & \mathbf{f} \\ \mathbf{c} & \mathbf{c} & \mathbf{d} & f \\ c & d & d & e \end{pmatrix} = -\begin{pmatrix} a & a & c & f \\ c & d & d & e \\ c & d & e & f \end{pmatrix} - \begin{pmatrix} a & a & d & f \\ c & c & e & f \\ c & d & d & e \end{pmatrix}$$

$$-2\begin{pmatrix} a & a & c & e \\ c & c & e & f \\ c & d & d & e \end{pmatrix} - 2\begin{pmatrix} a & a & d & e \\ c & c & f & f \\ c & d & d & e \end{pmatrix}$$

$$-2\begin{pmatrix} a & a & c & c \\ c & d & d & e \\ d & e & f & f \end{pmatrix} - 2\begin{pmatrix} a & a & c & d \\ c & d & d & e \\ c & e & f & f \end{pmatrix}.$$

THEOREM 3 (THE DOTTED STRAIGHTENING ALGORITHM). *The standard compact tableaux form a basis for the algebra of compact tableaux imposed by the dotted bracket correspondence.*

PROOF: The proof follows from Grasshans et. al. (1987), p. 27, and again was obtained independently by McMillan. Since the proof at this point is both short and enlightening, we include it for the benefit of the reader.

If T is a nonstandard tableau, we apply Lemma 1, obtaining an equal sum of tableaux. We set aside those tableaux in the sum which are standard and apply the lemma again to those which are nonstandard. We continue this process iteratively. Since there are a finite number of tableaux on the letters of T having the same shape as T, and since the smallest tableau among these,

$$\begin{pmatrix} a & a & \ldots & a & b & b & \ldots & b & c & \ldots \\ c & \ldots & c & d & d & \ldots & d & e & e & \ldots \\ & & & & \vdots & & & & & \end{pmatrix}$$

is standard, this process must end with T realized as a sum of standard tableaux.

It remains to show that the standard compact tableaux are independent. To this end, we first observe that the expansion of a compact standard tableau is a linear combination of ordinary standard tableaux, and that each ordinary standard tableau arises in this fashion from a unique compact standard tableau. Now suppose that T_i are standard compact tableaux on letters $a, b, c, \ldots$ and $\sum \alpha_i T_i = 0$. The tableau-dotted bracket correspondence gives:

$$0 = \sum_i \alpha_i T_i = \sum_i sign(T_i)\alpha_i [\overset{\bullet}{a}_1 \ldots][\overset{\blacktriangle}{b}_j \ldots] \ldots .$$

Expanding the dottings,

$$0 = \sum_i sign(T_i)\alpha_i \sum_{\sigma,\tau,\ldots} sign(\sigma\tau\cdots)[a_{\sigma 1}\ldots][b_{\tau j}\ldots]\ldots ,$$

which we can write as a sum of ordinary tableaux on letters $a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2}, \ldots$,

$$0 = \sum_i sign(T_i)\alpha_i \sum_{\sigma,\tau,\ldots} sign(\sigma\tau\cdots) sign(T_{i\sigma\tau\ldots}) T_{i\sigma\tau\ldots} .$$

The tableaux of this sum are distinct and standard. The independence of these ordinary standard tableaux is well-known, see for example Hodge and Pedoe (1947), hence $\alpha_i = 0$, for all i. Therefore the standard compact tableaux are independent. Note that the standard compact tableaux that we have obtained, when rewritten as expanded dotted bracket expressions, are precisely those that would have been obtained by the ordinary straightening algorithm. This fact is obvious just from the observation that they are indeed standard in the usual sense, and by the fact that ordinary standard products of brackets form a basis of all bracket expressions, we must have obtained precisely these.

EXAMPLE. In the examples, a tableau with boldface letters indicates that this tableau will be substituted for in the next step, using Lemma 1 on the boldface letters. We now completely straighten a non-standard tableau.

$$\begin{aligned}
\begin{pmatrix} a & a & \mathbf{c} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & d \end{pmatrix} &= -\begin{pmatrix} a & a & b & \mathbf{d} \\ \mathbf{b} & \mathbf{c} & \mathbf{c} & \mathbf{d} \end{pmatrix} - 2\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} \\
&\quad - 2\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} - 2\begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix} \\
&= 2\begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} \\
&\quad - 2\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} - 2\begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix} \\
&= -\begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix}.
\end{aligned}$$

EXAMPLE. The superbracket, as previously presented, straightens as follows:

$$\begin{pmatrix} a & a & d & e \\ b & b & d & f \\ c & c & e & f \end{pmatrix} - \begin{pmatrix} a & b & d & d \\ a & c & e & e \\ b & c & f & f \end{pmatrix}$$

$$= \begin{pmatrix} a & a & b & b \\ c & c & d & e \\ d & e & f & f \end{pmatrix} - \begin{pmatrix} a & a & b & c \\ b & c & d & d \\ e & e & f & f \end{pmatrix} - \begin{pmatrix} a & a & b & c \\ b & d & d & e \\ c & e & f & f \end{pmatrix} + \begin{pmatrix} a & a & b & d \\ b & c & c & e \\ d & e & f & f \end{pmatrix}.$$

The equivalence of dotted bracket expressions can be well concealed by bracket or tableau syzygies. To determine whether two dotted bracket expressions in the same set of points are in fact equal we straighten their difference and see if we get 0.

EXAMPLE. Let $f(a_1a_2, b_1b_2, c_1c_2, d_1d_2) = [a_1a_2d_1d_2][b_1b_2c_1c_2] - [a_1a_2c_1c_2][b_1b_2d_1d_2]$ and $g(a_1a_2, b_1b_2, c_1c_2, d_1d_2) = [a_1a_2\dot{b}_1\overset{\blacktriangle}{d}_1][\dot{b}_2c_1c_2\overset{\blacktriangle}{d}_2] - [a_1a_2b_1b_2][c_1c_2d_1d_2]$. To test whether $f = g$, we straighten the equivalent compact tableau expression of $f - g$.

$$\begin{pmatrix} a & a & \mathbf{d} & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & \mathbf{c} \end{pmatrix} - \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} + \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

$$= -\begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} - \begin{pmatrix} a & a & c & d \\ b & b & c & d \end{pmatrix} - \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

$$-\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} - \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} + \begin{pmatrix} a & a & b & d \\ b & c & c & d \end{pmatrix} + \begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}$$

$$= -\begin{pmatrix} a & a & c & \mathbf{d} \\ \mathbf{b} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{pmatrix} - \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} - 2\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix}$$

$$= \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix} + 2\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} - \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}$$

$$-2\begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix} \qquad = 0.$$

The van der Waerden syzygies and the straightening algorithm developed above may be regarded as a model for the relations and straightening algorithm presented in terms of the superalgebra of Grosshans, et al. (1987), specifically for the case of positive letters and negative places. Their divided power of a positive letter corresponds to our use of repeated occurrences of that letter in our compact tableaux.

In implementing the dotted straightening algorithm we order the compact tableaux of the dotted bracket expression to be straightened and do the iterative straightening on the largest nonstandard tableau in the queue. This way we never repeat the processing of a nonstandard tableau. Both the ordinary and dotted straightening algorithms have time complexity which is quadratic in the total number of tableaux (modulo our ordering conventions) which are of the same shape as the input tableaux. It is clear that if we have a reasonable amount of dotting, there are considerably fewer compact tableaux than ordinary tableaux, and thus that the dotted straightening algorithm is much faster given a dotted bracket polynomial as input. For example, working with 2×4 tableaux, there are 35 ordinary tableaux on 8 distinct letters, but if our compact tableaux are on 4 letters occurring twice each, then there are only 10 compact tableaux of the same shape.

The dotted straightening algorithm may be used in the the Cayley Factorization algorithm of White (1991), in this volume. In particular, the first step of that algorithm is to find the extensors, or sets of points in which the bracket polynomial is antisymmetric, and then to actually rewrite the polynomial so that it is explicitly dotted in those sets of points. Since this dotting has to be determined in any case, we do realize the efficiency improvement of the previous paragraph.

EXAMPLE. For a particular set of dotted points in a particular rank space we can list the basis for the space of all dotted bracket expressions. In rank four the linear invariant functions of the two-extensors $a_1a_2, b_1b_2, c_1c_2, d_1d_2$ are linear combinations of the functions:

$$[a_1a_2b_1b_2][c_1c_2d_1d_2], \qquad [a_1a_2\overset{\bullet}{b_1}\overset{\blacktriangle}{c_1}][\overset{\bullet}{b_2}\overset{\blacktriangle}{c_2}d_1d_2], \qquad [a_1a_2c_1c_2][b_1b_2d_1d_2],$$

as it is easy to verify that the three tableaux,

$$\begin{pmatrix} a & a & b & b \\ c & c & d & d \end{pmatrix}, \quad \begin{pmatrix} a & a & b & c \\ b & c & d & d \end{pmatrix}, \quad \begin{pmatrix} a & a & c & c \\ b & b & d & d \end{pmatrix},$$

are the only 2×4 standard tableaux from the multiset $\{a, a, b, b, c, c, d, d\}$.

An interesting problem is determining the number of standard (compact) tableaux of a specified shape, with entries from a particular multiset. There is no known generalization of the hook length formula of Frame, et al. (1954), to the problem of counting the number of standard tableaux with specified repeated letters. This number is known as a Kostka number; see McDonald (1979).

Suppose we have a bracket polynomial that is antisymmetric in the sets of points (or extensors) $x_1 x_2 \ldots x_{k_1}, y_1 y_2 \ldots y_{k_2}, \ldots, z_1 z_2 \ldots z_{k_j}$. It is straightforward by summing over the signed permutations of the polynomial to construct a dotted bracket expression for this invariant. From this we can realize our invariant as a compact tableau expression. It is, however, arbitrary which letters of the tableaux we associate with the tensor arguments of the function. Assuming we use the Roman alphabet in alphabetical order, we can associate the a's with any tensor we wish. We establish the tableau expression and then straighten it. In standard tableaux the a's must appear in the first row. In the dotted bracket expression corresponding to the standard tableaux the vector factors of the tensor associated with the letter a are in the first bracket of each term. We say that the bracket expression is **rectified** in the extensor a if all occurrences of a are in the same bracket for every monomial. We have established the following.

COROLLARY 1. *A dotted bracket expression can be rectified in any one of its extensor arguments.*

In the last example, the three dotted bracket expressions are each rectified in the extensors a and d.

There is another interesting observation made evident by the fact that the linear dimension of the space of all tableaux is constant despite the choice of how tableau letters are associated with extensor arguments of linear invariant functions. The number of standard tableaux of a particular shape with entries from a multiset with k_1 occurrences of one letter, and k_2 occurrences of a second letter, k_3 occurrences of a third letter, etc., is independent of which letter occurs k_1 times and which occurs k_2 times, etc. For instance, the number of 3×4 standard tableaux on letters $\{a, a, a, b, b, c, c, d, e, e, e, f\}$,

$$\begin{pmatrix} a & a & a & b \\ b & c & c & d \\ e & e & e & f \end{pmatrix}, \quad \begin{pmatrix} a & a & a & b \\ b & c & d & e \\ c & e & e & f \end{pmatrix}, \quad \begin{pmatrix} a & a & a & c \\ b & b & c & e \\ d & e & e & f \end{pmatrix}, \ldots$$

is the same as the number of standard tableaux on letters $\{a, a, b, b, b, c, d, d, d, e, f, f\}$,

$$\begin{pmatrix} a & a & b & b \\ b & c & d & d \\ d & e & f & f \end{pmatrix}, \quad \begin{pmatrix} a & a & b & c \\ b & b & d & e \\ c & d & f & f \end{pmatrix}, \quad \begin{pmatrix} a & a & b & d \\ b & b & c & e \\ d & d & f & f \end{pmatrix} \ldots .$$

In general the count depends only on the shape and the unordered set of multiplicities, $\{k_1, k_2, k_3, \ldots\}$, of the letters in the tableaux.

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