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# Changing upper irredundance by edge addition

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**Abstract**

Denote the upper irredundance number of a graph $G$ by $\mathrm{IR}(G)$. A graph $G$ is IR-edge-addition-sensitive if its upper irredundance number changes whenever an edge of $\bar{G}$ is added to $G$. Specifically, $G$ is IR-edge-critical ($\mathrm{IR}^+$-edge-critical, respectively) if $\mathrm{IR}(G+e) < \mathrm{IR}(G)$ ($\mathrm{IR}(G+e) > \mathrm{IR}(G)$, respectively) for each edge $e$ of $\bar{G}$. We show that if $G$ is IR-edge-addition-sensitive, then $G$ is either IR-edge-critical or $\mathrm{IR}^+$-edge-critical. We obtain properties of the latter class of graphs, particularly in the case where $\beta(G) = \mathrm{IR}(G) = 2$ (where $\beta(G)$ denotes the vertex independence number of $G$). This leads to an infinite class of $\mathrm{IR}^+$-edge-critical graphs where $\mathrm{IR}(G) = 2$.


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## 1. Introduction

We generally follow the notation and terminology of [10]. In a graph $G=(V_G,E_G)$, if $S,T \subseteq V_G$, then the set of all edges of $G$ with one endvertex in $S$ and the other in $T$, is denoted by $E_G(S,T)$. Further, $N_G(v)=\{u \in V_G : uv \in E_G\}$ and $N_G[v]=N_G(v)\cup\{v\}$ denote the *open* and *closed neighbourhoods*, respectively, of $v \in V_G$. The *closed neighbourhood of* $S \subseteq V_G$, denoted by $N_G[S]$, is the set $\bigcup_{s\in S} N_G[s]$. For $s \in S$, $\mathrm{pn}_G(s,S) = N_G[s] -$

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Table 1
Existence of critical graphs

| | ir | $\gamma$ | $i$ | $\beta$ | $\Gamma$ | IR |
|---|---|---|---|---|---|---|
| $\pi$-critical | Yes | yes | Yes | Yes | Yes | Yes |
| $\pi^+$-critical | No | No | No | No | Yes | No |
| $\pi$-edge-critical | Yes | Yes | Yes | Yes | Yes | Yes |
| $\pi^+$-edge-critical | No | No | No | No | Yes | ? |
| $\pi$-ER-critical | Yes | Yes | Yes | Yes | Yes | Yes |
| $\pi^-$-ER-critical | ? | No | Yes | No | No | No |

$N_G[S-\{s\}]$ is the *private neighbourhood of* $s$ relative to $S$ and its elements are called the *private neighbours* of $s$ *relative to* $S$. If $x \in \mathrm{pn}_G(s,S) - S$, then $x$ is said to be an *external private neighbour of* $s$ *relative to* $S$, or shortly an $S$*-epn of* $s$. If confusion is unlikely we omit the subscript $G$ from the above notation. If $\mathrm{pn}(s,S) \subseteq N[v]$, where $v \in V-S$, we say that $v$ *annihilates* $s$ (relative to $S$).

The vertex $s \in S \subseteq V$ is *irredundant* in $S$ if $\mathrm{pn}(s,S) \neq \phi$, and $S$ is an *irredundant set* of $G$ if each $s \in S$ is irredundant in $S$; otherwise $S$ is *redundant*. Clearly, $S$ is an irredundant set of $G$ if and only if for each $s \in S$, $s$ is isolated in $\langle S \rangle$ (the subgraph of $G$ induced by $S$), or $s$ has an $S$-epn. The *lower* (*upper*) *irredundance number* $\mathrm{ir}(G)$ ($\mathrm{IR}(G)$) of $G$ is the smallest (largest) cardinality of a maximal irredundant set of $G$. An IR-*set* of $G$ is an irredundant set $S$ of $G$ with $|S| = \mathrm{IR}(G)$. The lower and upper domination numbers $\gamma(G)$ and $\Gamma(G)$, the independent domination number $i(G)$ and the independence number $\beta(G)$, together with $\mathrm{ir}(G)$ and $\mathrm{IR}(G)$, are also called the *domination parameters*.

Carrington, Harary and Haynes [2] (also see [10, Chapter 5]) surveyed the problems of characterising graphs for which the domination number $\gamma$ changes/does not change whenever a vertex is removed or an edge is removed or added. Domination critical graphs, i.e., graphs for which $\gamma$ decreases whenever an edge is added, are well-studied and are surveyed in [11].

For each of the six domination parameters $\pi$, six types of criticality were studied in [6]. The graph $G$ is

*$\pi$-critical* if $\pi(G-v) < \pi(G)$ for all $v \in V_G$,

*$\pi^+$-critical* if $\pi(G-v) > \pi(G)$ for all $v \in V_G$,

*$\pi$-edge-critical* if $\pi(G+e) < \pi(G)$ for all $e \in E_{\bar{G}} \neq \phi$,

*$\pi^+$-edge-critical* if $\pi(G+e) > \pi(G)$ for all $e \in E_{\bar{G}} \neq \phi$,

*$\pi$-ER-critical* if $\pi(G-e) > \pi(G)$ for all $e \in E_G \neq \phi$,

*$\pi^-$-ER-critical* if $\pi(G-e) < \pi(G)$ for all $e \in E_G \neq \phi$.

We refer the reader to [6–9] for previous studies of some of the above-mentioned types of criticality, and only summarise the known existence or non-existence of the six types of criticality for the six domination parameters in Table 1.

IR-edge-critical graphs are characterised in [7] as precisely the graphs $K_a + bK_1$ or $K_a + (bK_1 \cup (K_c \times K_2))$ for $a, b \geqslant 0$ and $c \geqslant 3$ (where $+$ denotes join, $\cup$ disjoint union, and $\times$ the cartesian product, and where $V_{K_0} = \phi$).

Here we consider graphs $G$ such that for each $e \in E_{\bar{G}}$, $\mathrm{IR}(G+e) > \mathrm{IR}(G) = k$. These graphs are called $k$-IR$^+$-*edge-critical*.

Dunbar et al. [5] initiated the study of graphs for which IR changes on the addition of any edge, the so-called IR-*edge-addition-sensitive*, abbreviated IR-*EA-sensitive*, graphs. They characterised the bipartite IR-EA-sensitive graphs and showed that they are IR-edge-critical. They conjectured that **any** IR-EA-sensitive graph is IR-edge-critical, i.e. that IR$^+$-edge-critical graphs do not exist.

In this paper we prove that any IR-EA-sensitive graph is either IR-edge-critical or IR$^+$-edge-critical and disprove the conjecture of [5]. More specifically, we obtain structural properties of IR$^+$-edge-critical graphs, particularly in the case where $\beta(G)=2$. This enables us to define an infinite class of 2-IR$^+$-edge-critical graphs.

## 2. Preliminary results

Suppose $S$ is an irredundant set of the graph $G$. Let $C$, $B$ and $R$ denote the sets of vertices of $V - S$ which are adjacent to at least two vertices, exactly one vertex and no vertices of $S$, respectively, i.e.,

$$R = V - N[S],$$

$$B = \left(\bigcup_{s \in S} \mathrm{pn}(s, S)\right) - S,$$

$$C = N(S) - B. \tag{1}$$

For each $s \in S$, let

$$B(s) = \mathrm{pn}(s, S) - S, \tag{2}$$

i.e., $B(s)$ is the set of $S$-epns of $s$. If $s = u_i$ for some integer $i$, as will be the case in Section 4, then we abbreviate $B(s)$ to $B_i$, i.e.,

$$B_i = \mathrm{pn}(u_i, S). \tag{3}$$

Furthermore, let $Z = \{z \in S : z \text{ is an isolated vertex of } \langle S \rangle\}$. The following proposition gives a necessary and sufficient condition for an irredundant set to be maximal irredundant.

**Proposition 1** (Cockayne et al. [4]). *An irredundant set $S$ of $G$ is maximal irredundant if and only if for each $v \in N[R]$ there exists $s_v \in S$ such that $\phi \neq \mathrm{pn}(s_v, S) \subseteq N[v]$, i.e., such that $v$ annihilates $s_v$.*

Let $S$ be irredundant with $S - Z = \{u_1, \ldots, u_k\}$. Then for each $i \in \{1, \ldots, k\}$, $B_i \neq \phi$; let $v_i \in B_i$ be arbitrary. The private neighbour property implies that $N(v_i) \cap S = \{u_i\}$, that is, $E_G(S, \{v_1, \ldots, v_k\}) = E_G(\{u_1, \ldots, u_k\}, Z \cup \{v_1, \ldots, v_k\}) = E_G(\{u_1, \ldots, u_k\}, \{v_1, \ldots, v_k\}) =$

$\{u_iv_i\colon i=1,\ldots,k\}$. We call this set of edges a *one-to-one matching* between $\{u_1,\ldots,u_k\}$ and $\{v_1,\ldots,v_k\}$. It is clear that $Z\cup\{v_1,\ldots,v_k\}$ is an irredundant set of $G$ of cardinality $|S|$.

The above observation leads to the following result, which characterises irredundant sets of cardinality three. (See [3] for a generalisation of this result.)

**Proposition 2** (Brewster et al. [1]). *The graph $G$ has an irredundant set of cardinality three if and only if $\bar{G}$ has a triangle or an induced* 6*-cycle.*

It is easy to find a graph $G$ such that $\mathrm{IR}(G+e) > \mathrm{IR}(G)$ for some $e\in E_{\bar{G}}$. However, graphs $G$ such that $\mathrm{IR}(G+e) > \mathrm{IR}(G)$ for *each* $e\in E_{\bar{G}}$ are much harder to find and up to now were not known to exist. For an example of the first type, let $V_G = U\cup W$ (disjoint union), where $U=\{u_1,\ldots,u_m\}$, $W=\{w_1,\ldots,w_m\}$, $\langle U\rangle=\langle W\rangle=K_m$, $m\geqslant 3$, and $E_G(U,W)=\{u_1w_1,u_2w_2\}$. Then $\mathrm{IR}(G)=2$, $\mathrm{IR}(G+u_3w_3)=3$ and in general, if $G_k$ is the graph obtained from $G$ by adding the edges $u_3w_3,\ldots,u_kw_k$, $3\leqslant k\leqslant m$, then $\mathrm{IR}(G_k)=k$, where $\{u_1,\ldots,u_k\}$ and $\{w_1,\ldots,w_k\}$ are IR-sets of $G_k$. Other examples of such graphs are given in [5]. Also note that if $i,j\leqslant k$, $i\neq j$, then $\mathrm{IR}(G_k+u_iw_j)=k-1$; the addition of more edges of this type reduces IR even more. We will show in Proposition 5 that the addition of the edge $u_1w_k$, for example, leaves $\mathrm{IR}(G_{k-1})$ unchanged.

The upper irredundance number thus has the unusual property that is not monotonic: it can be increased or decreased by edge addition. It is easy to see that the addition of an edge reduces IR by at most one (the private neighbourhood of at most one vertex is destroyed). The next result of [5,7] shows that IR increases by at most one whenever an edge is added.

**Proposition 3** (Dunbar et al. [5] and Grobler and Mynhardt [7]). *Suppose* $\mathrm{IR}(G+uv) > \mathrm{IR}(G)$ *for some* $uv\in E_{\bar{G}}$. *For every* IR*-set $S$ of $G+uv$, (without loss of generality) $u\in S$, $\mathrm{pn}_{G+uv}(u,S)=\{v\}$ and $S-\{u\}$ is an* IR*-set of $G$.*

**Corollary 4.** *If* $\mathrm{IR}(G+e) > \mathrm{IR}(G)$ *for some* $e\in E_{\bar{G}}$, *then* $\mathrm{IR}(G+e)=\mathrm{IR}(G)+1$.

## 3. Criticality of IR-EA-sensitive graphs

In Section 2, we discussed a class of graphs where some edge additions increase IR while others decrease IR. We now show that for such a graph IR must also be invariant under some edge additions.

**Proposition 5.** *If $G$ is* IR*-EA-sensitive, then it is either* IR*-edge-critical or* $\mathrm{IR}^+$*-edge-critical.*

**Proof.** Suppose to the contrary that $G$ is IR-EA-sensitive and for edges $e=uv$ and $f=xp$ of $\bar{G}$, $\mathrm{IR}(G+e) > \mathrm{IR}(G)$ while $\mathrm{IR}(G+f) < \mathrm{IR}(G)$. By Proposition 3, $G+e$ has an irredundant set $S$ containing $u$ and $S'=S-\{u\}$ is irredundant in $G$. By Corollary 4, $|S'|=\mathrm{IR}(G) > \mathrm{IR}(G+f)$ and so $S'$ is redundant in $G+f$. Therefore (without loss of generality) $x \in S'$ and $\{p\} = \mathrm{pn}_G(y, S')$ for some $y \in S' - \{x\}$ (possibly $p = y$). Using Proposition 3 and the private neighbour property we deduce that

$$e \text{ and } f \text{ are not adjacent.} \tag{4}$$

Now consider the edge $g = xv$. By Proposition 3, $\{v\} = \mathrm{pn}_{G+e}(u, S)$ and so $g \in E_{\bar{G}}$. Since $g$ is adjacent to both $e$ and $f$, the statement (4) implies that the addition of $g$ neither increases nor decreases the upper irredundance number, a contradiction. □

## 4. Critical graphs with $\beta = 2 \leqslant \mathrm{IR}$

Note that if $\beta(G) = 1$, then $G$ is complete. Therefore, in studying $\mathrm{IR}^+$-critical graphs we assume that $\beta(G) \geqslant 2$ and so $\mathrm{IR}(G) \geqslant 2$. In the remainder of the paper we consider $\mathrm{IR}^+$-edge-critical graphs $G$ with $\beta(G) = 2$. We need the following simple result about irredundant sets of a graph $G$ with $\beta(G) = 2$.

**Proposition 6.** *If* $\mathrm{IR}(G) > \beta(G) = 2$, *then* $\mathrm{IR}(G) = \max\{r : K_r \times K_2$ *is an induced subgraph of* $G\}$.

**Proof.** We first prove that if $\beta(G) = 2$, then the subgraph of $G$ induced by each irredundant set of cardinality at least three is complete. Suppose $S = \{u_1, u_2, \ldots, u_r\}$, $r \geqslant 3$, is an irredundant set of $G$ such that $u_1u_2 \notin E_G$. Since $\beta(G) = 2$, $\langle\{u_1, u_2, u_3\}\rangle$ contains at least one edge; without loss of generality say $u_2u_3 \in E_G$. Thus by the private neighbour property we may assume that $B_3 \neq \phi$; let $v_3 \in B_3$. Then neither $u_1$ nor $u_2$ is adjacent to $v_3$, hence $\{u_1, u_2, v_3\}$ is independent, a contradiction.

Since $\langle S\rangle$ is complete and $r \geqslant 2$, it follows that each $u_i \in S$ has an $S$-epn and so $B_i \neq \phi$ for each $i \in \{1, \ldots, r\}$. Choose $v_i \in B_i$ and let $T = \{v_1, \ldots, v_r\}$. Then there is a one-to-one matching between $S$ and $T$ in $G$, hence $E_G(S, T) = \{u_iv_i\colon i = 1, \ldots, r\}$ and $T$ is an IR-set of $G$. Thus $\langle T\rangle$ is complete and $\langle S \cup T\rangle = K_r \times K_2$. □

We consider the structure of a graph $G$ with $\beta(G) = 2$ and $\mathrm{IR}(G + e) > \mathrm{IR}(G)$ for some $e \in E_{\bar{G}}$. Recall that any IR-set of $G + e$ contains one of the endvertices of $e$ (Proposition 3).

**Proposition 7.** *Let* $u_0$, $v_0$ *be vertices of a graph* $G$ *with* $\beta(G) = 2$ *such that* $\mathrm{IR}(G + u_0v_0) = \mathrm{IR}(G) + 1$. *For each* IR-*set* $S' = \{u_0, u_1, \ldots, u_k\}$ *of* $G' = G + u_0v_0$, *the following conditions hold.*

(i) $\mathrm{pn}_{G'}(u_0, S') = \{v_0\}$.
(ii) $S = S' - \{u_0\}$ *is an* IR-*set of* $G$.
(iii) $\langle S'\rangle_G$ *is complete.*
(iv) *There is an* IR-*set* $T' = \{v_0, v_1, \ldots, v_k\}$ *of* $G'$ *with* $\langle T'\rangle_G \cong K_{k+1}$ *and* $S' \cap T' = \phi$ *such that* $E_G(S', T') = \{u_iv_i\colon i = 1, \ldots, k\}$.

(v) *Define* $R$, $B$, $C$ *and* $B_i$ *with respect to* $S$ *as in* (1) *to* (3), *and define* $R'$, $B'$, $C'$ *and* $B'_i$ *with respect to* $S'$ *and* $G'$ *in a similar way.*

$$R = R' \cup \{v_0\},\ u_0 \in C \subseteq C' \cup \{u_0\} \text{ and } B'_i \subseteq B_i \text{ for each } i \in \{1,\ldots,k\}.$$

(vi) *If* $k \geqslant 3$, *then* $\langle B \cup R\rangle_G$ *is complete, while if* $k = 2$, *it is possible that some (or all) edges between* $B_1$ *and* $B_2 - B'_2$, *or between* $B_2$ *and* $B_1 - B'_1$, *are missing, but* $G$ *contains all other edges possible between vertices in* $B \cup R$.

**Proof.** By Proposition 3, (i) and (ii) hold, while (iii) and (iv) follow from Proposition 6 (since $\mathrm{IR}(G + u_0v_0) > \beta(G + u_0v_0) = 2$).

(v) Since $\mathrm{pn}_{G'}(u_0, S') = \{v_0\}$ (Proposition 3), $v_0$ is not adjacent to any vertex of $S$ in $G$ and thus $R = R' \cup \{v_0\}$. Since $\langle S'\rangle_G$ is complete and $k \geqslant 2$, it follows that $u_0$ is adjacent to at least two vertices of $S$; hence $u_0 \in C$. Further, any $c \in C$ is adjacent to at least two vertices in $S \subseteq S'$. Thus $C - \{u_0\} \subseteq C'$; that is, $C \subseteq C' \cup \{u_0\}$. If $x \in B'_i = \mathrm{pn}_{G'}(u_i, S')$ for some $i \in \{1,\ldots,k\}$, then $x$ is not adjacent (in $G'$) to any vertex in $S' - \{u_i\}$ and so $x$ is adjacent to $u_i$ in $G$ but not to any vertex in $S - \{u_i\}$. Hence $B'_i \subseteq B_i$.

(vi) In $G$, $u_0$ is not adjacent to any vertex in $R \cup B'$ and thus $\langle R \cup B'\rangle$ is complete since $\beta(G) = 2$. For $i \in \{1,\ldots,k\}$, since $u_i$ is not adjacent in $G$ to any vertex in $(B \cup R) - B_i$, it follows that $\langle (B \cup R) - B_i\rangle$ is complete. Hence $G$ contains all edges between vertices in $B \cup R$, except possibly some edges between $B_i - B'_i$ and $B_j$, for some $i \neq j$. Suppose $k \geqslant 3$. The fact that $\langle (B \cup R) - B_i\rangle$ is complete for $i \in \{1, 2, 3\}$ then implies that $\langle B \cup R\rangle$ is complete. $\square$

Thus if $G$ with $\beta(G) = 2$ is $k$-$\mathrm{IR}^+$-edge-critical, then for any $e \in E_{\bar{G}}$, $G + e$ contains $K_{k+1} \times K_2$ as induced subgraph while $G$ does not. The characterisation below of $k$-$\mathrm{IR}^+$-edge-critical graphs with $k \geqslant \beta(G) = 2$ in terms of the structure of their complements follows directly from Proposition 6, and will be used in Section 5 to construct 2-$\mathrm{IR}^+$-edge-critical graphs. Denote the graph obtained from $K_{n,n}$ by removing the edges of a 1-factor by $L_{n,n}$; note that $G$ has an induced $K_n \times K_2$ if and only if $\bar{G}$ has an induced $L_{n,n}$.

**Corollary 8.** *The graph* $G$ *is* $k$-$\mathrm{IR}^+$-*edge-critical with* $\beta(G) = 2$ *if and only if* $\bar{G}$ *is triangle-free and has no induced* $L_{k+1,k+1}$, *but* $\bar{G} - e$ *has an induced* $L_{k+1,k+1}$ *for each* $e \in E_{\bar{G}} \neq \phi$.

## 5. 2-IR$^+$-edge-critical graphs

In this section, we exhibit an infinite class of graphs $G$ with $\mathrm{IR}(G) = 2$ and $\mathrm{IR}(G + e) = 3$ for each $e \in E_{\bar{G}}$. We use the structural characterisation of such graphs obtained from Corollary 8 by noting that $L_{3,3} \cong C_6$. (This result also follows easily from Proposition 2.)

**Corollary 9.** *The graph $G$ is 2-IR$^+$-edge-critical if and only if $\bar{G}$ is triangle-free and has no induced 6-cycles, but $\bar{G} - e$ has an induced 6-cycle for each $e \in E_{\bar{G}}$.*

**Theorem 10.** *The graph $\overline{C_m \times C_n}$ is 2-IR$^+$-edge-critical if and only if $m, n \notin \{3, 4, 6\}$.*

**Proof.** Let $V_{C_m \times C_n} = \{v_{i,j}\colon 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n\}$, where for each $j$, $\langle\{v_{i,j}\colon 1 \leqslant i \leqslant m\}\rangle \cong C_m$ and for each $i$, $\langle\{v_{i,j}\colon 1 \leqslant j \leqslant n\}\rangle \cong C_n$. The arithmetic in the first (second) subscript is modulo $m$ ($n$, respectively). We consider four types of paths (of length at least one) in $C_m \times C_n$:

$\overrightarrow{P}$ is a path of the form $v_{ij}v_{(i+1)j}\dots v_{kj}$,

$\overleftarrow{P}$ is a path of the form $v_{ij}v_{(i-1)j}\dots v_{kj}$,

$Q^{\uparrow}$ is a path of the form $v_{ij}v_{i(j+1)}\dots v_{il}$,

$Q^{\downarrow}$is a path of the form $v_{ij}v_{i(j-1)}\dots v_{il}$.

Paths of type $\overrightarrow{P}$ and $\overleftarrow{P}$ ($Q^{\uparrow}$ and $Q^{\downarrow}$) are collectively referred to as paths of type $P$ ($Q$, respectively).

Obviously $C_3 \times C_n$ has triangles, $C_6 \times C_n$ has induced 6-cycles and $v_{11}v_{12}v_{22}v_{32}v_{31}v_{41}$ is an induced 6-cycle of $C_4 \times C_n$. Thus by Corollary 9 the complements of these graphs are not 2-IR$^+$-edge-critical.

Now consider the triangle-free graphs $C_m \times C_n$, $m, n \notin \{3, 4, 6\}$. (These graphs have, amongst others, induced $m$-cycles, $n$-cycles and 4-cycles $v_{ij}v_{i(j+1)}v_{(i+1)(j+1)}v_{(i+1)j}$.) Suppose there is an induced 6-cycle $H$. Without loss of generality we may assume that $v_{11} \in V(H)$ and $H$ is of the form $\overrightarrow{P}Q^{\uparrow}PQ\dots$ (at least two $P$s and two $Q$s), where the first vertex of any path of the cycle is the same as the last vertex of the preceding path. Let $i'$ and $j'$ be the largest integers such that $v_{i'j}, v_{ij'} \in V(H)$ for some $i, j$. We may also assume that $i' \leqslant j'$. By the choice of $m$ and $n$ and since $H \cong C_6$, it follows that $2 \leqslant i' \leqslant j'$, and the cases $i' = j' = 2$ and $i' = j' = 3$ are both impossible. Therefore $i' = 2$, $j' = 3$ and $H$ has vertex sequence $v_{11}v_{21}v_{22}v_{23}v_{13}v_{12}$. But $v_{12}$ and $v_{22}$ are adjacent, a contradiction. Hence $C_m \times C_n$ has no induced $C_6$.

If $e \in E_{C_m \times C_n}$, then $e = v_{ij}v_{(i+1)j}$ or $e = v_{ij}v_{i(j+1)}$ for some $i, j$; without loss of generality say the former. Then $v_{i(j-1)}v_{(i+1)(j-1)}v_{(i+1)j}v_{(i+1)(j+1)}v_{i(j+1)}v_{ij}$ is an induced 6-cycle in $C_m \times C_n$. By Corollary 9, $\overline{C_m \times C_n}$, $m, n \notin \{3, 4, 6\}$, is 2-IR$^+$-edge-critical. $\square$

If $G$ and $H$ are $k$-IR$^+$-edge-critical, then so are $G + H$ (every vertex of $G$ joined to every vertex of $H$) and $G + K_n$. Since $\beta(G + H) = \max\{\beta(G), \beta(H)\}$, the graphs $(\overline{C_m \times C_n}) + (\overline{C_p \times C_q})$ and $(\overline{C_m \times C_n}) + K_s$, etc., for appropriate values of $m$, $n$, $p$ and $q$, are also 2-IR$^+$-edge-critical with $\beta = 2$. However, we do not know whether all 2-IR$^+$-edge-critical graphs are of this type; this will be an interesting but difficult problem to investigate.

The *circulant* $C_n\langle a_1, a_2, \dots, a_l\rangle$, $0 < a_1 < a_2 < \cdots < a_l < n$, is the graph with vertex set $V = \{0, 1, \dots, n-1\}$ and edge set $E = \{\{i, i+j\}\colon i \in V \text{ and } j \in \{a_1, a_2, \dots, a_l\}\}$ (arithmetic modulo $n$). Note that if $k$ is even, then $L_{k+1,k+1} \cong C_{2k+2}\langle 1, 3, \dots, k-1\rangle$

and if $k$ is odd, then $L_{k+1,k+1} \cong H$, where $H$ is the graph $C_{2k+2}\langle 1,3,\ldots,k-2\rangle$ together with every second edge of the (even) cycle $0,k,2k,\ldots,0$ (arithmetic modulo $2k+2$). Thus, to find $k$-IR$^+$-edge-critical graphs with $\beta = 2$ and $k \geqslant 3$ (see Corollary 8), it may help to look for graphs somehow constructed using circulants, perhaps of the type $C_m\langle 1,3,\ldots,k-1\rangle \times C_n\langle 1,3,\ldots,k-1\rangle$ for some $m$ and $n$, with additional edges added. Unfortunately all efforts so far have failed. We also do not have examples of $k$-IR$^+$-edge-critical graphs with $\beta > 2$.

The graphs $\overline{C_m \times C_n}$, $m,n \notin \{3,4,6\}$, are also relevant to the study of upper domination, and answer a question from [7]. Recall that $\Gamma(G)$ is the maximum cardinality amongst the minimal dominating sets of $G$ and that $\Gamma(G) \leqslant \mathrm{IR}(G)$ for all $G$. It is easy to see that any IR-set of $\overline{C_m \times C_n} + e$ is dominating, and so $\overline{C_m \times C_n}$ is also $\Gamma^+$-edge-critical. However, since the removal of a vertex cannot increase the irredundance number (any IR-set of $G - v$ is irredundant in $G$), the removal of a vertex does not increase $\Gamma$. Hence the classes of $\Gamma^+$-edge-critical graphs and $\Gamma^+$-critical graphs do not coincide.

We do not yet know whether there exist IR$^+$-edge-critical graphs that are not $\Gamma^+$-edge-critical. More specifically, do there exist IR$^+$-edge-critical graphs with $\beta = 2$ that are not $\Gamma^+$-edge-critical? The answer to this question would be negative if $\beta(G)=2$ implied that $\mathrm{IR}(G)=\Gamma(G)$. But even for $\beta(G)=2$, the difference $\mathrm{IR}(G)-\Gamma(G)$ can be arbitrary—consider the graph $G_k$ discussed in Section 2 after Proposition 2 and note that if $k < m$, then $\beta(G_k) = \Gamma(G_k) = 2$ and $\mathrm{IR}(G_k) = k$.

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