Some families of strongly clean rings

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Abstract

A ring R with identity is called strongly clean if every element of R is the sum of an idempotent and a unit that commute with each other. For a commutative local ring R and for an arbitrary integer n ≥ 2, the paper deals with the question whether the strongly clean property of \( M_n(R[[x]]) \), \( M_n\left(R\left(\frac{x}{(x^2)}\right)\right) \), and \( M_n(RC_2) \) follows from the strongly clean property of \( M_n(R) \). This is ‘Yes’ if n = 2 by a known result.

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1. Introduction

Throughout the paper, R is an associative ring with identity. A ring R is called clean if every element of R is the sum of an idempotent and a unit [15], and R is called strongly clean if every element of R is the sum of an idempotent and a unit that commute with each other [16]. Strongly clean rings include local rings and strongly \( \pi \)-regular rings (see [2]), where a ring R is strongly \( \pi \)-regular if for every \( a \in R \), the chain \( aR \supseteq a^2R \supseteq \cdots \) terminates (or equivalently, for every \( a \in R \) the chain \( Ra \supseteq Ra^2 \supseteq \cdots \) terminates by [8]).

By [10], a ring R is clean if and only if the \( n \times n \) matrix ring \( M_n(R) \) is clean for all \( n \geq 1 \). It was a question in [16] whether the matrix ring over a strongly clean ring is again strongly clean. The answer is ‘No’ by [18] where it was shown that for the localization \( \mathbb{Z}_{(2)} \) of \( \mathbb{Z} \) at (2), \( M_2(\mathbb{Z}_{(2)}) \)

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is not strongly clean. This fact motivated the authors of [3,5,6] to consider the question: when is the matrix ring over a strongly clean ring still strongly clean? Among others, the following results are observed in [5,6]:

(1) For any prime \( p \), \( M_2(\hat{\mathbb{Z}}_p) \) is strongly clean but \( M_2(\mathbb{Z}_p) \) is not strongly clean where \( \hat{\mathbb{Z}}_p \) is the ring of \( p \)-adic integers and \( \mathbb{Z}_p \) is the localization of \( \mathbb{Z} \) at the prime ideal \( (p) \).

(2) For any commutative local ring \( R \), \( M_2(R) \) is strongly clean if and only if \( M_2(R[[x]]) \) is strongly clean if and only if \( M_2(RC_2) \) is strongly clean.

More recently, the authors of [3] have proved that for a commutative local ring \( R \), \( M_n(R) \) is strongly clean if and only if every monic polynomial of degree \( n \) in \( R[x] \) has a so called ‘SRC’ factorization. Generally, if \( M_2(R) \) has a ring property \( (P) \), one may think that \( M_n(R) \) also has the property \( (P) \). However, by [3, Example 20], there is a commutative local ring \( R \) such that \( M_2(R) \) is strongly clean but \( M_3(R) \) is not. Hence, it is a new and interesting question whether \( M_n(R[[x]]) \), \( M_n\left(\frac{R[x]}{(x^n)}\right) \), and \( M_n(RC_2) \) are all necessarily strongly clean whenever \( M_n(R) \) is strongly clean for any \( n \geq 2 \). This is the main topic of the paper. Some other strongly clean matrix rings are also identified. For example, if \( R \) is a right duo strongly \( \pi \)-regular ring, then \( M_n((RG)[[x]]) \) and \( M_n\left(\frac{(RG)[x]}{(x^n)}\right) \) are strongly clean for all \( n, k \geq 1 \) and all locally finite groups \( G \).

As usual, we use \( U(R) \) and \( J(R) \) to denote the group of units and the Jacobson radical of \( R \) respectively. We write \( \mathbb{Z} \) for the integers, \( \mathbb{Z}_n \) for the integers modulo \( n \), \( \mathbb{N} \) for the positive integers, and \( C_n \) for the cyclic group of order \( n \). The group ring of a group \( G \) over a ring \( R \) is denoted \( RG \).

2. Matrix rings over commutative local rings

Let \( n \geq 2 \) and let \( R \) be a commutative local ring such that \( M_n(R) \) is strongly clean. It is proved that both \( M_n(R[[x]]) \) and \( M_n\left(\frac{R[x]}{(x^n)}\right) \) are strongly clean and that \( M_n(RC_2) \) is strongly clean when \( 2 \in U(R) \) or \( 2 = 0 \) in \( R \). The unsettled situation is when \( 0 \neq 2 \in J(R) \). All the proofs rely on a result of [3], quoted as Lemma 2.2, below.

For a field \( F \), the monic greatest common divisor of polynomials \( h(t) \) and \( g(t) \) in \( F[t] \) is denoted \( \gcd(h(t), g(t)) \). For a ring homomorphism \( \theta : R \rightarrow S \), we define \( \theta' : R[x] \rightarrow S[x] \) by \( \theta'(\sum r_i x^i) = \sum \theta(r_i) x^i \). We let \( \eta_R : R \rightarrow R/J(R) \) be the natural ring homomorphism, i.e., \( \eta_R(r) = \bar{r} = r + J(R) \).

**Definition 2.1** [3]. Let \( R \) be a commutative local ring. A factorization \( h(t) = h_0(t)h_1(t) \) in \( R[t] \) of a monic polynomial \( h(t) \) is said to be an SRC factorization if \( h_0(0), h_1(1) \in U(R) \) and \( \overline{h_0(t)}, \overline{h_1(t)} \) are co-prime in the PID \( \overline{R}[t](=R/J(R)[t]) \). The ring \( R \) is an \( n \)-SRC ring if every monic polynomial of degree \( n \) in \( R[t] \) has an SRC factorization.

**Lemma 2.2** [3, Theorem 12]. Let \( R \) be a commutative local ring. Then \( R \) is an \( n \)-SRC ring if and only if \( M_n(R) \) is strongly clean.

**Lemma 2.3.** Let \( \theta : R \rightarrow S \) be a ring epimorphism. If \( R \) is an \( n \)-SRC ring, then \( S \) is an \( n \)-SRC ring.
**Proof.** The ring $S$ is commutative local since $R$ is an $n$-SRC ring. The following diagram is commutative where $\theta : R/J(R) \rightarrow S/J(S)$, $r + J(R) \mapsto \theta(r) + J(S)$, is an isomorphism:

$$
\begin{array}{ccc}
R & \xrightarrow{\theta} & S \\
\eta_R \downarrow & & \downarrow \eta_S \\
R/J(R) & \xrightarrow{\bar{\theta}} & S/J(S).
\end{array}
$$

It induces the commutative diagram with $\bar{\theta}'$ an isomorphism:

$$
\begin{array}{ccc}
R[t] & \xrightarrow{\theta'} & S[t] \\
\eta'_R \downarrow & & \downarrow \eta'_S \\
R/J(R)[t] & \xrightarrow{\bar{\theta}'} & S/J(S)[t].
\end{array}
$$

Let $h'(t) \in S[t]$ be a monic polynomial of degree $n$. Then there exists a monic polynomial $h(t) \in R[t]$ of degree $n$ such that $\theta'(h(t)) = h'(t)$. Since $R$ is an $n$-SRC ring, there exists an SRC factorization $h(t) = h_0(t)h_1(t)$ in $R[t]$. Let $\theta'(h_i(t)) = h_i'(t)$, $i = 0, 1$. Then $h'(t) = h_0'(t)h_1'(t)$ with $h_i'(t) = \theta'(h_i(t)) \in U(S)$. By the commutativity of the second diagram, $\bar{\theta}' \eta'_R(h_i(t)) = \eta'_S(h_i'(t))$ for $i = 0, 1$. Because $\bar{\theta}'$ is an isomorphism and $\gcd(\eta'_R(h_0(t)), \eta'_R(h_1(t))) = 1$, we get $\gcd(\eta'_S(h_0'(t)), \eta'_S(h_1'(t))) = 1$. So $\theta'(t) = h_0'(t)h_1'(t)$ is an SRC factorization in $S[t]$. Hence $S$ is an $n$-SRC ring. □

For a ring epimorphism $\theta : R \rightarrow S$, $S$ being an $n$-SRC does not imply that $R$ is an $n$-SRC. For example, let $\theta : \mathbb{Z}(p) \rightarrow \mathbb{Z}_p$ be the natural ring epimorphism. Then $\mathbb{M}_n(\mathbb{Z}(p))$ is not strongly clean for any $n > 1$ by [5, Corollary 1.9]. So $\mathbb{Z}(p)$ is not an $n$-SRC ring by Lemma 2.2, but $\mathbb{Z}_p$ is certainly an $n$-SRC.

Let $R$ be a commutative ring. For $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ in $R[x]$, the $(n + m) \times (n + m)$ determinant

$$
\mathfrak{R}(f, g) = \begin{vmatrix}
a_n & a_{n-1} & \cdots & a_0 & \cdots & a_0 \\
a_n & a_{n-1} & \cdots & a_0 & \cdots & a_0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
b_m & b_{m-1} & \cdots & b_0 & \cdots & b_0 \\
b_m & b_{m-1} & \cdots & b_0 & \cdots & b_0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
b_m & b_{m-1} & \cdots & b_0 & \cdots & b_0
\end{vmatrix}
\begin{array}{c}
m \\
\end{array}
\begin{array}{c}
n \\
\end{array}
$$

is called the resultant of $f(x)$ and $g(x)$ (see [4] or [12]).

**Lemma 2.4** [4, Lemma 2, p. 321]. Let $E$ be an algebraically closed field. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n(a^n \neq 0)$, and $g(x) = b_0 + b_1x + \cdots + b_mx^m(b_m \neq 0)$ be two polynomials in $E[x]$ such that $f(\alpha_i) = 0$ and $g(\beta_j) = 0$ where $\alpha_i$ and $\beta_j \in E$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then $\mathfrak{R}(f, g) = a_0^n\mathfrak{R}(\alpha_1, \alpha_2, \ldots, \alpha_n) = b_m^n\mathfrak{R}(\beta_1, \beta_2, \ldots, \beta_m)$.

**Lemma 2.5.** Let $R$ be a commutative local ring, $\eta_R : R \rightarrow \frac{R}{J(R)}$ be the natural ring homomorphism, $A = (r_{ij}) \in \mathbb{M}_n(R)$, and $\overline{A} = (\overline{r}_{ij}) \in \mathbb{M}_n\left(\frac{R}{J(R)}\right)$. Then det $A \in U(R)$ if and only if det $\overline{A} \neq 0$. 

Proof. This is [12, I.D.8, p. 26]. □

**Theorem 2.6.** Let $R$ be a commutative local ring and let $n \geq 1$. Then $R$ is an $n$-SRC ring if and only if so is $R[[x]]$.

**Proof.** “$\Rightarrow$”. Clearly $R[[x]]$ is a commutative local ring with $J(R[[x]]) = J(R) + xR[[x]]$. Define $\theta : R[[x]] \rightarrow R$ by $\theta(\sum_{i \geq 0} r_i x^i) = r_0$, and $\overline{\theta} : \frac{R[[x]]}{J(R[[x]])} \rightarrow \frac{R}{J(R)}$ by $\overline{\theta}(r + J(R[[x]])) = \theta(r + J(R)) = r + J(R)$, $r \in R$. Then $\theta$ is an epimorphism, $\overline{\theta}$ is an isomorphism, and the following diagram is commutative:

$$
\begin{array}{ccc}
R[[x]] & \xrightarrow{\theta} & R \\
\eta_{R[[x]]} & & \eta_R \\
\frac{R[[x]]}{J(R[[x]])} & \xrightarrow{\overline{\theta}} & \frac{R}{J(R)}.
\end{array}
$$

And it induces the commutative diagram

$$
\begin{array}{ccc}
R[[x]][t] & \xrightarrow{\theta'} & R[t] \\
\eta'_{R[[x]]} & & \eta_R' \\
\frac{R[[x]]}{J(R[[x]])}[t] & \xrightarrow{\overline{\theta}'} & \frac{R}{J(R)}[t]
\end{array}
$$

with $\overline{\theta}'$ an isomorphism. Let $h(t) = t^n + \sum_{i=0}^{n-1} f_i t^i \in R[[x]][t]$ with $f_i = \sum_{j \geq 0} r_{ij} x^j \in R[[x]]$.

**Case I.** If $h(0) \in U(R[[x]])$, then let $h_0(t) = h(t)$, $h_1(t) = 1$; and if $h(1) \in U(R[[x]])$, then let $h_0(t) = 1$, $h_1(t) = h(t)$. In either case, $h(t)$ has a trivial SRC factorization in $R[[x]][t]$.

**Case II.** If $h(0) = f_0 \in J(R[[x]])$ and $h(1) = 1 + \sum_{i=0}^{n-1} f_i \in J(R[[x]])$, then $r_{00} \in J(R)$ and $1 + \sum_{i=0}^{n-1} r_{i0} t^i$, $h'(0) = r_{00} \in J(R)$, and $h'(1) = 1 + \sum_{i=0}^{n-1} r_{i0} \in J(R)$. Since $R$ is $n$-SRC, there exist $h'_0(t) = t^k + \sum_{i=0}^{k-1} a_{0i} t^i$ and $h'_1(t) = t^{n-k} + \sum_{i=0}^{n-k-1} b_{0i} t^i$ in $R[t]$ such that $h'_0(0) \in U(R)$, $h'_1(1) \in U(R)$, $\gcd(\eta'_R(b'_0(t)), \eta'_R(h'_1(t))) = 1$, and $h'(t) = h'_0(t) h'_1(t)$. Let $h_0(t) = t^k + \sum_{i=0}^{k-1} A_i t^i \in R[[x]][t]$ with $A_i = \sum_{j \geq 0} a_{ij} x^j$, and $h_1(t) = t^{n-k} + \sum_{i=0}^{n-k-1} B_i t^i \in R[[x]][t]$ with $B_i = \sum_{j \geq 0} b_{ij} x^j$.

We next prove that there exist $A_i$, $B_j \in R[[x]]$ ($i = 0, \ldots, k-1$ and $j = 0, \ldots, n-k-1$) such that $h(t) = h_0(t) h_1(t)$. We notice that

$$
h(t) = h_0(t) h_1(t)
\Leftrightarrow t^n + \sum_{i=0}^{n-1} r_{i0} t^i + \sum_{i=0}^{\infty} r_{ij} t^i x^j
\Leftrightarrow t^k + \sum_{i=0}^{k-1} a_{0i} t^i + \sum_{i=0}^{\infty} a_{ij} t^i x^j
\times t^{n-k} + \sum_{i=0}^{n-k-1} b_{0i} t^i + \sum_{i=0}^{\infty} b_{ij} t^i x^j
\Leftrightarrow \text{the conditions (P}_0\text{) and (P}_m\text{) hold for all m } \in \mathbb{N},$$
\[ (P_0): \left( t^k + \sum_{i=0}^{k-1} a_i t^i \right) \left( t^{n-k} + \sum_{i=0}^{n-k-1} b_i t^i \right) = t^n + \sum_{i=0}^{n-1} r_i t^i, \]

\[ (P_m): \left( t^k + \sum_{i=0}^{k-1} a_i t^i \right) \left( \sum_{i=0}^{n-k-1} b_{im} t^i \right) + \sum_{j=1}^{m-1} \left( \sum_{i=0}^{k-1} a_{ij} t^i \right) \left( \sum_{i=0}^{n-k-1} b_{i,m-j} t^i \right) = \sum_{i=0}^{n-1} r_{im} t^i. \]

Notice that by the choice of \( h_0(t) \) and \( h_1(t) \), \( (P_0) \) holds for suitable \( a_{i0} \) (0 \( \leq i \leq k-1 \)) and \( b_{i0} \) (0 \( \leq i \leq n-k-1 \)). Assume that for \( s \geq 1 \), there exist \( a_{ij} \) (0 \( \leq i \leq k-1 \), 0 \( \leq j \leq s-1 \)) and \( b_{ij} \) (0 \( \leq i \leq n-k-1 \), 0 \( \leq j \leq s-1 \)) in \( R \) such that \( (P_m) \) hold for all \( m = 0, 1, \ldots, s-1 \). We next show that there exist \( a_{is} \) (0 \( \leq i \leq k-1 \)) and \( b_{js} \) (0 \( \leq j \leq n-k-1 \)) in \( R \) such that \( (P_s) \) holds. Note that \( (P_s) \) is equivalent to

\[ (*) \quad \left( t^k + \sum_{i=0}^{k-1} a_i t^i \right) \left( \sum_{i=0}^{n-k-1} b_{is} t^i \right) + \sum_{j=1}^{s-1} \left( \sum_{i=0}^{k-1} a_{ij} t^i \right) \left( \sum_{i=0}^{n-k-1} b_{i,s-j} t^i \right) = \sum_{i=0}^{n-1} r_{is} t^i, \]

where all \( r_{is} \) are known elements of \( R \). Thus \( (*) \) is equivalent to:

\[ (**) \quad \begin{align*}
&b_{n-k-1,s} + a_{k-1,s} = r'_{n-1,s}, \\
&a_{k-1,0} b_{n-k-1,s} + b_{n-k-2,s} + b_{n-k-1,0} a_{k-1,s} + a_{k-2,s} = r'_{n-2,s}, \\
&\vdots \\
&a_{00} b_{1s} + a_{10} b_{0s} + b_{10} a_{0s} + b_{00} a_{1s} = r'_{1s}, \\
&a_{00} b_{0s} + b_{00} a_{0s} = r'_{0s}.
\end{align*} \]

As a linear system in variables \( a_{is} \) (0 \( \leq i \leq k-1 \)) and \( b_{js} \) (0 \( \leq j \leq n-k-1 \)), the matrix form of \( (**) \) is \( A X = B \) where

\[
A^T = \begin{pmatrix}
1 & b_{n-k-1,0} & \cdots & \cdots & b_{00} \\
1 & b_{n-k-1,0} & \cdots & \cdots & b_{00} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & a_{k-1,0} & \cdots & \cdots & a_{00} \\
1 & a_{k-1,0} & \cdots & \cdots & a_{00} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & a_{k-1,0} & \cdots & \cdots & a_{00}
\end{pmatrix},
\]

\[
X^T = \begin{pmatrix}
a_{k-1,s} & a_{k-2,s} & \cdots & a_{0s} & b_{n-k-1,s} & b_{n-k-2,s} & \cdots & b_{0s}
\end{pmatrix}.
\]
A commutative local ring is called Henselian if \( R[x] \) satisfies Hensel’s lemma [1,13], i.e., for any monic polynomial \( f(x) \in R[x] \), if \( \overline{f}(x) = \overline{g}(x)\overline{h}(x) \) with \( \overline{g}(x), \overline{h}(x) \in \frac{R}{\mathfrak{m}(R)}[x] \) monic and coprime, then there exist monic polynomials \( g(x) \) and \( h(x) \) in \( R[x] \) such that \( f(x) = g(x)h(x) \), \( \overline{f'(x)} = \overline{g'(x)} = \overline{h'(x)} \). Hence matrix rings over Henselian rings are strongly clean.

**Example 2.8.** If \( R \) is Henselian and \( m, s, n_1, \ldots, n_s \in \mathbb{N} \), then \( \mathbb{M}_n(R[[x_1, x_2, \ldots, x_s]]) \) are strongly clean. The conclusion holds, in particular, for \( R \) any complete local ring such as a field or \( \mathbb{Z}_p \), for any prime \( p \).
It was proved in [6] that for a commutative local ring $R$, $\mathbb{M}_2(R)$ is strongly clean if and only if so is $\mathbb{M}_2(RC_2)$. Next, we extend this result from 2 to an arbitrary positive integer $n$. The methods used here depend on the fact that the group is cyclic of order two. Let $n \geq 2$ be an integer and let $G$ be a cyclic group of order greater than two. It is unknown when $\mathbb{M}_n(RG)$ is strongly clean (even if $R$ is a commutative local ring).

**Theorem 2.9.** Let $R$ be a commutative local ring with $2 \in U(R)$ or $\text{char } R = 2$. Then $\mathbb{M}_n(R)$ is strongly clean if and only if so is $\mathbb{M}_n(RC_2)$.

**Proof.** “$\Leftarrow$” holds because $\mathbb{M}_n(R)$ is an image of $\mathbb{M}_n(RC_2)$.

“$\Rightarrow$”. If $2 \in U(R)$, then $RC_2 \cong R \times R$ by [5, Lemma 11]. So we have $\mathbb{M}_n(RC_2) \cong \mathbb{M}_n(R) \oplus \mathbb{M}_n(R)$ is strongly clean.

Now assume that $\text{char } R = 2$. Then $RC_2$ is commutative local. We can assume $n \geq 2$. Write $C_2 = \{1, g\}$ and let $f(x) = x^n + \sum_{i=0}^{n-1}(r_i + r'_i)g x^i \in (RC_2)[x]$ such that $f(0) = r_0 + r'_0g \in J(RC_2)$ and $f(1) = 1 + \sum_{i=0}^{n-1}(r_i + r'_i)g \in J(RC_2)$. Let $\omega : RC_2 \to R$, $a + bg \mapsto a + b$, be the augmentation map. As in the proof of Theorem 2.6, we have two commutative diagrams with $\bar{\omega}$ and $\bar{\omega}'$ isomorphisms:

$$
\begin{array}{ccc}
RC_2 & \xrightarrow{\omega} & R \\
\eta_{RC_2} \downarrow & & \downarrow \eta_R \\
\frac{RC_2}{J(RC_2)} & \xrightarrow{\bar{\omega}} & \frac{R}{J(R)}, \\
\end{array}
$$

$$
\begin{array}{ccc}
RC_2[x] & \xrightarrow{\omega'} & R[x] \\
\eta'_{RC_2} \downarrow & & \downarrow \eta'_R \\
\frac{RC_2}{J(RC_2)}[x] & \xrightarrow{\bar{\omega}'} & \frac{R}{J(R)}[x]. \\
\end{array}
$$

Since $\mathbb{M}_n(R)$ is strongly clean, $f'(x) := \omega'(f(x)) = x^n + \sum_{i=0}^{n-1}(r_i + r'_i)x^i$ has a non-trivial SRC factorization $f'(x) = f'_0(x)f'_1(x)$ in $R[x]$. Write $f'_0(x) = a_0 + a_1x + \cdots + a_{m-1}x^{m-1} + x^m$ and $f'_1(x) = b_0 + b_1x + \cdots + b_{n-m-1}x^{n-m-1} + x^{n-m}$ where $1 \leq m < n$. Next we show that there exist $y_i, z_i \in R$ ($i = 0, \ldots, m - 1, j = 0, \ldots, n - m - 1$) such that

$$
\begin{align*}
f_0(x) &= x^m + \sum_{i=0}^{m-1}[y_i + (a_i - y_i)g]x^i, \\
f_1(x) &= x^{n-m} + \sum_{i=0}^{n-m-1}[z_i + (b_i - z_i)g]x^i, \\
f(x) &= f_0(x)f_1(x). \\
\end{align*}
$$

(2.2)

The equality $f(x) = f_0(x)f_1(x)$ is equivalent to

$$
\begin{align*}
x^n + \sum_{i=0}^{n-1}r_ix^i &= \left(x^m + \sum_{i=0}^{m-1}y_ix^i\right)\left(x^{n-m} + \sum_{i=0}^{n-m-1}z_ix^i\right) \\
&\quad + \left[\sum_{i=0}^{m-1}(a_i - y_i)x^i\right]\left[\sum_{i=0}^{n-m-1}(b_i - z_i)x^i\right], \\
&= \left[x^m + \sum_{i=0}^{m-1}y_ix^i\right]\left[x^{n-m} + \sum_{i=0}^{n-m-1}z_ix^i\right]. \\
\end{align*}
$$

(2.3)
\[
\sum_{i=0}^{n-1} r'_i x^i = \left( x^m + \sum_{i=0}^{m-1} y_i x^i \right) \left[ \sum_{i=0}^{n-m-1} (b_i - z_i) x^i \right] \\
+ \left( x^{n-m} + \sum_{i=0}^{m-1} z_i x^i \right) \left[ \sum_{i=0}^{m-1} (a_i - y_i) x^i \right].
\]

Clearly, the second equality of (2.3) follows from \( f'(x) = f'_0(x) f'_1(x) \) and from the first equality of (2.3). So it suffices to show that there exist \( y_i, z_j \in R \) \((i = 0, \ldots, m - 1, j = 0, \ldots, n - m - 1)\) that make the first equality of (2.3) hold. The first equality of (2.3) is equivalent to

\[
\begin{align*}
\sum y_0 z_0 + (a_0 - y_0)(b_0 - z_0) &= r_0, \\
y_0 z_1 + y_1 z_0 + (a_0 - y_0)(b_1 - z_1) + (a_1 - y_1)(b_0 - z_0) &= r_1, \\
& \vdots \\
y_m - 2 + y_{m-1} z_{m-1} + z_{m-2} + (a_{m-1} - y_{m-1})(b_{n-1} - z_{m-1}) &= r_{n-2}, \\
y_{m-1} + z_{m-1} &= r_{n-1},
\end{align*}
\]

which, since \( \text{char}(R) = 2 \), is equivalent to

\[
\begin{align*}
c_0 &:= r_0 + a_0 b_0 = a_0 z_0 + b_0 y_0, \\
c_1 &:= r_1 + a_0 b_1 + a_1 b_0 = a_1 z_0 + a_0 z_1 + b_1 y_0 + b_0 y_1, \\
& \vdots \\
c_{n-2} &:= r_{n-2} + a_{m-1} b_{n-1} = z_{m-2} + a_{m-1} z_{m-1}, \\
& + y_{m-2} + b_{n-1} y_{m-1}, \\
c_{n-1} &:= r_{n-1} = z_{n-1} + y_{m-1}.
\end{align*}
\]

As a linear system in variables \( y_i \) \((i = 0, \ldots, m - 1)\) and \( z_i \) \((i = 0, \ldots, n - m - 1)\), the matrix form of (2.4) is

\[\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
\begin{bmatrix}
\begin{array}{ccccccc}
b_{n-1} & \cdots & \cdots & b_1 & b_0 \\
b_{n-1} & \cdots & \cdots & b_1 & b_0 \\
\vdots & & & \vdots & & & \vdots \\
1 & b_{n-1} & \cdots & \cdots & b_1 & b_0 \\
a_{m-1} & \cdots & \cdots & a_1 & a_0 \\
a_{m-1} & \cdots & \cdots & a_1 & a_0 \\
\vdots & & & \vdots & & & \vdots \\
a_{m-1} & \cdots & \cdots & a_1 & a_0 \\
1 & a_{m-1} & \cdots & \cdots & a_1 & a_0
\end{array}
\end{bmatrix}
\begin{bmatrix}
y_{m-1} \\
y_{m-2} \\
y_0 \\
z_{m-1} \\
z_{m-2} \\
z_0
\end{bmatrix}
= 
\begin{bmatrix}
c_{n-1} \\
c_{n-2} \\
c_{0}
\end{bmatrix}.
\]

An argument similar to the proof of Theorem 2.6 shows that \( A \) is invertible. So \( AX = C \) is solvable. This shows the existence of the \( y_i \) and \( z_j \) such that \( f'(x) = f'_0(x) f'_1(x) \). Hence \( \mathbb{M}_n(R C_2) \) is strongly clean. \( \square \)

**Proposition 2.10.** Let \( R \) be a commutative local ring with \( 0 \neq 2 \in J(R) \) and let \( \mathbb{M}_3(R) \) be strongly clean. If for any \( m, n \in R \) and \( u \in U(R) \), \( 4x^3 - 2mx^2 + ux + n = 0 \) is solvable in \( R \), then \( \mathbb{M}_3(R C_2) \) is strongly clean.
Proof. The two diagrams in the proof of Theorem 2.9 are still valid. Let \( f(x) = (r_0 + r_0'g) + (r_1 + r_1'g)x + (r_2 + r_2'g)x^2 + x^3 \in (RC_2)[x] \) with \( f(0) = r_0 + r_0'g \in J(RC_2) \) and \( f(1) = (r_0 + r_0'g) + (r_1 + r_1'g) + (r_2 + r_2'g) + 1 \in J(RC_2) \). Then \( f'(x) = \omega'(f(x)) = (r_0 + r_0'g) + (r_1 + r_1')x + (r_2 + r_2')x^2 + x^3 \in R[x] \) with \( f'(0) = r_0 + r_0' \in J(R) \) and with \( f'(1) = (r_0 + r_0') + (r_1 + r_1') + (r_2 + r_2') + 1 \in J(R) \). Since \( \mathbb{M}_3(R) \) is strongly clean, \( f'(x) \) has a non-trivial SRC-factorization \( f'(x) = f'_0(x)f'_1(x) \) in \( R[x] \). We can assume that \( \{f'_0(x), f'_1(x)\} = \{a_0 + x, b_0 + b_1x + x^2\} \). Then

\[
\begin{align*}
  r_0 + r_0' &= a_0b_0, \\
  r_1 + r_1' &= a_0b_1 + b_0, \\
  r_2 + r_2' &= a_0 + b_0. 
\end{align*}
\]

(2.5)

Next we show that there exist \( y_0, z_0, z_1 \in R \) such that \( f(x) = f_0(x)f_1(x) \) and \( f'_i(x) = \omega'(f_i(x)) \) (for \( i = 0, 1 \)), where

\[
\{f_0(x), f_1(x)\} = \left\{ [y_0 + (a_0 - y_0)g] + x, [z_0 + (b_0 - z_0)g] + [z_1 + (b_1 - z_1)g]x + x^2 \right\}.
\]

The condition \( f(x) = f_0(x)f_1(x) \) is equivalent to

\[
\begin{align*}
  r_0 &= y_0z_0 + (a_0 - y_0)(b_0 - z_0), \\
  r_1 &= y_0z_1 + (a_0 - y_0)(b_1 - z_1) + z_0, \\
  r_2 &= z_1 + y_0, \\
  r'_0 &= z_0(a_0 - y_0) + y_0(b_0 - z_0), \\
  r'_1 &= z_1(a_0 - y_0) + y_0(b_1 - z_1) + b_0 + z_0, \\
  r'_2 &= b_1 - z_1 + a_0 - y_0. 
\end{align*}
\]

(2.6)

Since the first three equalities of (2.6) plus (2.5) clearly imply the last three equalities of (2.6), it suffices to show that there exist \( y_0, z_0, z_1 \in R \) such that the first three equalities of (2.6) hold true. Rewrite the first three equations of (2.6) as

\[
\begin{align*}
  2y_0z_0 - b_0y_0 - a_0z_0 &= r_0 - a_0b_0, \\
  2y_0z_1 - b_1y_0 + z_0 - a_0z_1 &= r_1 - a_0b_1, \\
  z_1 &= r_2 - y_0. 
\end{align*}
\]

(2.7)

Clearly (2.7) is equivalent to

\[
\begin{align*}
  4y_0^2 - 2mz_0^2 + uy_0 + n &= 0, \\
  z_0 &= 2y_0^2 - (2r_2 - b_1 + a_0)y_0 + a_0(r_2 - b_1) + r_1, \\
  z_1 &= r_2 - y_0. 
\end{align*}
\]

(2.8)

where \( m = (2r_2 + 2a_0 - b_1), u = (4a_0r_2 - 2a_0b_1 + 2r_1 - b_0 - a_0b_1 + a_0^2), \) and \( n = -a_0^2r_2 + a_0^2b_1 - a_0r_1 + a_0b_0 - r_0 \). As in (last part of) the proof of Theorem 2.6, \( b_0 - a_0b_1 + a_0^2 = \Re(f'_0(x), f'_1(x)) \in U(R) \). So \( u \in U(R) \). By hypothesis, the first equation of (2.8) is solvable for \( y_0 \) in \( R \). Hence (2.8) is solvable for \( y_0, z_0 \) and \( z_1 \) in \( R \). So \( \mathbb{M}_3(RC_2) \) is strongly clean. \( \square \)

Corollary 2.11. If \( R \) is Noetherian Henselian, then \( \mathbb{M}_3(RC_2), \mathbb{M}_3((RC_2)[x]) \), and \( \mathbb{M}_3 \left( \frac{(RC_2)[x]}{(x^2)} \right) \) are strongly clean.

Proof. We first show that \( \mathbb{M}_3(RC_2) \) is strongly clean. By Theorem 2.9 and Proposition 2.10, it suffices to show that for any \( m, n \in R \) and \( u \in U(R) \), \( h(x) = 4x^3 - 2mx^2 + ux + n \) has a root in \( R \). Let \( h'_1(x) = x + \frac{u}{4} \) and \( h'_0(x) = u \). Then \( \eta'_R(h(x)) = \eta'_R(h'_0(x))\eta'_R(h'_1(x)) \). By Hensel's Lemma [9, Theorem 7.18], there exist \( h_1(x) = x + s_3 \) and \( h_0(x) \) in \( R[x] \) such that
Let \( R \) be a ring whose prime factor rings are Artinian. Then every finite extension of \( R \) is strongly \( \pi \)-regular.

Note that, by [17], there exists a strongly \( \pi \)-regular ring \( R \) such that \( \mathbb{M}_{2}(R) \) is not strongly \( \pi \)-regular. A ring \( R \) is called right duo if every right ideal is an ideal.

**Corollary 3.2.** Let \( R \) be a ring whose prime factor rings are Artinian, \( G \) be a locally finite group, and \( n, k \geq 1 \). Then \( \mathbb{M}_{n}(RG) \) is strongly \( \pi \)-regular, and \( \mathbb{M}_{n}((RG)[[x]]) \) and \( \mathbb{M}_{n}((RG)[[x]]) \) are strongly clean. The conclusion holds, in particular, for \( R \) right duo and strongly \( \pi \)-regular.

**Proof.** Without loss of generality, we may assume that \( G \) is a finite group. Then \( \mathbb{M}_{n}(RG) \) is a finite extension of \( RG \) and \( RG \) is a finite extension of \( R \). Thus, it follows from Theorem 3.1 that \( \mathbb{M}_{n}(RG) \) is strongly \( \pi \)-regular. Moreover, \( \mathbb{M}_{n}((RG)[[x]]) \cong \mathbb{M}_{n}(RG)[[x]] \) is strongly clean by [7, Corollary 2], and \( \mathbb{M}_{n}((RG)[[x]]) \cong \mathbb{M}_{n}(RG) \) is strongly clean by [7, Corollary 4].

If \( R \) is right duo and strongly \( \pi \)-regular, then every prime factor ring of \( R \) is again right duo strongly \( \pi \)-regular, so it must be a strongly \( \pi \)-regular domain. Hence it is a field. □

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**References**


\[ \eta_{R}'(h_{1}(x)) = \eta_{R}'(h_{1}'(x)), \eta_{R}'(h_{0}(x)) = \eta_{R}'(h_{0}'(x)), \text{ and } h(x) = h_{1}(x)h_{0}(x). \] So \( h(x) \) has a solution \( x = -s_{3} \in R \). Hence \( \mathbb{M}_{3}(RC_{2}) \) is strongly clean. If \( 2 \in U(R) \), then \( RC_{2} \cong R \times R \), so \( \mathbb{M}_{3}((RC_{2})[[x]]) \cong \mathbb{M}_{3}(R[[x]]) \times \mathbb{M}_{3}(R[[x]]) \) is strongly clean by Theorem 2.7. If \( 2 \in J(R) \), then \( RC_{2} \) is again commutative local by [14], so \( \mathbb{M}_{3}((RC_{2})[[x]]) \) is strongly clean by Theorem 2.7. Thus, \( \mathbb{M}_{3}((RC_{2})[[x]]) \) is strongly clean in either case. Hence \( \mathbb{M}_{3}((RC_{2})[[x]]) \) is strongly clean because it is an image of \( \mathbb{M}_{3}((RC_{2})[[x]]) \). □

For a commutative local ring \( R \) with \( 0 \neq 2 \in J(R) \) and for \( n > 2 \), we have been unable to answer if \( \mathbb{M}_{n}(R) \) being strongly clean implies that \( \mathbb{M}_{n}(RC_{2}) \) is strongly clean.

3. Some other matrix rings

Since every strongly \( \pi \)-regular ring is strongly clean, another interesting approach would be to determine the strongly \( \pi \)-regular rings \( R \) for which the matrix rings over \( R \) are strongly clean. Let \( S \) be a ring and \( R \) be a subring of \( S \) such that they share the same identity. The ring \( S \) is called a finite extension of \( R \) if \( S \), as an \( R \)-module, is generated by a finite set \( X \) of generators.

**Theorem 3.1** [11]. Let \( R \) be a ring whose prime factor rings are Artinian. Then every finite extension of \( R \) is strongly \( \pi \)-regular.