Oscillation of First-Order Impulsive Differential Equations with Advanced Argument

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Abstract—For first-order linear impulsive differential equations with an advanced argument, various criteria for oscillation and nonoscillation of solutions of these equations are found. Some results of which improve the known theorems. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The theory of oscillation of impulsive differential equations with deviating arguments is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of nonimpulsive differential equations with deviating arguments. Many evolution processes in nature are characterized by the fact that at certain moments of time, they experience an abrupt change of state. That was the reason for the development of the theory of impulsive differential equations and impulsive functional differential equations (see [1]).

The purpose of this paper is to study oscillation and nonoscillation of the solutions of impulsive differential equations with advanced argument. For the theory of oscillation of nonimpulsive differential equations and impulsive differential equations with retarded arguments, we refer to the monographs [2,3] and papers [4–10].

Unlike differential equations with retarded argument, those with advanced argument are very rare in the literature. In studying an electrodynamic systems, Schulman investigated (see [11, p. 443]) the second-order nonimpulsive differential equation

\[ x''(t) + wx(t) = \frac{1}{2} \alpha x(t - \tau) + \frac{1}{2} \beta x(t + \sigma) + \varphi(t), \]

where \( \alpha, \beta \) are constants and \( \tau, \sigma \) are positive constants. When \( \alpha = 0 \), the above equation becomes

\[ x''(t) + wx(t) = \frac{1}{2} \beta x(t + \sigma) + \varphi(t), \]
which is a differential equation with advanced argument. Györi and Ladas [2], and Erbe, Kong and Zhang [3] investigated, respectively, oscillations of the following nonimpulsive differential equations with advanced arguments

\[ x'(t) = p(t)x(t + \tau), \]
\[ x^{(n)}(t) = p^n x(t + n\tau), \quad n \geq 1, \]

and

\[ x'(t) = \sum_{i=1}^{m} p_i(t)x(t + \tau_i(t)), \]

where \( \tau > 0, \tau_i(t) > 0, i = 1, 2, \ldots, m \) (see [3, p. 46, p. 260; 4, p. 69]). Bainov, Dimitrova and Simeonov investigated in [12] and [13] oscillations of impulsive differential equations with advanced argument.

Let \( N = \{1, 2, 3, \ldots\} \). Consider the impulsive differential equation with an advanced argument

\[ y'(t) = p(t)y(t + \tau), \quad t \neq t_k, \]
\[ y(t_k^+) - y(t_k) = b_ky(t_k), \quad k \in N, \]

under the following hypotheses:

(A1) \( 0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots \) are fixed points with \( \lim_{k \to \infty} t_k = \infty \);

(A2) \( p \in ([t_0, \infty), R) \) is locally summable function, \( \tau > 0 \) is constant;

(A3) \( b_k \in (-\infty, -1) \cup (-1, \infty) \) are constants for \( k \in N = \{1, 2, \ldots\} \).

**Definition 1.** A function \( y \in ([t_0, \infty), R) \) is said to be a solution of (1) on \([t_0, \infty)\) if the following conditions are satisfied:

(i) \( y(t) \) is absolutely continuous on each interval \((t_k, t_{k+1}), k \in N, \) and \((t_0, t_1); \)

(ii) for any \( t_k \in [t_0, \infty), y(t_k^+) \) and \( y(t_k^-) \) exists and \( y(t_k^+) = y(t_k^-) \), \( k \in N; \)

(iii) for \( t \neq t_k, k \in N, y(t) \) satisfies \( y'(t) = p(t)y(t + \tau), \) a.e. (almost everywhere) and for each \( t = t_k, y(t_k^+) - y(t_k) = b_ky(t_k), k \in N. \)

**Definition 2.** A solution of (1) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

Recently, Bainov and Dimitrova [12] established the following results for oscillation of solutions of (1) under the assumptions that \( p \in C([t_0, \infty), [0, \infty)), \tau > 0, \) and \( \{t_k\} \) satisfies (A1). They introduced the following conditions.

(H1) \( 0 < \tau < t_1. \)

(H2) There exists a positive constant \( T > \tau \) such that \( t_{k+1} - t_k \geq T, k \in N. \)

(H3) There exists a constant \( M \) such that for any \( k \in N = \{1, 2, \ldots\}, \) the inequality \( 0 \leq M \leq b_k \) is valid.

**Theorem A.** (See [12, Theorem 1].) Let the following conditions hold:

(i) Conditions (H1) and (H2) are satisfied;

(ii) \( \limsup_{k \to \infty} [(1 + b_k) \int_{t_k-\tau}^{t_k} p(s) ds] > 1. \)

Then all solutions of (1) are oscillatory.

**Theorem B.** (See [12, Theorem 2].) Let the following conditions hold:

(i) Conditions (H1)-(H3) are satisfied;

(ii) \( \liminf_{t \to \infty} \int_{t}^{t+\tau} p(s) ds > 1/(e(1 + M)). \)

Then all solutions of (1) are oscillatory.

In the next section, we will give several new criteria on oscillation and nonoscillation of solutions of (1). Our results improve noticeably Theorems A and B.
2. OSCILLATION CRITERIA

Together with (1), we also consider a nonimpulsive differential equation with an advanced argument

$$x'(t) = P(t)x(t + \tau),$$

where

$$P(t) = \prod_{t \leq t_k < t + \tau} (1 + b_k)p(t), \quad t \geq t_0.$$ 

Here and in the sequel, we assume that a product equals to unit if the number of factors is equal to zero.

In this section, first we establish a fundamental theorem that enables us to reduce oscillation properties of solutions of (1) to corresponding properties of (2).

**REMARK 1.** Let \(\{b_{m_k}\}, k \in \mathbb{N}\) be a subsequence of \(\{b_k\}\) and \(b_{m_k} \leq -1, k \in \mathbb{N}\). Since \(y(t_{m_k}^+)\)

$$y(t_{m_k}^+) = (1 + b_{m_k})y^2(t_{m_k}) \leq 0,$$

it follows that solution \(y(t)\) of (1) is not eventually of constant sign. It implies that \(y(t)\) is oscillatory. Hence, in the sequel, we suppose that \(b_k > -1, k \in \mathbb{N}\).

**THEOREM 1.** Assume that \((A_1)-(A_3)\) hold. Then all solutions of (1) are oscillatory if and only if all solutions of (2) are oscillatory.

**PROOF.** Let \(y(t)\) be a nonoscillatory solution of (1). Without loss of generality, we suppose that \(y(t)\) is eventually positive. Then there exists a \(T \geq t_0\) such that \(y(t) > 0\) for \(t \geq T\). From Remark 1, \(b_k > -1, k \in \mathbb{N}\). Set \(x(t) = \prod_{T \leq t_k < t}(1 + b_k)^{-1}y(t)\). Hence, \(x(t) > 0\) for \(t \geq T\). Since \(y(t)\) is absolutely continuous on each interval \((t_k, t_{k+1}]\), and in view of \(y(t_k^+) = (1 + b_k)y(t_k)\), it follows that for \(t_k \geq T\),

$$x(t_k^+) = \prod_{T \leq t_j \leq t_k} (1 + b_j)^{-1}y(t_k^+) = \prod_{T \leq t_j < t_k} (1 + b_j)^{-1}y(t_k) = x(t_k),$$

which implies that \(x(t)\) is continuous on \([T, \infty)\) and it is easy to prove that \(x(t)\) is also absolutely continuous on \([T, \infty)\). Thus, we obtain that for \(t \geq T\),

$$x'(t) - P(t)x(t + \tau) = \prod_{T \leq t_k < t} (1 + b_k)^{-1}y'(t) - P(t) \prod_{T \leq t_k < t + \tau} (1 + b_k)^{-1}y(t + \tau)$$

$$= \prod_{T \leq t_k < t} (1 + b_k)^{-1}y(t) - \prod_{T \leq t_k < t + \tau} (1 + b_k)p(t) \prod_{T \leq t_k < t + \tau} (1 + b_k)^{-1}y(t + \tau)$$

$$= \prod_{T \leq t_k < t} (1 + b_k)^{-1}(y(t) - p(t)y(t + \tau)) = 0, \quad \text{a.e.,}$$

which means that \(x(t)\) is a positive solution of (2).

Conversely, without loss of generality, suppose that \(x(t)\) is an eventually positive solution of (2) and \(x(t) > 0\) for \(t \geq T \geq t_0\). Set \(y(t) = \prod_{T \leq t_k < t}(1 + b_k)x(t)\). As \(x(t)\) is absolutely continuous on \([T, \infty), y(t)\) is absolutely continuous on each interval \((t_k, t_{k+1}], t_k \geq T\) and for \(t \geq T\),

$$y'(t) - p(t)y(t + \tau) = \prod_{T \leq t_k < t} (1 + b_k)x'(t) - p(t) \prod_{T \leq t_k < t + \tau} (1 + b_k)x(t + \tau)$$

$$= \prod_{T \leq t_k < t} (1 + b_k) \left(x'(t) - p(t) \prod_{t \leq t_k < t + \tau} (1 + b_k)x(t + \tau)\right) = 0, \quad \text{a.e.}$$

On the other hand, for all \(t \geq T\),

$$y(t_k^+) = \lim_{t \to t_k^-} \prod_{T \leq t_j < t} (1 + b_j)x(t) = \prod_{T \leq t_j < t_k} (1 + b_j)x(t_k)$$
and
\[ y(t_k) = \prod_{T \leq t_j < t_k} (1 + b_j)x(t_k) > 0. \]
Thus, for every \( t_k \geq T \), we have
\[ y(t_k^+) = (1 + b_k)y(t_k) > 0, \]
which together with (4) implies that \( y(t) \) is a positive solution of (1). The proof of Theorem 1 is complete.

From Theorem 1, we obtain the following result.

**Corollary 1.** Assume that \((A_1)-(A_3)\) hold.

(i) If \( x(t) \) is a solution of (2) on \([\sigma, \infty)\), \( \sigma \geq 0 \), then \( y(t) = \prod_{\sigma \leq t_k < t} (1 + b_k)x(t) \) is a solution of (1) on \([\sigma, \infty)\).

(ii) If \( y(t) \) is a solution of (2) on \([\sigma, \infty)\), \( \sigma \geq 0 \), then \( x(t) = \prod_{\sigma \leq t_k < t} (1 + b_k)^{-1}y(t) \) is a solution of (2) on \([\sigma, \infty)\).

The following results provide two explicit sufficient conditions for the oscillation of all solutions of (1).

**Theorem 2.** Assume that \((A_1)-(A_3)\) hold and there exists a sequence of intervals \( \{(\xi_n, \eta_n)\} \) such that \( \lim_{n \to \infty} \xi_n = \infty \) and \( \eta_n - \xi_n > \tau \) for all \( n \geq N \geq 1 \). If \( p(t) \geq 0 \) for all \( t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n) \) and
\[ \limsup_{t \to \infty} \int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1 + b_k)p(s) \, ds > 1, \quad \text{for } t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau), \]
then all solutions of (1) are oscillatory.

**Proof.** Let \( y(t) \) be a nonoscillatory solution of (1). Without loss of generality, we suppose that \( y(t) > 0 \) for \( t \geq T \geq t_0 \). From Theorem 1, (2) has also a positive solution \( x(t) \) on \([T, \infty)\). Thus, for \( t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau) \), \( P(t) = \prod_{t \leq t_k < t+\tau} (1 + b_k)p(t) \geq 0 \), and hence,
\[ x'(t) > 0, \quad \text{a.e. for } t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau), \]
which implies \( x(t) \) is nondecreasing in \( \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau) \). Integrating (2) from \( t \) to \( t + \tau \), we obtain that for \( t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau) \),
\[ x(t) - x(t + \tau) + \int_t^{t+\tau} P(s)x(s + \tau) \, ds = 0. \]
By using the nondecreasing character of \( x \), we derive that
\[ x(t) + x(t + \tau) \left( \int_t^{t+\tau} P(s) \, ds - 1 \right) \leq 0, \quad \text{for } t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau), \]
which contradicts (5). The proof of Theorem 2 is complete.

**Theorem 3.** Assume that \((A_1)-(A_3)\) hold and there exists a sequence of intervals \( \{(\xi_n, \eta_n)\} \) with \( \lim_{n \to \infty} (\eta_n - \xi_n) = \infty \). Besides,
\[ p(t) \geq 0, \quad \text{for all } t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau), \]
and
\[ \liminf_{t \to \infty} \int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1 + b_k)p(s) \, ds > \frac{1}{e}, \quad \text{for } t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau), \]
then all solutions of (1) are oscillatory.

**Proof.** Let \( y(t) \) be a nonoscillatory solution of (1). Without loss of generality, we suppose that \( y(t) > 0 \) for \( t \geq T \geq t_0 \). From Theorem 1, (2) has also a positive solution \( x(t) \) on \([T, \infty)\). Thus, for \( t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau) \), \( P(t) = \prod_{t \leq t_k \leq t+\tau} (1 + b_k)p(t) \geq 0 \), and hence,
\[ x'(t) \geq 0 \quad \text{and} \quad x(t) \leq x(t + \tau), \quad \text{for } t \in \bigcup_{n=N}^{\infty} (\xi_n, \eta_n - \tau). \]

From (6) and (7), there exist \( \gamma > 1/e \) and \( N_1 \geq N \) such that
\[ \int_t^{t+\tau} P(s) \, ds \geq \gamma > \frac{1}{e}, \quad \text{for } t \in \bigcup_{n=N_1}^{\infty} (\xi_n, \eta_n - \tau). \]
Moreover, for every \( t^* \in \bigcup_{n=N_1}^{\infty} (\xi_n, \eta_n - \tau) \), there exists \( t \in \bigcup_{n=N_1}^{\infty} (\xi_n, \eta_n - \tau) \) such that
\[ \int_{t^*}^{t} P(s) \, ds \geq \gamma \quad \text{and} \quad \int_{t}^{t^* + \tau} P(s) \, ds \geq \gamma. \]
Integrating (2), first from \( t^* \) to \( t \) and then from \( t \) to \( t^* + \tau \), and using (8) and (9), we obtain, respectively,
\[ x(t) - x(t^*) = \int_{t^*}^{t} P(s)x(s + \tau) \, ds \geq x(t + \tau) \int_{t^*}^{t} P(s) \, ds \geq \frac{\gamma}{2}x(t + \tau) \]
and
\[ x(t^* + \tau) - x(t) \geq \int_{t}^{t^* + \tau} P(s)x(s + \tau) \, ds \geq x(t + \tau) \int_{t}^{t^* + \tau} P(s) \, ds \geq \frac{\gamma}{2}x(t + \tau). \]
From these relations, we derive that
\[ x(t) \geq \frac{\gamma^2}{4}x(t + \tau), \quad \text{for } t \in \bigcup_{n=N_1}^{\infty} (\xi_n + \tau, \eta_n - 2\tau), \]
and
\[ 1 \leq \frac{x(t + \tau)}{x(t)} < \frac{4}{\gamma^2}, \quad \text{for } t \in \bigcup_{n=N_1}^{\infty} (\xi_n + \tau, \eta_n - 2\tau). \]
Besides, by \( \lim_{n \to \infty} (\eta_n - \xi_n) = \infty \) and \( e\gamma > 1 \), there exists \( N_2 \geq N_1 \) such that for all \( t \geq N_2 \),
\[ (e\gamma)^{N_2} > \frac{4}{\gamma^2} \quad \text{and} \quad \eta_n - \xi_n > (N_2 + 2)\tau. \]
Since, by (2) and our assumptions, for every \( t \in \bigcup_{n=N_2}^{\infty} (\xi_n + \tau, \eta_n - 2\tau) \),
\[ x'(t) = P(t)x(t + \tau) \geq P(t)x(t), \quad \text{a.e.} \]
It is easy to see that for all \( t \in \bigcup_{n=N_2}^{\infty} (\xi_n + \tau, \eta_n - 2\tau) \),
\[ x(t + \tau) \geq x(t) \exp \left( \int_{t}^{t+\tau} P(s) \, ds \right) \geq (e\gamma)x(t). \]
Next, for every \( t \in \bigcup_{n=N_2}^{\infty} (\xi_n + \tau, \eta_n - 3\tau) \), we obtain
\[
x'(t) = P(t)x(t + \tau) \geq P(t)e^\gamma x(t), \quad \text{a.e.,}
\]
and hence, we have that for all \( t \in \bigcup_{n=N_2}^{\infty} (\xi_n + \tau, \eta_n - 3\tau) \),
\[
x(t + \tau) \geq x(t) \exp \left( e^\gamma \int_t^{t+\tau} P(s) \, ds \right) \geq (e^\gamma)^2 x(t).
\]
Following in this way, we conclude that for \( t \in \bigcup_{n=N_2}^{\infty} (\xi_n + \tau, \eta_n - (N_2 + 1)\tau) \),
\[
\frac{x(t + \tau)}{x(t)} \geq (e^\gamma)^{N_2} > \frac{4}{\tau^2},
\]
which contradicts (10). The proof of Theorem 3 is complete.

For reference in the sequel, we also list the following hypotheses:

\((A_1')\) \( p \in ([t_0, \infty), [0, \infty)) \) is locally summable function and \( \tau > 0 \) is constant;
\((A_2')\) \( \beta_k \in (-1, \infty) \) are constants for all \( k \in N \) and there exists a constant \( M \) such that
\[-1 < \beta_k < b_k.
\]
We now can apply immediately Theorems 2 and 3 to obtain the following results.

**Theorem 4.** Assume that \((A_1), (A_2'), (A_3)\) hold and
\[
\lim_{t \to \infty} \sup \int_t^{t+\tau} \prod_{s \leq t_k < s + \tau} (1 + \beta_k)p(s) \, ds > 1,
\]
then all solutions of (1) are oscillatory.

**Theorem 5.** Assume that \((A_1), (A_2'), (A_3)\) hold and
\[
\lim_{t \to \infty} \inf \int_t^{t+\tau} \prod_{s \leq t_k < s + \tau} (1 + \beta_k)p(s) \, ds > \frac{1}{e},
\]
then all solutions of (1) are oscillatory.

**Remark 2.** Since
\[
\lim_{t \to \infty} \sup \int_t^{t+\tau} \prod_{s \leq t_k < s + \tau} (1 + \beta_k)p(s) \, ds = \lim_{(t-\tau) \to \infty} \int_{t-\tau}^{t} \prod_{s \leq t_k < s + \tau} (1 + \beta_k)p(s) \, ds,
\]
it follows that Theorem 4 improves Theorem A. It is easy to see that Theorem 5 improves Theorem B.

From Theorems 4 and 5, we can obtain the following corollary.

**Corollary 2.** Assume that \((A_1), (A_2'), (A_3)\) hold and there exists a positive integer \( m \) such that
\[
m(t_{k+1} - t_k) \geq \tau \quad \text{for all} \ k \in N.
\]
If one of the following conditions satisfies:

(i) \( \limsup_{t \to \infty} (1 + M)^m \int_t^{t+\tau} p(s) \, ds > 1; \)

(ii) \( \liminf_{t \to \infty} (1 + M)^m \int_t^{t+\tau} p(s) \, ds > 1/e. \)

Then all solutions of (1) are oscillatory.

**Remark 3.** It is easy to find that Corollary 1 improves noticeably Theorems 3 and 4 in [12].
3. NONOSCILLATION CRITERIA

In this section, our aim is to obtain explicit conditions for nonoscillation of impulsive differential equation (1).

**THEOREM 6.** Assume that $(A_1), (A_2), (A_3)$ with $b_k > -1$ hold. Then the following statements are equivalent.

(i) Equation (1) has a nonoscillatory solution.

(ii) Integral equation

\[ u(t) = P(t) \exp \left( \int_t^{t+\tau} u(s) \, ds \right) \]  

has a nonnegative solution on $[T, \infty)$, $T \geq t_0$, where $P(t)$ is defined by (3).

(iii) The sequence $\{ u_k(t) \}$ of locally summable functions is convergent on $[T, \infty)$, a.e. $T \geq t_0$, where

\[
\begin{align*}
 u_1(t) &= P(t) = \prod_{t \leq t_k \leq t+\tau} (1 + b_k)p(t), \quad t \geq T, \\
 u_{k+1}(t) &= P(t) \exp \left( \int_t^{t+\tau} u_k(s) \, ds \right), \quad t \geq T, \quad k \in N.
\end{align*}
\]  

**Proof.**

(i) $\Rightarrow$ (ii). Let $y(t)$ be a nonoscillatory solution of (1). By Theorem 1, equation (2) has also a nonoscillatory solution $z(t)$. Without loss of generality, we suppose that $x(t) > 0$ for $t \geq T \geq t_0$. Set

\[ u(t) = \frac{x'(t)}{x(t)}, \quad t \geq T. \]

Thus, $u(t)$ is defined a.e. on $[T, \infty)$. Now we claim that $u(t)$ is a solution of (11) on $[T, \infty)$. Indeed, from (13), we see that $x(t) = x(T) \exp(\int_T^t u(s) \, ds)$ and so

\[ \frac{x(t+\tau)}{x(t)} = \exp \left( \int_t^{t+\tau} u(s) \, ds \right), \quad t \geq T. \]

By dividing both sides of (2) by $x(t)$ and using (14), we obtain

\[ u(t) = P(t) \exp \left( \int_t^{t+\tau} u(s) \, ds \right). \]

This proves (ii).

(ii) $\Rightarrow$ (i). Let $u(t)$ be a nonnegative solution on $[T, \infty)$. Set $x(t) = \exp(\int_T^t u(s) \, ds)$. It is easy to check that $x(t)$ is a positive solution of (2) on $[T, \infty)$. By Corollary 1, $y(t) = \prod_{T \leq t_k \leq t+\tau} (1 + b_k)x(t)$ is a positive solution of (1) on $[T, \infty)$. The proof that (ii) $\Rightarrow$ (i) is complete.

(ii) $\Rightarrow$ (iii). Let $u(t)$ be a nonnegative solution of (11). Clearly, $u_1(t) = P(t) \leq P(t) \exp(\int_T^{t+\tau} u_1(s) \, ds) = u_2(t) \leq u(t)$, $t \geq T$. By a simple induction, we prove

\[ u_k(t) \leq u_{k+1}(t) \leq u(t), \quad t \geq T, \quad k \in N. \]

Hence, the sequence $\{ u_k(t) \}$ has a pointwise limiting function $\tilde{u}(t)$, that is,

\[ \lim_{k \to \infty} u_k(t) = \tilde{u}(t) \leq u(t), \quad \text{a.e. for } t \geq T, \]

which means that (iii) is proved.

(iii) $\Rightarrow$ (ii). Let $\lim_{k \to \infty} u_k(t) = u(t)$, a.e. $t \geq T$. From (12), we have that $u_k(t) \leq u(t)$ on $[T, \infty)$, $k \in N$. Hence, $u_k(t)$, $k \in N$ are uniformly bounded on $[t, t+\tau]$ for all $t \geq T$ and $k \in N$. So by the Lebesgue’s dominated convergence theorem, from (12), we obtain (11). The proof that (iii) $\Rightarrow$ (ii) is complete.

The next result is a sufficient condition for nonoscillation of solutions of (1).
THEOREM 7. Assume that \((A_1), (A_2'), (A_3)\) with \(b_k > -1\) hold and there exists a \(T \geq t_0\) such that for all \(t \geq T\)

\[
\int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1 + b_k)p(s) \, ds \leq \frac{1}{e},
\]
then (1) has a nonoscillatory solution.

PROOF. Now, we prove that (16) implies that statement (iii) of Theorem 6 holds. In fact, from (12) and (16),

\[
\int_t^{t+\tau} u_1(s) \, ds = \int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1 + b_k)p(s) \, ds \leq \frac{1}{e}, \quad t \geq T.
\]

Thus, using again (16), we get

\[
u_2(t) = P(t) \exp \left( \int_t^{t+\tau} u_1(s) \, ds \right) \leq P(t) \exp \left( \frac{1}{e} \right) \leq P(t)e, \quad t \geq T.
\]

By induction, we obtain

\[
u_k(t) \leq u_{k+1}(t) = P(t) \exp \left( \int_t^{t+\tau} u_k(s) \, ds \right) \leq P(t) \exp \left( e \int_t^{t+\tau} P(s) \, ds \right) \leq P(t)e, \quad t \geq T, \quad k \in \mathbb{N},
\]
which implies that \(\{u_k(t)\}\) is convergent on \([T, \infty)\). By Theorem 6, equation (1) has a nonoscillatory solution. The proof of Theorem 7 is complete.

The following result is an immediate consequence of Theorems 5 and 7 applied to the autonomous impulsive differential equation with advanced argument

\[y'(t) = py(t + \tau), \quad t \neq t_k, \]
\[y(t_k^+) - y(t_k) = by(t_k), \quad k \in \mathbb{N}, \quad (1')\]
where \(t_{k+1} - t_k = \tau > 0, b > -1, \) and \(p > 0\) are constants.

COROLLARY 3. All solutions of (1') are oscillatory if and only if \((1 + b)p\tau > 1/e\).

4. EXAMPLES AND MORE REMARKS

4.1. Two Examples

In order to show the applications of our results, we present here the following examples. The results established in [12] and other known criteria are not applicable to these examples.

EXAMPLE 1. Let \(t_k = \sigma + km\tau, m\) is a positive integer, \(p(t) \geq 0\) is a locally summable function and \(\tau > 0, b_k \in (-1, \infty), k \in \mathbb{N},\) are constants.

Consider impulsive differential equation (1). Since \(t_{k+1} - t_k = m\tau,\) there is at most one point of impulsive effect on each \([t, t + \tau), t \geq \sigma.\) So,

\[
\int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1 + b_k)p(s) \, ds = (1 + b_k) \int_t^{t+\tau} p(s) \, ds, \quad \text{if } t_k \in [t, t + \tau),
\]
or

\[
\int_t^{t+\tau} \prod_{s \leq t_k < s+\tau} (1 + b_k)p(s) \, ds = \int_t^{t+\tau} p(s) \, ds, \quad \text{if some } t_k \notin [t, t + \tau), \quad k \in \mathbb{N}.
\]
(i) Let
\[ d_1 = \limsup_{t \to \infty} \left( (1 + b_k) \int_t^{t+\tau} p(s) \, ds, \, t_k \leq t_k < t + \tau \right) \]
and
\[ d_2 = \limsup_{t \to \infty} \int_t^{t+\tau} p(s) \, ds. \]
If \( d > 1 \), where \( d = \max\{d_1, d_2\} \), then by Theorem 4, all solutions of (1) are oscillatory.

(ii) Let
\[ c_1 = \liminf_{t \to \infty} \left( (1 + b_k) \int_t^{t+\tau} p(s) \, ds \right) \]
and
\[ c_2 = \liminf_{t \to \infty} \int_t^{t+\tau} p(s) \, ds. \]
If \( c > 1/\varepsilon \), where \( c = \min\{c_1, c_2\} \), then by Theorem 5, all solutions of (1) are oscillatory.

(iii) If there is \( T \geq t_0 \) such that
\[ Q(t) \leq \frac{1}{\varepsilon}, \quad \text{for all } t \geq T, \]
where
\[ Q(t) = \max \left\{ (1 + b_k) \int_t^{t+\tau} p(s) \, ds, \, t \leq t_k < t + \tau; \int_t^{t+\tau} p(s) \, ds \right\}, \quad t \geq T, \]
then by Theorem 7, (1) has a nonoscillatory solution on \([T, \infty)\).

**Example 2.** Let \( t_k = k, \ b_k = 1/k, \ k \in \mathbb{N}, \ \tau = 1, \ p(t) = 1/|t|, \ t \geq t_0 > 0 \), where \([\cdot]\) denotes the greatest-integer function. Consider the impulsive differential equation (1). Since there is just a point of impulsive effect on \([t, t + \tau), \ t \geq t_0\).

\[ \int_t^{t+1} \prod_{s \leq t_k < s+1} (1 + b_k)p(s) \, ds = \int_t^{t+1} \left( 1 + \frac{1}{k} \right) \frac{1}{k} \, ds = \left( 1 + \frac{1}{k} \right) \frac{1}{k}, \quad k \leq t < k + 1, \quad k \in \mathbb{N}, \]

where the corresponding nonimpulsive differential equation (2) is
\[ x'(t) = \left( 1 + \frac{1}{k} \right) \frac{1}{k} x(t+1), \quad k \leq t < k + 1, \quad k \in \mathbb{N}. \quad (17)_k \]
The “characteristic equation” of \((17)_k\) is
\[ \lambda = \left( 1 + \frac{1}{k} \right) \frac{1}{k} e^\lambda, \quad k \leq t < k + 1, \quad k \in \mathbb{N}. \quad (18)_k \]
Choose a \( K \) such that for all \( k \geq K \),
\[ \left( 1 + \frac{1}{k} \right) \frac{1}{k} \leq \frac{1}{\varepsilon}. \]
Then for a fixed \( k \geq K \), \((18)_k\) has a real root \( \lambda_k \) such that
\[ \lambda_k = \left( 1 + \frac{1}{k} \right) \frac{1}{k} e^{\lambda_k}. \]
Thus, \( x_k(t) = c_k e^{\lambda_k t} \) is a positive solution of \((17)_k\) on \([k, k + 1)\) where \(c_k, k \geq K\), is a positive constant. Set

\[
x(t) = c_{k+m} e^{\lambda_{k+m} t}, \quad k + m \leq t < k + (m + 1), \quad m = 0, 1, 2, \ldots, \quad k \geq K,
\]

we can choose constants \(c_{k+1}, c_{k+2}, \ldots\) such that \(x(t)\) is continuous at \(k + 1, k + 2, \ldots\). In fact, suppose that \(c_k\) is a positive constant. To choose \(c_{k+1}\), let

\[
c_{k+1} = c_k e^{\lambda_k (k+1)} = c_{k+1} e^{\lambda_{k+1} (k+1)},
\]

that is, \(c_{k+1} = c_k e^{\lambda_k (k+1)} e^{\lambda_{k+1} (k+1)}\), then \(x(t)\) is continuous at \(t = k + 1\). Thus, by induction, we can obtain \(c_{k+1}, c_{k+2}, c_{k+3}, \ldots\) such that \(x(t)\) is continuous on \([K, \infty)\) and it is easy to check that \(x(t)\) is a positive solution on \([K, \infty)\). By Corollary 1, \(y(t) = \prod_{k \leq t < k}(1 + 1/k)x(t)\) is a positive solution of \((1)\) on \([K, \infty)\).

4.2. Generalization for Several Advanced Arguments

The results of this paper are extensible to more general impulsive equations with advanced arguments, for example,

\[
y'(t) = \sum_{i=1}^{m} p_i(t)y(t + \tau_i),
\]

\[
y(t^+_k) - y(t_k) = b_k y(t_k), \quad k \in N.
\]

4.3. Generalization for Inequalities

The results of Section 2 of this paper can be formulated in a more general form about the inequalities

\[
y'(t) - p(t)y(y + \tau) \geq 0, \quad t \neq t_k,
\]

\[
y(t^+_k) - y(t_k) = b_k y(t_k), \quad k \in N,
\]

and

\[
y'(t) - p(t)y(y + \tau) \leq 0, \quad t \neq t_k,
\]

\[
y(t^+_k) - y(t_k) = b_k y(t_k), \quad k \in N,
\]

(see [12, Corollary 1]).

REFERENCES