Matrix decomposition MFS algorithms for elasticity and thermo-elasticity problems in axisymmetric domains

Andreas Karageorghis*, Yiorgos-Sokratis Smyrlis

Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

Received 25 August 2006

Abstract

In this work, we propose an efficient matrix decomposition algorithm for the Method of Fundamental Solutions when applied to three-dimensional boundary value problems governed by elliptic systems of partial differential equations. In particular, we consider problems arising in linear elasticity in axisymmetric domains. The proposed algorithm exploits the block circulant structure of the coefficient matrices and makes use of fast Fourier transforms. The algorithm is also applied to problems in thermo-elasticity. Several numerical experiments are carried out.

MSC: primary 35E05; 35J55; 65N35; secondary 35J40; 41A30; 65N38

Keywords: Method of fundamental solutions; Elliptic systems; Cauchy–Navier equations; Thermo-elastostatics; Circulant matrices; Fast Fourier transform; Fast algorithms

1. Introduction

The Method of Fundamental Solutions (MFS) is a Trefftz-type technique in which the solution is approximated by a linear combination of fundamental solutions of the underlying partial differential operator with singularities (sources) placed outside the domain of the problem under consideration. This method is applicable when the fundamental solution of the underlying differential operator is known. The singularities of the fundamental solutions can be either free or preassigned. In the former, the solution of the discrete problems requires the solution of a nonlinear least-squares problem whereas in the latter, it requires the solution of a linear system.

In the current study, we use the MFS with the singularities fixed on a prescribed surface known as the pseudo-boundary. The coefficients in the linear combinations of fundamental solutions are determined by collocating the boundary conditions.

In recent years, the MFS has become popular mainly because of its simplicity and ease of implementation. It has been successfully applied to a large variety of physical problems. A review of such applications as well as the advantages of the MFS over other methods can be found in [10,14,15,22,23].

* This work was supported by University of Cyprus Grant #8037-3/312-21005.
* Corresponding author.
E-mail addresses: andreask@ucy.ac.cy (A. Karageorghis), smyrlis@ucy.ac.cy (Y.-S. Smyrlis).
The MFS has been widely used for the solution of problems in linear elasticity. The first application of the MFS for elasticity problems can be found in the 1964 paper [26], whereas a theoretical analysis and density results for problems of linear elasticity may be found in [24,42]. The solution of anisotropic elasticity problems was considered in [2,30]. In [31], inverse problems in planar elasticity were considered whereas axisymmetric elastostatics problems are studied in [17,38]. Recently, the MFS has been applied to the computation of stress intensity factors in linear elastic fracture mechanics [3,18]. The MFS was applied to thermo-elasticity problems in [1,27]. Further applications of the MFS to elasticity problems can be found in [8,12,33–37].

In this work, our goal is to propose efficient MFS discretizations with fixed sources and boundary collocation for the solution of the Cauchy–Navier equations in axisymmetric domains. This is achieved by developing efficient matrix decomposition algorithms (MDAs); MDAs are fast direct methods which reduce the solution of a problem to the solution of a set of independent problems of lower order. Such algorithms have been developed for the solution of linear systems resulting from finite difference, finite element, spectral and orthogonal spline collocation methods. Surveys of MDAs can be found in [4,5]. Efficient MDAs for the approximations of functions and their derivatives using radial basis functions are proposed in [16]. MDAs similar to the ones proposed in this study were used in the context of the boundary element method for harmonic problems in [32,39] and for linear elasticity problems in [40,41].

The MDAs developed in this work are nontrivial extensions of the corresponding algorithms developed for harmonic and biharmonic problems in axisymmetric domains [11,19,45,47] and make use of the block circulant nature of the coefficient matrices arising in the MFS discretization. The efficiency of the algorithms is improved further with the use of fast Fourier transforms (FFTs). Such algorithms for two-dimensional linear elasticity problems were proposed in [20].

The paper is organized as follows. In Section 2, we give a description of the MFS for general elliptic systems. In Section 3, we formulate the MFS for three-dimensional elasticity problems and develop an efficient MDA for such problems in axisymmetric domains. In Section 4, we test our algorithm on several examples. An extension of the proposed MDA is applied to thermo-elasticity problems in Section 5. Finally, in Section 6 we provide some concluding remarks.

2. The MFS for elliptic systems

2.1. Fundamental solutions of linear systems

Let \( \mathcal{L}u = 0 \) be a \( d \times d \) linear homogeneous system of partial differential equations, where

\[
\mathcal{L} u = \begin{pmatrix} \mathcal{L}_{11} & \cdots & \mathcal{L}_{1d} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{d1} & \cdots & \mathcal{L}_{dd} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{d} \mathcal{L}_{1j} u_j \\ \vdots \\ \sum_{j=1}^{d} \mathcal{L}_{dj} u_j \end{pmatrix},
\]

(2.1)

where \( u = (u_1, \ldots, u_d) \in \mathbb{R}^d \), \( \mathcal{L} = (\mathcal{L}_{ij})_{i,j=1}^{d} \) and \( \mathcal{L}_{ij} = \sum_{|\alpha| \leq m} a_{ij}^\alpha D^\alpha \) are scalar partial differential operators in \( \mathbb{R}^d \) with constant coefficients. Alternatively, \( \mathcal{L} = \sum_{|\alpha| \leq m} A_{2} D^\alpha \), with constant coefficient matrices \( A_{2} = (a_{ij}^\alpha)_{i,j=1}^{d} \).

A fundamental solution of \( \mathcal{L} \) is a matrix \( E = (e_{ij})_{i,j=1}^{d} \), where \( e_{ij} \) are real-valued functions, smooth in \( \mathbb{R}^d \setminus \{0\} \), satisfying

\[
\mathcal{L}^\ast E = \delta \mathcal{I} = \delta \mathcal{I},
\]

in the sense of distributions, where \( \delta \) is the Dirac measure with unit mass at the origin and \( \mathcal{I} \) is the identity matrix in \( \mathbb{R}^{d \times d} \). Equivalently,

\[
\int_{\mathbb{R}^d} E(x - y) \mathcal{L}^\ast \psi(y) \, dy = \psi(x)
\]

for every \( x \in \mathbb{R}^d \) and \( \psi = (\psi_1, \ldots, \psi_d) \) with \( \psi_i \in C_0^\infty(\mathbb{R}^d) \), where \( \mathcal{L}^\ast \) is the adjoint operator of \( \mathcal{L} \). If \( f_1, \ldots, f_d \) are measurable functions and

\[
u_i(x) = \sum_{j=1}^{d} \int_{\mathbb{R}^d} e_{ij}(x - y) f_j(y) \, dy, \quad i = 1, \ldots, d,
\]

(2.2)
then \( \mathcal{L}u = f \), where \( f = (f_1, \ldots, f_d) \) and \( u = (u_1, \ldots, u_d) \), provided that the integrals on the right-hand side of (2.2) are meaningful. Formulæ (2.2) can be written in vector form as

\[
\mathbf{u}(x) = \int_{\mathbb{R}^n} E(x - y) f(y) \, dy,
\]

or in the simpler form \( \mathbf{u} = E \ast \mathbf{f} \).

2.2. The MFS formulation

In the MFS for second order\(^1\) elliptic systems, the components of the approximate solution \( \mathbf{u}^N = (u_1^N, \ldots, u_d^N) \) is a linear combination of the form

\[
\begin{aligned}
\mathbf{u}_i^N(x) &= \sum_{k=1}^d \sum_{j=1}^N c^{k}_j e_{ik}(x - y_j), \\
&= \sum_{k=1}^d e_k^T \mathbf{c}_j \mathbf{e}_j,
\end{aligned}
\]

\[
\text{where } e_k, \; k = 1, \ldots, d, \text{ are the columns of the matrix } E.
\]

---

\(^1\) In the case of a higher order elliptic partial differential operator \( \mathcal{L} \), the linear combinations of fundamental solutions of \( \mathcal{L} \) are not dense in the space of all solutions of the same operator. For example, in the case of the biharmonic operator one needs to also include the fundamental solution of the Laplace operator. See \([6,42]\).

3. Three-dimensional elasticity problems

We consider the boundary value problem in $\mathbb{R}^3$ governed by the Cauchy–Navier equations of elasticity

\[(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} = 0 \quad \text{in } \Omega, \quad (3.1a)\]

\[u_i = f_i \quad \text{on } \partial \Omega. \quad (3.1b)\]

Here we are using the indicial tensor notation in terms of the displacements $u_1, u_2$ and $u_3$. System (3.1a) often appears in the literature as

\[\Delta^e u = \mu \Delta u + (\lambda + \mu) \text{grad div } u = 0.\]

The operator $\Delta^e$ is known as the Lamé operator. The constants $\lambda$ and $\mu$, which are positive, are known as the Lamé constants.

3.1. MFS formulation

The matrix $G(x) = (g_{ij}(x))_{i,j=1}^3$ with

\[g_{ij}(x) = -\frac{3\mu + \lambda}{8\pi \mu (2\mu + \lambda)} \delta_{ij} \frac{x_i x_j}{|x|^3} - \frac{\mu + \lambda}{8\pi \mu (2\mu + \lambda)} \frac{x_i x_j}{|x|^3}, \quad (3.2)\]

where $\delta_{ij}$ is the Kronecker delta and $|x|^2 = x_1^2 + x_2^2 + x_3^2$, is a fundamental solution of $\Delta^e$, i.e., $\Delta^e G = \delta^e$. Here $\delta^e$ is the identity matrix in $\mathbb{R}^3 \times \mathbb{R}^3$. Alternatively, expression (3.2) can be written as

\[G(x) = -\frac{1}{8\pi \mu (2\mu + \lambda)} \left( \frac{3\mu + \lambda}{|x|} \delta^e + \frac{\mu + \lambda}{|x|^3} x \cdot x^T \right), \quad (3.2')\]

where $x \cdot x^T = (x_i x_j)_{i,j=1}^3 \in \mathbb{R}^3$.

In the MFS [1,24,25,34,42] the displacements $u = (u_1, u_2, u_3)$ at the point $P \in \mathbb{R}^3$ are approximated by

\[u_{M,N}^i(c, Q; P) = \sum_{m=1}^M \sum_{n=1}^N G(P - Q_{m,n}) c_{m,n}, \quad (3.3)\]

or equivalently

\[u_{i, M,N}^i(c, Q; P) = \sum_{m=1}^M \sum_{n=1}^N \sum_{j=1}^3 c_{m,n}^j g_{ij}(P - Q_{m,n}), \quad i = 1, 2, 3, \]

where $c_{m,n} = (c_{m,n}^1, c_{m,n}^2, c_{m,n}^3) \in \mathbb{R}^3$, $Q = (Q_{m,n})_{m=1,\ldots,M}^{n=1,\ldots,N}$ with $Q_{m,n} = (x_{m,n}^Q, y_{m,n}^Q, z_{m,n}^Q) \in \mathbb{R}^3$ being the coordinates of the singularities (sources) which lie outside $\overline{\Omega}$. In Eq. (3.3), we use a double summation to indicate that the surface of a three-dimensional region is two-dimensional and the position of each source may be described by two parameters.

Note that the Cauchy–Navier equations can also be written as

\[\Delta u + \frac{1}{1 - 2\nu} \text{grad div } u = 0,\]

where $\nu$ is Poisson’s ratio. The Lamé constants can be expressed as

\[\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)},\]

where $E$ is the modulus of elasticity.

Expression (3.2) is due to Lord Kelvin (see [29]). For further details and the derivation of (3.2) see [28].
Let $\Omega$ be an open and bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary. Linear combinations of the form (3.3), with the singularities lying on a prescribed pseudo-boundary $\partial \Omega'$ of $\Omega$, are dense, with respect to the uniform norm of $C(\overline{\Omega})$, in the set of solutions of Cauchy–Navier system in $\Omega$ which extend continuously to $\overline{\Omega}$. For a proof and extensions see [42].

3.2. Axisymmetric surface discretization

In our work, the domain $\Omega \subset \mathbb{R}^3$ is axisymmetric, formed by the rotation of a region $\Omega' \subset \mathbb{R}^2$ about the $z$-axis.

The singularities $Q_{m,n}$ are fixed on the boundary $\partial \tilde{\Omega}$ of a solid $\tilde{\Omega}$ surrounding $\Omega$. The solid $\tilde{\Omega}$ is generated by the rotation of the planar domain $\tilde{\Omega}'$ which is similar to $\Omega'$. A set of $MN$ collocation points $\{P_{k,\ell}\}_{k=1,\ell=1}^{M,N}$ is chosen on $\partial \tilde{\Omega}$ in the following way: we first choose $N$ points on the boundary $\partial \Omega'$ of $\Omega'$. These can be described by their polar coordinates $(r_P^\ell, z_P^\ell), \ell = 1, \ldots, N$, where $r_P^\ell$ denotes the vertical distance of the point $P_\ell$ from the $z$-axis and $z_P^\ell$ denotes the $z$-coordinate of the point $P_\ell$. The points on $\partial \Omega$ are taken to be

$$P_{k,\ell} = (r_P^\ell \cos \psi_k, r_P^\ell \sin \psi_k, z_P^\ell),$$

where

$$\psi_k = \frac{2(k-1)\pi}{M}, \quad k = 1, \ldots, M. \tag{3.4}$$

Similarly, we choose a set of $MN$ singularities $\{Q_{m,n}\}_{m=1,n=1}^{M,N}$ on $\partial \tilde{\Omega}$ by taking

$$Q_{m,n} = (r_Q^m \cos \varphi_m, r_Q^m \sin \varphi_m, z_Q^m),$$

where

$$\varphi_m = \frac{2(m-1+\sigma)\pi}{M}, \quad m = 1, \ldots, M, \tag{3.5}$$

where the $N$ points $Q_m$ are chosen on the boundary $\partial \tilde{\Omega}'$ of $\tilde{\Omega}'$. The parameter $\sigma \in [0, \frac{1}{2})$ indicates a rotation of the sources by an angle $2\pi \sigma/M$ with respect to the boundary points in the azimuthal direction. Such a rotation has been used with other MFS approximations and can improve the accuracy of the method [11,45,46]. A typical distribution of singularities on an axisymmetric pseudo-boundary surrounding an axisymmetric solid is shown in Fig. 2.

Fig. 2. Typical distribution of sources on an axisymmetric pseudo-boundary of an axisymmetric domain.
3.3. Generation of linear system

In the MFS, the coefficients \( c \) are determined so that the boundary condition is satisfied at the boundary points \( \{ P_{k,\ell} \}_{k=1, \ell=1}^{M,N} \):

\[
\begin{align*}
\mathcal{A}_{1,1} \mathcal{A}_{1,2} \cdots \mathcal{A}_{1,N} & \begin{pmatrix} d^1 \\ d^2 \\ \vdots \\ d^N \end{pmatrix} = \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^N \end{pmatrix}.
\end{align*}
\]

Here

\[
\mathcal{A}_{\ell,\nu} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{pmatrix}
\]

and

\[
\mathcal{A}_{i,\nu}^k = g(P_{k,\ell} - Q_{\mu,\nu}) \in \mathbb{R}^{3 \times 3},
\]

\[
d^i = (d_1^i, \ldots, d_M^i) = (c_1^i, c_2^i, \ldots, c_M^i) \in \mathbb{R}^M,
\]

\[
f^i = (f_1^i, \ldots, f_M^i) = (f_1^i, f_2^i, \ldots, f_M^i) \in \mathbb{R}^M,
\]

and \( f_{i,\nu}^k = f_i(P_{k,\ell}) \), \( i = 1, 2, 3 \), \( k = 1, \ldots, M \) and \( \ell, \nu = 1, \ldots, N \).

3.4. Circulant structure of the system matrix

Clearly, the system matrix \( \mathcal{A} \) has no circulant structure. In fact, even the matrices \( \mathcal{A}_{\ell,\nu} \) in (3.8) are not block circulant. However, by a suitable transformation, system (3.7) can be written in an equivalent form where the coefficient matrix has a block circulant structure.

Let

\[
R_\vartheta = \begin{pmatrix} \cos \vartheta & \sin \vartheta & 0 \\ - \sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We pre-multiply system (3.7) by the matrix

\[
\mathcal{R} = \mathcal{I}_N \otimes R
\]

where \( \mathcal{I}_N \) is the identity matrix in \( \mathbb{R}^N \), and

\[
R = \begin{pmatrix} R_{\vartheta_1} & 0 & \cdots & 0 \\ 0 & R_{\vartheta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{\vartheta_M} \end{pmatrix}
\]

where \( \vartheta_k = 2\pi(k - 1)/M \).
Clearly,
\[ R_2^2 = I_3, \quad R^2 = I_{3M} \quad \text{and} \quad R^2 = I_{3MN}. \]

System (3.7) becomes
\[ \mathcal{R} \tilde{A} \tilde{d} = \mathcal{R} \tilde{A} \mathcal{R} \mathcal{R} \tilde{d} = \mathcal{R} \tilde{f}, \]
or
\[ \tilde{A} \tilde{d} = \tilde{f}, \quad (3.9) \]
where
\[ \tilde{A} = \mathcal{R} \mathcal{A} \mathcal{R}, \quad \tilde{d} = \mathcal{R} \tilde{d} \quad \text{and} \quad \tilde{f} = \mathcal{R} \tilde{f}. \]

### 3.5. Matrix decomposition algorithm

The matrix \( \tilde{A} \) can be written as
\[
\tilde{A} = R \tilde{A} R = \begin{pmatrix}
\tilde{A}_{1,1} & \tilde{A}_{1,2} & \cdots & \tilde{A}_{1,N} \\
\tilde{A}_{2,1} & \tilde{A}_{2,2} & \cdots & \tilde{A}_{2,N} \\
& \vdots & \ddots & \vdots \\
\tilde{A}_{N,1} & \tilde{A}_{N,2} & \cdots & \tilde{A}_{N,N}
\end{pmatrix}
\]
where
\[
\tilde{A}_{\ell,\nu} = R\tilde{A}_{\ell,\nu} R = \begin{pmatrix}
\tilde{A}_{1,1} & \tilde{A}_{1,2} & \cdots & \tilde{A}_{1,M} \\
\tilde{A}_{2,1} & \tilde{A}_{2,2} & \cdots & \tilde{A}_{2,M} \\
& \vdots & \ddots & \vdots \\
\tilde{A}_{M,1} & \tilde{A}_{M,2} & \cdots & \tilde{A}_{M,M}
\end{pmatrix}
\]
and
\[
\tilde{A}_{\ell,\nu}^{k,\mu} = R_{\partial_k} \tilde{A}_{\ell,\nu}^{k,\mu} R_{\partial_{\mu}}. \quad (3.10)
\]

Using the fact that (for a proof see Lemma A.1 in the Appendix)
\[
R_{\partial_k} \tilde{A}_{\ell,\nu}^{k,\mu} R_{\partial_{\mu}} = R_{\partial_{k+j}} \tilde{A}_{\ell,\nu}^{k+j,\mu+j} R_{\partial_{\mu+j}}, \quad (3.11)
\]
for every \( j \), such that \( j, k+j, \mu+j \in \{1, \ldots, M\} \), it follows that the matrix \( \tilde{A} \) has a block circulant structure, i.e., if \( k-\mu = k'-\mu' \mod M \), then \( \tilde{A}_{\ell,\nu}^{k,\mu} = \tilde{A}_{\ell,\nu}^{k',\mu'} \). In particular,
\[
\tilde{A}_{\ell,\nu} = \text{circ}(\tilde{A}_{\ell,\nu}^{1,1}, \tilde{A}_{\ell,\nu}^{1,2}, \ldots, \tilde{A}_{\ell,\nu}^{1,M}).
\]
Equivalently, we have
\[
\tilde{A}_{\ell,\nu} = \sum_{m=1}^{M} \mathcal{P}^{m-1} \otimes \tilde{A}_{\ell,\nu}^{1,m},
\]
where the matrix $\hat{P}$ is the matrix $\hat{P} = \text{circ}(0, 1, 0, \ldots, 0) \in \mathbb{R}^{M \times M}$. Here we take $\hat{P}^0 = I_M$. Thus,

\[
\hat{A} = \left( \begin{array}{c|c|c} \sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{1,1}^{m} & \sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{1,2}^{m} & \cdots & \sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{1,N}^{m} \\
\sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{2,1}^{m} & \sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{2,2}^{m} & \cdots & \sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{2,N}^{m} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{N,1}^{m} & \sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{N,2}^{m} & \cdots & \sum_{m=1}^{M} \hat{P}^{m-1} \otimes \hat{A}_{N,N}^{m} \\
\end{array} \right)
\]

From [9], the matrices $\hat{P}^m$ can be diagonalized as

\[
\hat{P}^m = U^* \hat{E}^m U,
\]

where $\hat{E} = \text{diag}(\hat{\omega}, \hat{\omega}^2, \ldots, \hat{\omega}^{M-1})$,

\[
U^* = \frac{1}{M^{1/2}} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \hat{\omega} & \hat{\omega}^2 & \cdots & \hat{\omega}^{M-1} \\
1 & \hat{\omega}^2 & \hat{\omega}^4 & \cdots & \hat{\omega}^{2(M-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \hat{\omega}^{M-1} & \hat{\omega}^{2(M-1)} & \cdots & \hat{\omega}^{(M-1)(M-1)}
\end{pmatrix},
\]

and $\hat{\omega} = e^{2\pi i/M}$. Pre-multiplying (3.9) by $I_N \otimes U \otimes I_3$ yields

\[
\hat{A} \hat{d} = \hat{f},
\]

where

\[
\hat{A} = (I_N \otimes U \otimes I_3) \hat{A} (I_N \otimes U^* \otimes I_3),
\]

\[
\hat{d} = (I_N \otimes U \otimes I_3) \hat{d} = (\hat{d}_i)_{i=1}^{MN}, \quad \hat{f} = (I_N \otimes U \otimes I_3) \hat{f} = (\hat{f}_i)_{i=1}^{MN}.
\]

The matrix $\hat{A}$ can be written as

\[
\hat{A} = \sum_{m=1}^{M} \begin{pmatrix}
\hat{E}^{m-1} \otimes \hat{A}_{1,1}^{m} & \hat{E}^{m-1} \otimes \hat{A}_{1,2}^{m} & \cdots & \hat{E}^{m-1} \otimes \hat{A}_{1,N}^{m} \\
\hat{E}^{m-1} \otimes \hat{A}_{2,1}^{m} & \hat{E}^{m-1} \otimes \hat{A}_{2,2}^{m} & \cdots & \hat{E}^{m-1} \otimes \hat{A}_{2,N}^{m} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{E}^{m-1} \otimes \hat{A}_{N,1}^{m} & \hat{E}^{m-1} \otimes \hat{A}_{N,2}^{m} & \cdots & \hat{E}^{m-1} \otimes \hat{A}_{N,N}^{m}
\end{pmatrix}
\]

Since the matrix $\hat{E}$ is diagonal, then clearly each block $\hat{E}^{m-1} \otimes \hat{A}_{i,j}^{m}$ is of block diagonal form, i.e.,

\[
\hat{E}^{m-1} \otimes \hat{A}_{i,j}^{m} = \text{diag}(\hat{A}_{i,j}^{m}, \hat{\omega}^{m-1} \hat{A}_{i,j}^{m}, \ldots, \hat{\omega}^{(m-1)(M-1)} \hat{A}_{i,j}^{m}).
\]
and so are the $N^2$ blocks of size $3M \times 3M$ of the matrix $A$:

$$
\sum_{m=1}^{M} e^{m-1} \otimes \tilde{A}_{i,j}^{1,m} = \operatorname{diag} \left( \sum_{m=1}^{M} \tilde{A}_{i,j}^{1,m}, \sum_{m=1}^{M} \omega^{m-1} \tilde{A}_{i,j}^{1,m}, \ldots, \sum_{m=1}^{M} \omega^{(M-1)(m-1)} \tilde{A}_{i,j}^{1,m} \right) = \operatorname{diag}(\tilde{A}_{i,j}^{1}, \tilde{A}_{i,j}^{2}, \ldots, \tilde{A}_{i,j}^{M}),
$$

where

$$\tilde{A}_{i,j}^{\ell} = \sum_{m=1}^{M} \omega^{(\ell-1)(m-1)} \tilde{A}_{i,j}^{1,m}, \quad \ell = 1, \ldots, M,$$

(i, j = 1, \ldots, N). The coefficient matrix $\tilde{A}$ in system (3.13) consists of $N \times N$ blocks of size $3M \times 3M$. Each of these blocks is block diagonal consisting of $M$ blocks of size $3 \times 3$. Therefore, system (3.13) can be decomposed into the $M$ independent $3N \times 3N$ systems:

$$
\begin{pmatrix}
\tilde{A}_{1,1}^{\ell} & \tilde{A}_{1,2}^{\ell} & \cdots & \tilde{A}_{1,N}^{\ell} \\
\tilde{A}_{2,1}^{\ell} & \tilde{A}_{2,2}^{\ell} & \cdots & \tilde{A}_{2,N}^{\ell} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{A}_{N,1}^{\ell} & \tilde{A}_{N,2}^{\ell} & \cdots & \tilde{A}_{N,N}^{\ell}
\end{pmatrix}
\begin{pmatrix}
\tilde{d}_{1,\ell} \\
\tilde{d}_{2,\ell} \\
\vdots \\
\tilde{d}_{N,\ell}
\end{pmatrix}
=
\begin{pmatrix}
\tilde{f}_{1,\ell} \\
\tilde{f}_{2,\ell} \\
\vdots \\
\tilde{f}_{N,\ell}
\end{pmatrix},
\quad \ell = 1, \ldots, M,
$$

(3.15)

where

$$
\tilde{d}_{j,\ell} = (\tilde{d}_{3M(j-1)+3\ell-2} \tilde{d}_{3M(j-1)+3\ell-1} \tilde{d}_{3M(j-1)+3\ell}),
$$

$$
\tilde{f}_{j,\ell} = (\tilde{f}_{3M(j-1)+3\ell-2} \tilde{f}_{3M(j-1)+3\ell-1} \tilde{f}_{3M(j-1)+3\ell}),
$$

with $j = 1, \ldots, N$ and $\ell = 1, \ldots, M$.

Once $\tilde{d}$ has been computed, the vector $\tilde{d}$ can be calculated from

$$
\tilde{d} = (\mathcal{F}_N \otimes U^* \otimes \mathcal{F}_3) \tilde{d}.
$$

(3.16)

Finally, from

$$
d = \mathcal{R} \tilde{d}
$$

(3.17)

we recover $d$, the solution of system (3.7).

Note that, because of the block circulant structure of the matrix $A$, we do not need to construct all the entries of either matrix $A$ or matrix $\tilde{A}$. We only need to construct the submatrices $A_{1,v}^{1,m}$, $m = 1, \ldots, M$, $\ell, v = 1, \ldots, N$. For the construction of these we first need to construct the submatrices $A_{1,v}^{1,m}$, $m = 1, \ldots, M$, $\ell, v = 1, \ldots, N$.

System (3.7) can thus be solved efficiently with the following:

**Algorithm**

**Step 1:** Compute $\tilde{f} = \mathcal{R} f$.

**Step 2:** Compute the submatrices $\tilde{A}_{1,v}^{1,m}$, $m = 1, \ldots, M$, $\ell, v = 1, \ldots, N$ from (3.10).

**Step 3:** Compute $\tilde{f} = (\mathcal{F}_N \otimes U \otimes \mathcal{F}_3) \mathcal{F}$ and hence $\tilde{f}_{n,\ell}$, $n = 1, \ldots, N$, $\ell = 1, \ldots, M$.

**Step 4:** Construct the matrices $A_{i,j}^{\ell}$, $\ell = 1, \ldots, M$, $i, j = 1, \ldots, N$ from formula (3.14).

**Step 5:** Solve the $M$ systems of order $3N$ in (3.15).

**Step 6:** Compute $\tilde{d}$ from (3.16).

**Step 7:** Compute $d$ from (3.17).
Cost.

- In Steps 3 and 6, because of the form of the matrices \( U \) and \( U^* \), the operations can be carried out via FFTs and inverse FFTs at a cost of \( O(NM \log M) \).
- In Step 4, for each \( i, j = 1, \ldots, N, \ell = 1, \ldots, M, \) we need to perform an \( M \)-dimensional inverse FFT, in order to compute the nine entries of each \( \hat{A}^{\ell}_{i,j} \). This can be done at a cost of \( O(N^2 M \log M) \).
- In Step 5, we need to solve \( M \) complex linear systems of order \( 3N \). This is done using an LU-factorization with partial pivoting at a cost of \( O(MN^3) \) operations.
- The FFT operations are performed using the NAG4 routines C06FPF, C06FQF and C06FRF.

4. Numerical results

4.1. Axisymmetric solids considered

In our numerical experiments we considered the following axisymmetric solids.

4.1.1. Spherical domains

Consider the case where \( \Omega \subset \mathbb{R}^3 \) is the sphere of radius \( q \):

\[
\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < q^2\}.
\] (4.1)

The singularities \( \{Q_{m,n}\}_{m=1,n=1}^{M,N} \) are fixed on the boundary \( \partial \Omega \) of the sphere

\[
\tilde{\Omega} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < R^2\},
\]

where \( R > q \), and \( Q_{m,n} = (xQ_{m,n}, yQ_{m,n}, zQ_{m,n}) \) with

\[
Q_{m,n} = R(\sin \varphi_m \cos \psi_m, \sin \varphi_m \sin \psi_m, \cos \varphi_m),
\]

where \( \varphi_m \) is given in (3.5) and

\[
\varphi_n = \frac{n \pi}{N + 1}, \quad n = 1, \ldots, N.
\]

The MN collocation points \( \{P_{k,\ell}\}_{k=1,\ell=1}^{M,N} \) on \( \partial \Omega \) are given by

\[
P_{k,\ell} = q(\sin \chi_k \cos \psi_k, \sin \chi_k \sin \psi_k, \cos \chi_k),
\]

where \( \psi_k \) is given by (3.4).

Note that we avoid the points corresponding to \( \chi_k = 0 \) and \( \pi \) as they remain invariant under rotation in the azimuthal direction and would lead to singular matrices. (see Fig. 3(a.).)

4.1.2. Cylindrical domains

For the cylindrical domain

\[
\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < q^2, \quad -h < z < h\},
\] (4.2)

the MN singularities \( \{Q_{m,n}\}_{m=1,n=1}^{M,N} \) are given by

\[
Q_{m,n} = (r_Q \cos \varphi_m, r_Q \sin \varphi_m, z_Q),
\]

with \( \varphi_m \) as in (3.5) and \( (r_Q, z_Q) \), \( r_Q > 0, n = 1, \ldots, N, \) the polar coordinates of \( N \) points on the boundary of the rectangle \([0, R] \times [-H, H]\) with \( R > q \) and \( H > h \). The collocation points \( \{P_{k,\ell}\}_{k=1,\ell=1}^{M,N} \) are taken to be

\[
P_{k,\ell} = (r_P \cos \psi_k, r_P \sin \psi_k, z_P),
\]

\footnote{Numerical Algorithms Group Library Mark 20, NAG Ltd, Wilkinson House, Jordan Hill Road, Oxford, UK, 2001.}
with \((r^P_\ell, z^P_\ell), \ r^P_\ell > 0, \ \ell = 1, \ldots, N,\) the polar coordinates of \(N\) points on the boundary of the rectangle \([0, q] \times [-h, h].\) (see Fig. 3(b).)

### 4.1.3. Toroidal domains

Consider the torus of radii \(\rho_1, \rho_2\) with \(\rho_2 > \rho_1:\)

\[
\Omega = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - \rho_2)^2 + z^2 < \rho_1^2\}
\]  

(4.3)

whose boundary \(\partial\Omega\) is given by the parametric equations

\[
x = \rho_2 \cos \chi^2 + \rho_1 \cos \chi^1, \quad y = \rho_2 \sin \chi^2 + \rho_1 \sin \chi^1, \quad z = \rho_1 \sin \chi^1,
\]

with

\[0 \leq \chi^1, \quad \chi^2 \leq 2\pi.
\]

In this case, \(\tilde{\Omega}\) is a torus embracing \(\Omega,\) and which has boundary \(\partial \tilde{\Omega}\) defined by the parametric equations

\[
x = \rho_2 \cos \tilde{\chi}^2 + R_1 \cos \tilde{\chi}^1, \quad y = \rho_2 \sin \tilde{\chi}^2 + R_1 \sin \tilde{\chi}^1, \quad z = R_1 \sin \chi^1,
\]

where

\[\rho_1 < R_1 < \rho_2, \quad 0 \leq \chi^1, \quad \tilde{\chi}^2 \leq 2\pi.
\]

The singularities \(\{Q_{m,n}\}^{M,N}_{m=1,n=1}\) on \(\partial \tilde{\Omega}\) have coordinates

\[Q_{m,n} = (\rho_2 \cos \tilde{\chi}_n^2 + R_1 \cos \tilde{\chi}_n^1, \rho_2 \sin \tilde{\chi}_n^2 + R_1 \sin \tilde{\chi}_n^1, R_1 \sin \chi^1),\]

where \(\tilde{\chi}_n^2\) is as in (3.5), and

\[\chi_m^1 = \frac{2(m - 1)\pi}{M}, \quad m = 1, \ldots, M.
\]  

(4.4)

The \(MN\) collocation points \(\{P_{k,\ell}\}^{M,N}_{k=1,\ell=1}\) on \(\partial \Omega\) have coordinates

\[P_{k,\ell} = (\rho_2 \cos \chi_k^2 + \rho_1 \cos \chi_k^1 \cos \chi_1^1, \rho_2 \sin \chi_k^2 + \rho_1 \sin \chi_k^1 \cos \chi_1^1, \rho_1 \sin \chi_1^1),\]

where \(\chi_k^2\) is as in (3.4). (see Fig. 3(c).)

### 4.2. Numerical experiments

We considered boundary value problems of the form (3.1) with the function \(f = (f_1, f_2, f_3)\) corresponding to the exact solutions
Example 1. \( u_1 = 0.3e^{3x+4y} \cos(0.5z), \quad u_2 = 0.4e^{3x+4y} \cos(0.5z), \quad u_3 = -0.5e^{3x+4y} \sin(0.5z). \)

Example 2. \( u_1 = (3 - 4v)/r + (x - 10)^2/r^3, \quad u_2 = (x - 10)y/r^3, \quad u_3 = (x - 10)z/r^3, \) with \( r^2 = (x - 10)^2 + y^2 + z^2. \)

In all the numerical examples reported in this study, we calculated the maximum error on the boundary of \( \Omega \) for each of the components of the solution:

\[
E_i = \max_{x \in \partial \Omega} |u_i^{M,N}(x) - u_i(x)|, \quad i = 1, 2, 3,
\]

and

\[
E = \max\{E_1, E_2, E_3\}.
\]

We only study the error on the boundary because the error \( e^{M,N} = u^{M,N} - u \) satisfies the Cauchy–Navier system which obeys the following maximum principle:

If \( \Omega \) is bounded domain in \( \mathbb{R}^3 \) with smooth boundary, there exists a constant \( H \) depending on \( \Omega \), such that, for every solution \( u = (u_1, u_2, u_3) \in C^{2}(\Omega; \mathbb{R}^3) \cap C(\overline{\Omega}; \mathbb{R}^3) \) of (3.1a) the following inequality holds:

\[
\max_{x \in \partial \Omega} |u_i(x)| \leq H \max_{x \in \partial \Omega} |u_i(x)|, \quad i = 1, 2, 3.
\]

This result is due to Fichera [13]. (see also [7, Chapter 4].)

4.2.1. Sphere

In the case of the sphere, the error was calculated on a grid of boundary points defined by

\[
x_{i,j} = q \sin \vartheta \cos \varphi_i, \quad y_{i,j} = q \sin \vartheta \sin \varphi_i, \quad z_{i,j} = q \cos \vartheta,
\]

where \( \varphi_i = 2(i - 1)\pi/L, \quad i = 1, \ldots, L \) and \( \vartheta_j = j\pi/(L + 1), \quad j = 1, \ldots, L \) and \( L \) is taken to be appropriately large.

In particular, we considered the sphere with \( q = 1 \). In Fig. 4, we present the logarithm of the maximum error \( E \) versus the logarithm of the distance \( \varepsilon = R - q \) in the case \( \varepsilon = 0 \) and for \( N(=M) = 16, 24, 32, 48 \) in Example 1. This figure reveals that the MFS approximation is poor when \( \varepsilon = R - q \) is either very small or very large. Similar observations were reported in the case of Laplace’s equation [43,45]. The behavior of \( E \) is very similar in the case when \( \varepsilon \neq 0 \). In Fig. 5, we present the corresponding results for Example 2. The behavior of these results is very similar to the behavior of the results for Example 1. The poor results for large \( \varepsilon \) are due to ill-conditioning as has been reported several times in the literature (see, for example [21,45]), whereas the poor results for small \( \varepsilon \) are due to the near-singularity of the fundamental solutions (see for example [44]).

In Fig. 6, we present the logarithm of the maximum error \( E_1 \) versus the angular parameter \( \varepsilon \). Because of the symmetry of the problem about \( \varepsilon = \frac{1}{2} \), we only consider the values \( 0 \leq \varepsilon \leq \frac{1}{2} \). We present the four cases \( \varepsilon = R - q = .05, .1,.2,.5 \) for \( N(=M) = 12, 16, 24, 32, 48 \) and 64 in Example 1. In all cases, we observe that as \( N \) increases, the error \( E_1 \) has a tendency to have a minimum for \( \varepsilon \approx \frac{1}{2} \). Also, the larger the distance \( \varepsilon \), the less-dependent \( E_1 \) is on \( \varepsilon \). Similar observations were reported in the case of axisymmetric harmonic and biharmonic problems [11,45]. The corresponding results for Example 2 are presented in Fig. 7 and are very similar to the results for Example 1. The results for \( E_2 \) and \( E_3 \) are very similar to those for \( E_1 \).

4.2.2. Cylinder

We considered the cylinder with \( h = 1, \quad q = 1 \). We only present results for Example 1 as the results for Example 2 are very similar. In Fig. 8, we present the logarithm of \( E \) versus the logarithm of the distance \( \varepsilon \) in the case \( \varepsilon = 0 \) and for \( N(=M) = 16, 24, 32 \) and 48. This figure reveals that, as in the case of the sphere, the MFS approximation is poor when \( \varepsilon = H - h \) is either very small or very large. The behavior of \( E \) is very similar when \( \varepsilon \neq 0 \).

In Fig. 9, we present the logarithm of the error \( E_1 \) versus the angular parameter \( \varepsilon \). As before, because of the symmetry of the problem about \( \varepsilon = \frac{1}{2} \), we only considered the values \( 0 \leq \varepsilon \leq \frac{1}{2} \). We present the five cases \( \varepsilon = H - h = .9, .7, .5, .4 \) and .3 for various values of \( N(=M) \). As in the case of the sphere, as \( \varepsilon \) increases, the error \( E_1 \) decreases. Unlike the case of the sphere, there is little evidence of a minimum for \( \varepsilon \approx \frac{1}{2} \).
4.2.3. Torus

We considered the torus with \( \varrho_1 = \frac{1}{2}, \varrho_2 = 1 \). Here, we only present results for Example 2. In Fig. 10, we present the logarithm of the maximum error \( E_1 \) versus the angular parameter. We present the two cases \( \varepsilon = R_1 - \varrho_1 = .4 \) and \(.1 \) for various values of \( N(=M) \). Note that in this case, we are restricted in the choice of \( \varepsilon \). In particular, \( \varepsilon \) has to lie in \( (0, \frac{1}{2}) \). As in the case of the sphere, there is a clear indication that \( E_1 \) is optimized for \( \varphi \approx \frac{1}{4} \). For \( \varepsilon = .4, \ E_1 \) is less dependent on \( \varphi \). As was the case with the other two solids, as \( \varepsilon \) increases, the error \( E \) decreases.

5. Equations of the static theory of thermo-elasticity

5.1. The three-dimensional model

The displacements \( \mathbf{u} = (u_1, u_2, u_3) \) and the temperature \( \vartheta \) of a thermo-elastic medium are described by the system (see [1,27])

\[
\begin{align*}
\Delta^* \mathbf{u} &= \gamma \ \text{grad} \ \vartheta, \\
\Delta \vartheta &= 0,
\end{align*}
\]

in \( \Omega \), where \( \gamma \) is a positive constant, subject to the Dirichlet boundary conditions

\[
\begin{align*}
\mathbf{u} &= \mathbf{f}, \\
\vartheta &= g,
\end{align*}
\]

on the boundary \( \partial \Omega \). Eq. (5.1) constitute a \( 4 \times 4 \) elliptic system with unknowns \( \mathbf{u} \) and \( \vartheta \). It is readily seen that the corresponding Dirichlet problem for bounded domains, in which the displacements and the temperature are prescribed on the boundary, enjoys uniqueness.
A fundamental solution $E = (e_{ij})_{i,j=1}^4$ of (5.1) is given by (see [1,27])

$$e_{ij}(x) = -\frac{(1 - \delta_{i4})(1 - \delta_{j4})}{8\pi\mu(\lambda + 2\mu)} \left( (\lambda + \mu) \frac{x_i x_j}{|x|^3} + (\lambda + 3\mu)\frac{\delta_{ij}}{|x|} \right) - \frac{\gamma\delta_{j4}(1 - \delta_{i4}) x_i}{8\pi(\lambda + 2\mu) |x|} - \frac{\delta_{i4}\delta_{j4}}{4\pi|x|},$$

(5.3)

$i, j = 1, 2, 3, 4$. The matrix $E$ can be alternatively written in block form as

$$E(x) = \begin{pmatrix} G(x) & \eta \frac{x}{|x|} \\ 0 & -e_1(x) \end{pmatrix}$$

(5.4)

where $G(x)$ is the fundamental solution of the operator $\Delta^*$ given by (3.2), $\eta = -\gamma/8\pi(\lambda + 2\mu)$ and $e_1(x) = \frac{1}{4\pi|x|}$ is a fundamental solution of $-\Delta$.

5.2. MFS formulation

The temperature $\vartheta$ is a harmonic function and it can be thus approximated by linear combinations of the form

$$\vartheta^K(x) = \sum_{k=1}^K a_k e_1(x - y_k),$$

with $y_1, y_2, \ldots, y_K$ being the points on the boundary.

Fig. 5. Log-plot of maximum error $E$ versus distance $\varepsilon$ in Example 2 in the case of the sphere for different values of $N$. 

$N = 16$

$N = 24$

$N = 32$

$N = 48$
Fig. 6. Log-plot of maximum error $E_1$ versus the angular parameter $\alpha$ in Example 1 in the case of the sphere for different values of $N$ and $\varepsilon$.

Fig. 7. Log-plot of maximum error $E_1$ versus the angular parameter $\alpha$ in Example 2 in the case of the sphere for different values of $N$ and $\varepsilon$. 
where \( \{y_k\}_{k=1}^K \) lie on a prescribed pseudo-boundary \( \partial \Omega' \) and \( \{a_k\}_{k=1}^K \) are real constants. Eqs. (5.1a) now become

\[
A^* u = \gamma \text{grad} \sum_{k=1}^{K} a_k e_1(x - y_k) = -\frac{\gamma}{4\pi} \sum_{k=1}^{K} a_k \frac{x - y_k}{|x - y_k|^3},
\]

which are inhomogeneous. It can be easily verified that

\[
A^* \left( \frac{x}{|x|} \right) = -2(\lambda + 2\mu) \frac{x}{|x|^3},
\]

which allows us to obtain a particular solution \( u_p \) of (5.5):

\[
u_p(x) = \frac{\gamma}{8\pi(\lambda + 2\mu)} \sum_{k=1}^{K} a_k \frac{x - y_k}{|x - y_k|^3}.
\]

Clearly, \( A^*(u - u_p) = 0 \), and thus the difference \( v = u - u_p \) can be approximated by linear combinations of the form

\[
u^J(x) = \sum_{j=1}^{J} E(x - z_j) b_j,
\]

where \( E(x) \) is given by (3.2), the points \( \{z_j\}_{j=1}^J \) lie on \( \partial \Omega' \) and \( \{b_j\}_{j=1}^J \) are constant vectors in \( \mathbb{R}^3 \). (see [42, Theorem 8].)
Altogether, we have the following approximate solution:

\[
\vartheta^L(x) = \vartheta^K(x) = \sum_{k=1}^{K} a_k e_1(x - y_k), \quad (5.8a)
\]

\[
u^L(x) = \sum_{j=1}^{J} E(x - z_j) b_j + \frac{\gamma}{2\pi(\lambda + 2\mu)} \sum_{k=1}^{K} a_k \frac{x - y_k}{|x - y_k|} , \quad (5.8b)
\]

If we set \( \{x_\ell\}_{\ell=1}^{L} = \{y_k\}_{k=1}^{K} \cup \{z_j\}_{j=1}^{J} \), the vector \((u_1^L, u_2^L, u_3^L, \vartheta^L)\) is a linear combination of the columns of the matrices \(E(x - x_\ell), \ \ell = 1, \ldots, L\).

**Algorithm**

We next present the steps of the MFS algorithm for the solution of the thermo-elastostatic Dirichlet problem (5.1)–(5.2) for axisymmetric solids.

**Step 1:** Compute the coefficients \(\{a_\ell\}_{\ell=1}^{L}\) in (5.8a).

**Step 2:** Evaluate \(h = f - u_p\) at the points \(P_{k,\ell}\), where \(u_p\) is given by (5.6).

**Step 3:** Compute \(\nu^\ell\) in (5.7).

**Step 4:** Evaluate \(u^L\) from (5.8b).
Fig. 10. Log-plot of maximum error $E_1$ versus the angular parameter $\alpha$ in Example 2 in the case of the torus for different values of $N$ and $\varepsilon$.

Comments:

- In Step 1, we solve a Dirichlet problem for Laplace’s equation in an axisymmetric domain. This is done efficiently by using the MFS matrix decomposition algorithm described in [45].
- In Step 3, we solve a Dirichlet problem for the Cauchy–Navier equations of elasticity, given by (3.1) replacing $f$ in the boundary conditions by $f - u_p$, in an axisymmetric domain. This problem can be solved efficiently using the algorithm described in Section 3.5.

5.3. Numerical example

We considered the boundary value problem of the form (5.1)–(5.2) with the functions $f$ and $g$ corresponding to the exact solutions

**Example 3.**

$$\vartheta(x) = \frac{1}{|x-x_0|}, \quad u(x) = \frac{1}{2(\lambda + 2\mu)} \cdot \frac{x-x_0}{|x-x_0|}.$$ 

In our experiments we chose $x_0 = (8, 8, 8)$.

In our numerical experiments, we calculated the maximum error in $\overline{\Omega}$ for each of the components of the solution:

$$E_0 = \max_{x \in \overline{\Omega}} |\vartheta^L(x) - \vartheta(x)|,$$

$$E_i = \max_{x \in \overline{\Omega}} |u_i^L(x) - u_i(x)|, \quad i = 1, 2, 3,$$
and
\[ E = \max\{E_0, E_1, E_2, E_3\}. \]

In order to calculate the above quantities, \( \vartheta, \vartheta^L, u \) and \( u^L \) were calculated on a radially uniform \( \mathcal{L} \times \mathcal{L} \times \mathcal{L} \) grid in \( \overline{\Omega} \), where \( \mathcal{L} \) was taken to be appropriately large.

We considered the sphere with \( \varrho = 1 \). In Fig. 11, we present the logarithm of the maximum error \( E \) versus the logarithm of the distance \( \varepsilon = R - \varrho \) in the case \( \varnothing = 0 \) and for \( N(=M) = 16, 24, 32, 48 \) in Example 3. The results are consistent with the observations reported for the pure elasticity problem (Examples 1 and 2).

Finally, in Fig. 12, we present the logarithm of the maximum relative \( E_1 \) versus the angular parameter. We present the two cases \( \varepsilon = .2 \) and \( .05 \) for various values of \( N(=M) \). As in the pure elasticity problems, we observe that there is a clear indication that \( E_1 \) is optimized for \( \varnothing \approx \frac{1}{3} \) and that for \( \varepsilon = .2 \), \( E_1 \) is less dependent on \( \varnothing \).

6. Concluding remarks

In this work an efficient MDA–MFS algorithm was developed for the solution of the Cauchy–Navier system in axisymmetric domains. The linear system resulting from the MFS discretization does not possess a block circulant structure like its counterparts for the harmonic and biharmonic equations, due to the fact that its fundamental solutions are not radial. However, by a suitable transformation which rotates certain blocks in the coefficient matrix, one obtains a block circulant matrix. The resulting system is solved efficiently using an MDA and FFTs.

The algorithm developed in this study was extended to the more complicated problem of thermo-elasticity in axisymmetric domains.
Fig. 12. Log-plot of maximum error $E_1$ versus the angular parameter $\alpha$ in Example 3 in the case of the sphere for different values of $N$ and $\varepsilon$.

Note: It is noteworthy that the MDA–MFS for the Cauchy–Navier system is also readily applicable to Stokes equations for incompressible flow:

$$\begin{align*}
\mu \Delta u &= \text{grad} \, p, \\
\text{div} \, u &= 0,
\end{align*}$$

where $u$ is the velocity vector, $p$ is the pressure and $\mu$ is the dynamic viscosity coefficient. The fundamental solution of Stokes system (Stokeslets) is given by the matrix $G(x) = (g_{ij}(x))_{i,j=1}^{3}$ with

$$g_{ij}(x) = -\frac{1}{8\pi \mu} \delta_{ij} - \frac{1}{8\pi \mu} \frac{x_i x_j}{|x|^3}.$$ 

The MFS was applied to such problems in [48,49].

Acknowledgements

The authors wish to thank Christos Arvanitis for his assistance in technical matters in the preparation of this paper.

Appendix. Circulant structure of the system matrix

Eq. (3.11) is a direct consequence of the following lemma:

**Lemma A.1.** Let $x \in \mathbb{R}^3$ and

$$W_\varphi = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Then
\[ R_\partial x \cdot x^T R_\psi = R_{\partial + \varphi} (W_\varphi x) \cdot (W_\varphi x)^T R_{\psi + \varphi} , \]  \hspace{1cm} (A.1)
for every \( \partial, \psi, \varphi \in \mathbb{R} \).

**Proof.** We are using the notation
\[
x \cdot x^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\
 x_2 & x_2 & x_3 \\
 x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\
 x_1 x_2 & x_2^2 & x_2 x_3 \\
 x_1 x_3 & x_2 x_3 & x_3^2 \end{pmatrix}.
\]
Eq. (A.1) is a result of the fact that
\[
R_\partial W_\psi = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
\sin \theta & -\cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cos \varphi & \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix} = R_{\partial + \varphi} .
\] □

**References**


