Uniform or Mean Convergence of Hermite–Fejér Interpolation of Higher Order for Freud Weights

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In this paper we show the uniform or mean convergence of Hermite–Fejér interpolation polynomials of higher order based at the zeros of orthonormal polynomials with the typical Freud weight.

1. INTRODUCTION

Let \( W^2(x) = \exp(-x^m) \), where \( m = 2, 4, 6, \ldots \), be the typical Freud weight, and let the orthonormal polynomials \( p_n(W^2; x) \) with the weight \( W^2(x) \) be defined by the relation

\[
\int_{-\infty}^{\infty} p_i(x) p_j(x) W^2(x) \, dx = \delta_{ij}, \quad i, j = 0, 1, 2, \ldots
\]

where \( p_n(x) = p_n(W^2; x) = \gamma_n x^n + \cdots \) with \( \gamma_n > 0 \). We denote the zeros of \( p_n(x) \) by \( x_{kn}, k = 1, 2, \ldots, n \), where \( x_{1n} > x_{2n} > \cdots > x_{mn} \). Let \( r \) be a positive integer. For an arbitrary real valued continuous function \( f \in C(R) \), the unique Hermite–Fejér interpolation polynomial \( L_r(v, f; x) \in \mathcal{P}_{mn-1} \) of order \( r \) based at \( \{x_{kn}\} \) is defined by

\[
L_{m}(v; f; x_{kn}) = f(x_{kn}), \quad k = 1, 2, \ldots, n,
\]

\[
L_{r}^{(n)}(v; f; x_{kn}) = 0, \quad k = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, v-1,
\]

where \( \mathcal{P}_{m} \) is the set of all algebraic polynomials of degree \( \leq n \). The interpolation polynomial \( L_r(v, f; x) \) is written in the form
\[ L_n(f; x) = \sum_{k=1}^{n} f(x_{kn}) h_{kn}(v; x), \quad n = 1, 2, 3, \ldots \]  

(1.3)

where the polynomial \( h_{kn}(v; x) \in \Pi_{n-1} \) satisfies

\[ h_{kn}(v; x_{kn}) = \delta_{pk}, \quad p = 1, 2, \ldots, n, \]

\[ h_{kn}^{(r)}(v; x_{kn}) = 0, \quad p = 1, 2, \ldots, n, \quad r = 1, 2, \ldots, v - 1. \]

(1.4)

An explicit form of \( h_{kn}(v; x) \) is

\[ h_{kn}(v; x) = \ell_{kn}(x) \sum_{j=0}^{v-1} e_{jk}(x - x_{kn}), \quad k = 1, 2, \ldots, n. \]

(1.5)

where \( \ell_{kn}(x) \) are the Lagrange fundamental polynomials of degree exactly \( n - 1 \), that is, with \( \delta_{nk}(x) = \Pi_{n-1}^{x_{kn}}(x - x_{kn}) \)

\[ \ell_{kn}(x) = \delta_{nk}(x) / \{ (x - x_{kn}) \delta'(x_{kn}) \}, \quad k = 1, 2, \ldots, n. \]

(1.6)

and the coefficients \( e_{jk} \) can be calculated by (1.4) (see [3]). We see that \( L_n(f; x) = L_n(1, f; x) \) is the Lagrange interpolation polynomial and \( L_n(2, f; x) \) is the ordinary Hermite–Fejér interpolation polynomial. In [3] we showed the following.

**Theorem 1** [3, Corollary 1]. Let \( f \in C(R) \) be a uniformly continuous function on \( R \). Then, for every \( M > 0 \), the sequence of Hermite–Fejér interpolation polynomials of even order \( v \) converges uniformly to \( f \) in the interval \([-M, M]\), that is,

\[ \lim_{n \to \infty} \max_{-M \leq x \leq M} |L_n(v, f; x) - f(x)| = 0. \]

**Theorem 2** [3, Corollary 2]. Let \( v \) be an odd integer. For \( a \) and \( b \) with \( a < b \), there exists \( f \in C(R) \) such that

\[ \lim_{n \to \infty} \max_{a \leq x \leq b} |L_n(v, f; x)| = \infty. \]

Recently, for the Lagrange interpolation polynomial \( L_n(f; x) \), Lubinsky and Matjila [6] obtained the following nice result. Let \( W_\beta^\nu(x) = \exp(-|x|^\beta) \), \( \beta > 1 \), be a Freud weight, and let \( L_n(f; x) \) be the Lagrange interpolation polynomial based at the zeros \( \{x_{kn}\} \) of the orthonormal polynomial \( p_n(W_\beta; x) \). Let \( 1 < p < \infty \). For \( x \in R \) we define

\[ \langle x \rangle = \min \{1, x \}, \]

(1.7)

**Theorem 3** [6, Theorem 1.1]. Let \( A \in R \) and \( x > 0 \). Then, for

\[ \lim_{n \to \infty} \| (1 + |x|)^{-d} W_\beta(x) \{ L_n(f; x) - f(x) \} \|_{L_p(R)} = 0, \]

where \( d \) is a positive integer.
to hold for every continuous function \( f: \mathbb{R} \to \mathbb{R} \) satisfying
\[
\lim_{|x| \to \infty} (1 + |x|)^s W_{\alpha}^1(x) |f(x)| = 0,
\]
it is necessary and sufficient that
\[
\begin{align*}
A &> 1/p - \alpha & \text{if } 1 < p \leq 4; \\
A &> 1/p - \alpha + (\beta/6)(1 - 4/p) & \text{if } p > 4 \text{ and } \alpha = 1; \\
A &> 1/p - \alpha + (\beta/6)(1 - 4/p) & \text{if } p > 4 \text{ and } \alpha \neq 1.
\end{align*}
\]

Our purpose of this paper is to extend Theorem 3 to the Hermite–Fejér interpolation polynomials \( L_n(v; f; x) \) based at the zeros \( \{x_{kn}\} \) of the orthonormal polynomial \( p_n(W^2; x) \) defined by (1.1). Let \( \alpha, A \in \mathbb{R} \), and let us define
\[
\langle x \rangle = \begin{cases} 
1, & x < 1, \\
x, & x \geq 1.
\end{cases}
\]

First, we prove a uniform convergence theorem.

**Theorem 4.** Let \( v = 2, 4, 6, \ldots \), and let \( \alpha + \langle \frac{vm}{6} \rangle \geq \frac{vm}{6} \geq 0 \). We fix an arbitrary constant \( 1 > \eta \geq 0 \). If
\[
A + \alpha + \langle \frac{vm}{6} \rangle \geq 0, \\
A + \langle \frac{vm}{6} \rangle \geq 0,
\]
then, for every continuous function \( f: \mathbb{R} \to \mathbb{R} \) satisfying
\[
\lim_{|x| \to \infty} (1 + |x|)^{s + m - \eta + \langle \frac{vm}{6} \rangle} W^\nu(x) |f(x)| = 0,
\]
we have
\[
\lim_{n \to \infty} \|(1 + |x|)^{-(A + \alpha + \langle \frac{vm}{6} \rangle)} W^\nu(x) \{L_n(v; f; x) - f(x)\} \|_{C(\mathbb{R})} = 0.
\]

**Remark.** Let \( v = 2, 4, 6, \ldots \), and fix an arbitrary constant \( 1 > \eta \geq 0 \). If \( m = v = 2 \) and \( \alpha + 1/3 \geq 0 \), then for \( A + \min\{2/3, \langle \alpha + 1/3 \rangle \} \geq 0 \) we have for every continuous function \( f: \mathbb{R} \to \mathbb{R} \) satisfying
\[
\lim_{|x| \to \infty} (1 + |x|)^{s + 3 - \eta} W^\nu(x) |f(x)| = 0,
\]
the estimation
\[
\lim_{n \to \infty} \|(1 + |x|)^{-(A + 2)} W^\nu(x) \{L_n(v; f; x) - f(x)\} \|_{C(\mathbb{R})} = 0.
\]
If \(mv \neq 4\) and \(x \geq 0\), then for \(A + \langle x \rangle \geq 0\) we have for every continuous function \(f: \mathbb{R} \to \mathbb{R}\) satisfying
\[
\lim_{|x| \to \infty} (1 + |x|)^{\frac{m}{m+1} + (1 + \nu/8)m - \eta} W^m(x) |f(x)| = 0,
\]
the estimation
\[
\lim_{n \to \infty} ||(1 + |x|)^{-(d + \nu m/6)} W^m(x) \{L_n(v, f; x) - f(x)\}||_{C^1(\mathbb{R})} = 0.
\]

The following are the analogues of Theorem 3.

**Theorem 5.** Let \(v = 2, 3, 4, \ldots, 1 < p < \infty,\) and \(\alpha > 0\). Assume that

\[
A > 1/p \quad \text{if} \quad 1 < p \leq 4/v \quad (v < 4); \tag{1.11}
\]

\[
A > 1/p \quad \text{if} \quad (m/6)(v - 4/p) \leq \langle x \rangle, \quad p > 4/v; \tag{1.12}
\]

\[
A \geq 1/p - \langle x \rangle + (m/6)(v - 4/p)
\]

\[
\text{if} \quad (m/6)(v - 4/p) > \langle x \rangle, \quad p > 4/v. \tag{1.13}
\]

(Here, if \(4 \leq v\) we omit (1.11), and we set \(p > 1\) for (1.12) or (1.13).) Then, for every continuous function \(f: \mathbb{R} \to \mathbb{R}\) satisfying
\[
\lim_{|x| \to \infty} (1 + |x|)^{\frac{m}{m+1} + (1 + \nu m/8) m - \eta} W^m(x) |f(x)| = 0, \tag{1.14}
\]
we have
\[
\lim_{n \to \infty} ||(1 + |x|)^{-d} W^m(x) \{L_n(v, f; x) - f(x)\}||_{L^p(\mathbb{R})} = 0. \tag{1.15}
\]

**Theorem 6.** Let \(v = 3, 5, 7, \ldots, 1 < p < \infty,\) and \(\alpha > 0\). Assume that for every continuous function \(f: \mathbb{R} \to \mathbb{R}\) satisfying (1.14) we have (1.15). Then, the following inequalities hold.

\[
A > 1/p - \langle x + (v - 1)m/6 \rangle
\]

\[
\text{if} \quad 1 < p \leq 4/v \quad (v < 4); \tag{1.16}
\]

\[
A > 1/p - \langle x + (v - 1)m/6 \rangle + (m/6)(v - 4/p)
\]

\[
\text{if} \quad p > 4/v \quad \text{and} \quad x + (v - 1)m/6 = 1; \tag{1.17}
\]

\[
A \geq 1/p - \langle x + (v - 1)m/6 \rangle + (m/6)(v - 4/p)
\]

\[
\text{if} \quad p > 4/v \quad \text{and} \quad x + (v - 1)m/6 \neq 1. \tag{1.18}
\]
If we consider the case of $v = 3, 5, 7, \ldots$, then we have the following

**Corollary 7.** Let $v = 3, 5, 7, \ldots$, and let $(m/6)(v - 4/p) > 1$. Then, for (1.15) to hold for every continuous function $f: \mathbb{R} \to \mathbb{R}$ satisfying (1.14), it is necessary and sufficient that

$$
A \geq 1/p - 1 + (m/6)(v - 4/p).
$$

If $v = 3$, then we suppose $p > 4/3$.

2. PRELIMINARIES

The Hermite–Fejér interpolation polynomial $L_n(v; f; x)$ is defined by (1.2) and (1.3). The Lagrange fundamental polynomials $\ell_{kn}(x)$, $k = 1, 2, \ldots, n$, of degree exactly $n - 1$ are defined by (1.6), and the fundamental polynomials $h_{kn}(v; x)$, $k = 1, 2, \ldots, n$, of $L_n(v; f; x)$ are defined by (1.5) with (1.4). For $m > 0$, the $m$th Mhaskar–Rahmanov–Saff number $a_m(w)$ is the positive root of the equation

$$
u = (m/\pi)(a_u)^{m} \int_{-1}^{1} t^{m}(1 - t^{2})^{-1/2} dt$$

$$= (m/2)!((m - 1)!!/m!!)(a_u)^{m}. \quad (2.1)$$

Let $\gamma_1$ be the leading coefficient of $p_d(x) = \gamma_n x^n + \cdots$, and we set $b_n = \gamma_{n-1}/\gamma_n$. Furthermore, we also use the number $q_n = (2n/m)^1m$. Then, we see that

$$x_{1n} \sim a_n \sim b_n \sim q_n \sim n^{1/m} \quad (2.2)$$

as $n \to \infty$ (see (2.1), (2.3), and [5, (12.26)]), where for the positive functions $b(u)$ and $c(u)$, $b(u) \sim c(u)$ remarks that there exist $C_1, C_2 > 0$ independent of $u$ such that $C_1 \leq b(u)/c(u) \leq C_2$.

We need some fundamental lemmas. Let $C$ be a positive constant independent of $k$ and $n$. First, we denote the useful lemmas from [6].

**Lemma 2.1** [6, Theorem 2.1]. (a) For $n \geq 1$,

$$\|(x_{kn}/a_n) - 1\| \leq Cn^{-2/3}, \quad (2.3)$$

and uniformly for $n \geq 3$ and $2 \leq k \leq n - 1$,

$$x_{k-1,n} - x_{k+1,n} \sim (a_n/n)(\max\{n^{-2/3}, 1 - |x_{kn}/a_n|\})^{-1/2}. \quad (2.4)$$
(b) For $n \geq 1$,
\[ \sup_{x \in \mathbb{R}} |1 - |x||a_n|^{1/4} W(x) |p_n(x)| \sim a_n^{-1/2}. \tag{2.5} \]

and
\[ \sup_{x \in \mathbb{R}} W(x) |p_n(x)| \sim n^{1/6} a_n^{-1/2}. \tag{2.6} \]

(c) Uniformly for $n \geq 1$ and $1 \leq k \leq n$,
\[ W(x_{kn}) |p_n(x_{kn})| \sim na_n^{-3/2}(\max\{n^{-2/3}, 1 - |x_{kn}|/a_n\}^{1/4} \quad \text{(by [5, (1.19)])}. \tag{2.7} \]

(d) Let $0 < p \leq \infty$. There exists $C > 0$ such that for $n \geq 1$ and $P \in \Pi_n$,
\[ \|WP\|_{L_p(R)} \leq C \|WP\|_{L_p([-a_n, a_n])}. \tag{2.8} \]

**Lemma 2.2 [6, Theorem 2.2].**

(a) Given $0 < p < \infty$, we have for $n \geq 1$,
\[ \|WP\|_{L_p(R)} \sim \begin{cases} 1, & 0 < p < 4 \\ a_n^{1/p - 1/2} \times \{\log(1 + n)\}^{1/4}, & p = 4 \\ n^{1/8}(1 - 4/p), & p > 4. \end{cases} \tag{2.9} \]

(b) Uniformly for $n \geq 1$, $1 \leq k \leq n$, and $x \in \mathbb{R}$,
\[ |a_n(x)| \sim (a_n^{3/2}/n) W(x_{kn})(\max\{n^{-2/3}, 1 - |x_{kn}|/a_n\}^{1/4} \times |p_n(x)/(x - x_{kn})| \quad \text{(by (1.6), (2.7))}. \tag{2.10} \]

(c) Uniformly for $n \geq 1$, $1 \leq k \leq n$, and $x \in \mathbb{R}$,
\[ |W^{-1}(x_{kn}) W(x) / x_{kn}(x)| \leq C. \tag{2.11} \]

**Lemma 2.3 [3, Lemma 6, Lemma 14 (4.16)].** Let $e_{jk}$ be the coefficient of (1.5). Then, by (2.2) we have
\[ |e_{jk}| \leq C(n/a_n)^j, \quad j = 0, 1, ..., v - 1, \quad k = 1, 2, ..., n, \tag{2.12} \]

especially, for odd number $j$
\[ |e_{jk}| \leq CM_n(x_{kn})(n/a_n)^j, \quad k = 1, 2, ..., n, \tag{2.13} \]

where
\[ M_n(x_{kn}) = a_n^{-2} |x_{kn}| + |x_{kn}|^{m-1}, \quad k = 1, 2, ..., n. \tag{2.14} \]
Furthermore, we need a certain generalized Hermite-Fejér interpolation polynomial of $(\ell, v)$-order, $\ell = 0, 1, \ldots, v - 1$ (cf. [4]). For $f \in C^\ell(R)$ we define $L_\ell(\ell, v; f; x) \in \Pi_{m-1}$ by

$$L_\ell(\ell, v; f; x) = f^\ell(x_{kn}), \quad r = 0, 1, \ldots, \ell,$$

$$L_n(\ell, v; f; x) = 0, \quad r = \ell + 1, \ell + 2, \ldots, v - 1, \quad k = 1, 2, \ldots, n.$$ 

The polynomial $L_\ell(\ell, v; f; x)$ is written in the form

$$L_n(\ell, v; f; x) = \sum_{k=1}^{n} \sum_{s=0}^{\ell} f^{(s)}(x_{kn}) h_{skn}(v; x), \quad n = 1, 2, 3, \ldots,$$

where for the polynomial $h_{skn}(v; x) \in \Pi_{m-1}$

$$h_{skn}(v; x) = \delta_{js} \delta_{pk}, \quad s = 0, 1, \ldots, \ell, \quad j = s, s + 1, \ldots, v - 1, \quad p, k = 1, 2, \ldots, n. \quad (2.15)$$

An explicit form of $h_{skn}(v; x)$ is

$$h_{skn}(v; x) = \ell_{kn}^* (x) \sum_{j=s}^{\ell-1} e_{pk}(x - x_{kn})^j, \quad k = 1, 2, \ldots, n. \quad (2.16)$$

where the $\ell_{kn}(x)$ are the Lagrange fundamental polynomials.

We see that $L_\ell(0, v; f; x)$ is the Hermite-Fejér interpolation polynomial $L_n(v; f; x)$ of order $v$, and $L_n(v - 1, v; f; x)$ preserves any polynomial $P \in \Pi_{m-1}$, that is,

$$L_n(v - 1, v; f; x) = P(x), \quad x \in R. \quad (2.17)$$

From (2.15) we obtain the following.

**Lemma 2.4** [4, Lemma 3]. *For the coefficients $e_{pk}$ we have*

$$|e_{pk}| \leq C(n/\alpha_p)^{r-s}, \quad s = 0, 1, \ldots, \ell, \quad j = s, s + 1, \ldots, v - 1, \quad k = 1, 2, \ldots, n. \quad (2.18)$$

From (2.4) there exists a positive constant $\delta$ such that

$$\delta \alpha_n/n \leq x_{j-1,n} - x_{j+1,n}, \quad j = 1, 2, \ldots, n, \quad (2.19)$$

where

$$x_{1n} = x_{1n}(1 + n^{-2/3}), \quad x_{n+1,n} = x_{nn}(1 - n^{-2/3}) \quad (\text{cf. (2.3)}).$$
Therefore, if we set
\[ x_{kn} - x = t(k, x) \delta a_n / n, \quad k = 0, 1, \ldots, n + 1, \] (2.20)
then we see that
\[ t(n + 1, x) < t(n, x) < \cdots < t(1, x) < t(0, x), \]
and
\[ t(j - 1, x) - t(j + 1, x) \geq 1, \quad j = 1, 2, \ldots, n. \]

3. PROOF OF THEOREM 4

Throughout this section we assume that \( \pi + (vm/6) - vm/6 \geq 0, \lambda + vm/6 \geq 0, \) and \( D + (\pi + (vm/6) - vm/6) \geq 0, \) where \( \langle \cdot \rangle \) and \( \langle \cdot \rangle \) are defined by (1.7) and (1.8), respectively.

**Lemma 3.1.** Let \( \nu = 2, 4, 6, \ldots, \) and \( \varepsilon > 0, 1 > \eta \geq 0. \) If \( g \in C(R) \) satisfies
\[ (1 + |x|)^{\pi + m - \eta + (vm/6)} W^\nu(x) |g(x)| < \varepsilon, \quad x \in R, \] (3.1)
then we have
\[ \sum (x) = (1 + |x|)^{- (\lambda + vm/6)} W^\nu(x) \sum_{k=1}^{n} |g(x_{kn}) h_{kn}(x)| < C \varepsilon, \quad x \in R, \] (3.2)
where \( C \) is a positive constant independent of \( n \) and \( \varepsilon. \)

**Lemma 3.2.** Let \( \nu = 1, 2, 3, \ldots, \) and \( 1 > \eta \geq 0. \) If \( g \in C(R) \) satisfies that for a positive constant \( M(g), \)
\[ (1 + |x|)^{\pi + m - \eta + (vm/6)} W^\nu(x) |g(x)| < M(g), \quad x \in R, \]
where \( M(g) \) may depend on \( g, \) then, for every \( x \in R \) we have
\[ \sum (x) = (1 + |x|)^{- (\lambda + vm/6)} W^\nu(x) \sum_{k=1}^{n} |g(x_{kn}) h_{kn}^*(x)| < CM(g) \log(1 + n), \]
where
\[ h_{kn}^*(x) = |l_{kn}^*(x)| \sum_{j=0}^{n-1} (n/a_n)^j |x - x_{kn}|, \] (3.3)
and \( C \) is a positive constant independent of \( n \) and \( M(g). \)
Throughout the paper, the letter \( C \) denotes a positive constant which may differ at each different occurrence, even in the same chain of inequalities. Let \( \delta \) be the positive constant which is defined by (2.19). We often use the expression (2.20).

**Proof of Lemma 3.1.** (i) \( \) Let \( K = \{ k; |x - x_{kn}| < \delta a_n/n \} \). Then, the number of \( K \) is at most four. By (2.11)

\[
W^{-1}(x_{kn}) |W(x) f_{kn}(x)| \leq C, \quad x \in R,
\]

therefore, using (3.1) and (2.12)

\[
\sum_{k \in K} (1 + |x|)^{-(d + \langle \log |x| \rangle / (2a_n/n)^{1/4}} \sum_{k \in K} |W^{-1}(x_{kn}) W(x) f_{kn}(x)|^r |W^n(x_{kn}) g(x_{kn})|
\]

\[
\times \sum_{j=0}^{r-1} (a_n/n)^j (a_n/n)^j
\]

\[
\leq C \sum_{k \in K} (1 + |x_{kn}|)^{-(d + \langle \log |x| \rangle / (2a_n/n)^{1/4}} \quad \text{(by } |x| \sim |x_{kn}|)\]

\[
\leq C \sum_{k \in K} (1 + |x_{kn}|)^{-(d + \langle \log |x| \rangle / (2a_n/n)^{1/4}} \quad \text{(byCEL)}
\]

Consequently, we assume that \( |x - x_{kn}| \geq \delta a_n/n \) below. Using (2.10) (or (2.7)) we rewrite \( \sum(x) \) of (3.2) as

\[
\sum_{k=1}^{n} |W(x) p_n(x)/(W(x_{kn}) p_n(x_{kn}))|^r
\]

\[
\times |W^n(x_{kn}) g(x_{kn})| \sum_{j=0}^{r-1} |e^{j}(x_{kn})|^{-r}
\]

\[
\leq C(1 + |x|)^{-(d + \langle \log |x| \rangle / (2a_n/n)^{1/4}}
\]

\[
\times \sum_{k=1}^{n} |a_n|^{1/2} W(x) p_n(x)/(\max(n^{-2/3}, 1 - |x_{kn}|/a_n))^{1/4})|^r
\]

\[
\times (1 + |x_{kn}|)^{(\langle \log |x| \rangle / (2a_n/n)^{1/4}} \sum_{j=0}^{r-1} |e^{j}(x_{kn})|^{-r}.
\]
therefore, by (2.12), (2.13), and (2.14)

\[
\sum (x) \leq C \varepsilon (1 + |x|)^{-(d + \eta \varepsilon /2)} \\
\times \sum_{k=1}^{n} \left[ (x - x_{kn})^{-2} + |x - x_{kn}|^{-1} \left\{ a_{\eta}^{-2} |x_{kn}| + |x_{kn}|^{m-1} \right\} (a_{\eta}/n)^2 \\
\times \left( 1 + |x_{kn}| \right)^{-\left( m - \eta \right)} \left( 1 + |x_{kn}| \right)^{-\left( \eta + \left( \eta \varepsilon /4 \right) \right)} \\
\times |a_{\eta}^{1/2} W(x) p_{\eta}(x)| \left[ \max(n^{-2/3}, 1 - |x_{kn}|/(a_{\eta})) \right]^{1/4}.
\] \tag{3.4}

Let \( 0 < \beta < 1 \). We use (2.5) and (2.6).

(ii) We consider the sum \( \sum^2 (x) \) for the case of \( |x_{kn}| \leq \beta a_{n}, \ |x| \leq \beta a_{n} \).
By (3.4),

\[
\sum^2 (x) \leq C \varepsilon \sum_{k \neq n/2}^{2} \left[ t(k, x)^{-2} + |t(k, x)|^{-1} |n/2 - k|^{-1} \right] \\
\times \left( 1 + |x_{kn}| \right)^{-\left( \eta + \left( \eta \varepsilon /4 \right) \right)} \left( 1 + |x| \right)^{-\left( d + \eta \varepsilon /2 \right)} \\
\leq C \varepsilon.
\]

(iii) We consider the sum \( \sum^3 (x) \) for the case of \( |x_{kn}| \geq \beta a_{n}/2, \ |x| \geq \beta a_{n}/2 \). Let \( |x| \leq 2a_{n} \), then we see that \( |x| \sim |x_{kn}| \sim a_{n} \). By (3.4),

\[
\sum^3 (x) \leq C \varepsilon \sum_{k \neq n/2}^{3} \left[ t(k, x)^{-2} + |t(k, x)|^{-1} |n/2 - k|^{-1} \right] \\
\times \left( 1 + |x_{kn}| \right)^{-\left( d + \eta \varepsilon /4 - \eta \varepsilon /2 \right)} \\
\leq C \varepsilon a_{n}^{-\left( d + \eta \varepsilon /4 - \eta \varepsilon /2 \right)} \\
\leq C \varepsilon.
\]

If \( 2a_{n} \leq |x| \), then by (2.5) we see that \( |a_{n}^{1/2} W(x) p_{n}(x)| \leq C \). Therefore by (3.4),

\[
\sum^3 (x) \leq C \varepsilon \sum_{k \neq n/2}^{3} \left[ t(k, x)^{-2} + |t(k, x)|^{-1} |n/2 - k|^{-1} \right] \\
\times \left( 1 + |x_{kn}| \right)^{-\left( \eta + \left( \eta \varepsilon /4 \right) - \eta \varepsilon /2 \right)} \left( 1 + |x| \right)^{-\left( d + \eta \varepsilon /2 \right)} \\
\leq C \varepsilon.
\]
(iv) Let $|x_{kn}| \leq \beta a_n/2$, $\beta a_n \leq |x| \leq 2a_n$, and let us denote the sum with respect to these $x_{kn}$ and $x$ by $\sum^4 (x)$. By (3.4),

$$\sum^4 (x) \leq C\varepsilon (1 + |x|)^{-\delta} \sum \left[ a_n^{-2} + a_n^{-1} \right] (a_n/n)^2 (1 + |x_{kn}|)^{-(\alpha + 1 - \eta + \langle vm/6 \rangle)}$$

$$\leq C\varepsilon (1 + |x|)^{-\delta} \left( 1 + t \right)^{-(\alpha + 1 - \eta + \langle vm/6 \rangle)} dt \quad \text{(by (2.4))}$$

$$\leq C\varepsilon (1 + |x|)^{-\delta} a_n^{-m} \times \begin{cases} 1, & \alpha - \eta + \langle vm/6 \rangle > 0, \\ \log(1 + n), & \alpha - \eta + \langle vm/6 \rangle = 0, \\ a_n^{-(\alpha - \eta + \langle vm/6 \rangle)}, & \alpha - \eta + \langle vm/6 \rangle < 0. \end{cases}$$

If $A \geq 0$, then by $x + \langle vm/6 \rangle - \eta + m > 0$ we see that

$$\sum^4 (x) \leq C\varepsilon.$$

If $A < 0$, then we see that

$$\sum^4 (x) \leq C\varepsilon a_n^{-\delta + (\alpha - \eta + \langle vm/6 \rangle)} \times \begin{cases} 1, & \alpha - \eta + \langle vm/6 \rangle > 0, \\ \log(1 + n), & \alpha - \eta + \langle vm/6 \rangle = 0, \\ a_n^{-(\alpha - \eta + \langle vm/6 \rangle)}, & \alpha - \eta + \langle vm/6 \rangle < 0. \end{cases}$$

Since

$$A + m > A + 1 \geq 0,$$

$$A + m + \alpha - \eta + \langle vm/6 \rangle \geq A + \langle x + \langle vm/6 \rangle - vm/6 \rangle + vm/6 - \eta + m > 0,$$

we have

$$\sum^4 (x) \leq C\varepsilon.$$

(v) Let $|x_{kn}| \leq \beta a_n/2$, $2a_n \leq |x|$, and let us denote the sum with respect to these $x_{kn}$ and $x$ by $\sum^5 (x)$. By (3.4)

$$\sum^5 (x) \leq C\varepsilon (1 + |x|)^{-(\delta + \langle vm/6 \rangle)}$$

$$\times \sum \left[ a_n^{-2} + a_n^{-1} \right] (a_n/n)^2 (1 + |x_{kn}|)^{-(\alpha + 1 - \eta + \langle vm/6 \rangle)}$$

$$\leq C\varepsilon (1/n) \int_0^{\beta a_n} (1 + t)^{-(\alpha + 1 - \eta + \langle vm/6 \rangle)} dt \quad \text{(by $A + \langle vm/6 \rangle \geq 0$)}$$

$$\leq C\varepsilon a_n^{-m} \times \begin{cases} 1, & \alpha - \eta + \langle vm/6 \rangle > 0, \\ \log(1 + n), & \alpha - \eta + \langle vm/6 \rangle = 0, \\ a_n^{-(\alpha - \eta + \langle vm/6 \rangle)}, & \alpha - \eta + \langle vm/6 \rangle < 0. \end{cases}$$
Since \( x + \langle \nu m/6 \rangle - \eta + m > 0 \) we see that
\[
\sum_{k=1}^{6} \phi_k(x) \leq C e.
\]

(vi) Let \(|x| \leq \beta \alpha_n/2\), \( \beta \alpha_n \leq |x_{kn}| \), and let us denote the sum with respect to these \( x_{kn} \) and \( x \) by \( \sum_{n}^{6} \phi_k(x) \). By (3.4),
\[
\sum_{n}^{6} \phi_k(x) \leq C e (1 + |x|)^{-(d + \nu m/6)} \sum_{n}^{6} \phi_k(x_{kn}) \left(a_n^{-2} + a_n^{-1}\right) |(a_n/n)^2
\]
\[
\times (1 + |x_{kn}|)^{-\langle x + \nu m/6 \rangle - \eta + \nu m/6)}
\]
\[
\leq C e (1 + |x|)^{-(d + \nu m/6)} \leq \int_{0}^{A} (1 + t)^{-\langle x + \nu m/6 \rangle - \eta + \nu m/6)} d t
\]
\[
\leq C e a_n^{-m} \times \log(1 + n), \quad \frac{x - \eta + \nu m/6 + \nu m/6}{-\eta + \nu m/6 - \nu m/6} > 0,
\]
\[
\frac{x - \eta + \nu m/6 + \nu m/6}{-\eta + \nu m/6 - \nu m/6} < 0.
\]

Since \( x + \langle \nu m/6 \rangle - \nu m/6 - \eta + m > 0 \) we see that
\[
\sum_{n}^{6} \phi_k(x) \leq C e.
\]

Proof of Lemma 3.2. For odd number \( \nu \) we can use neither (2.13) nor (2.14). However, if we repeat the same method as the proof of Lemma 3.1, then by (2.12) we obtain the upper bound
\[
\sum_{n}^{6} \phi_k(x) = (1 + |x|)^{-(d + \nu m/6)} W^\nu(x) \sum_{n}^{6} \phi_k(x_{kn}) h_{kn}(x) < C M(g) \log(1 + n).
\]

Proof of Theorem 4. Let the assumptions of Theorem 4 be satisfied. By (1.9), there exists a polynomial \( P_\varepsilon(x) \) such that
\[
(1 + |x|)^{x + \nu m/6 + \nu m/6} W^\nu(x) |f(x) - P_\varepsilon(x)| < \varepsilon, \quad x \in R \quad (3.5)
\]
(cf. [2, p. 180]). By (2.17), for \( n \) large enough we have
\[
L_\varepsilon(v - 1, v, P_\varepsilon; x) = P_\varepsilon(x), \quad x \in R.
\]

By \( h_{kn}(v; x) = h_{kn}(v; x) \),
\[
(1 + |x|)^{-(d + \nu m/6)} W^\nu(x) \left[L_\varepsilon(v, f; x) - f(x)\right]
\]
\[
= (1 + |x|)^{-(d + \nu m/6)} W^\nu(x) \left[L_\varepsilon(v, f - P_\varepsilon; x) + P_\varepsilon(x) - f(x)\right]
\]
\[
+ \sum_{k=1}^{n} \phi_k(x_{kn}) h_{kn}(x).\]
By Lemma 3.1 and (3.5), it is easy to see

\[(1 + |x|)^{-(d + \eta + m/6)} W^\prime(x) \| [L_n(x, f - P_s; x)] + |P_s(x) - f(x)| \| \leq C_n.\]

Therefore, it is enough to show that

\[
\lim_{n \to \infty} \left(1 + |x|\right)^{-(d + \eta + m/6)} W^\prime(x) \sum_{k=1}^{n} \sum_{s=1}^{\ell} P_s^{(s)}(x_kn) h_{sk\theta}(x) = 0. \tag{3.6}
\]

By (2.16) and (2.18),

\[
(1 + |x|)^{-(d + \eta + m/6)} W^\prime(x) \left| \sum_{k=1}^{n} \sum_{s=1}^{\ell} P_s^{(s)}(x_kn) h_{sk\theta}(x) \right|
\leq C \left(1 + |x|\right)^{-(d + \eta + m/6)} W^\prime(x) \left| \sum_{k=1}^{n} \sum_{s=1}^{\ell} |P_s^{(s)}(x_kn) h_{sk\theta}(x)| \right|
\times \sum_{j=0}^{v-1} (n/a_n)^{j} |x - x_{kn}|^j
\leq C \left(1 + |x|\right)^{-(d + \eta + m/6)} W^\prime(x) \left| \sum_{k=1}^{n} \sum_{s=1}^{\ell} |P_s^{(s)}(x_kn) h_{sk\theta}(x)| \right|
\times \sum_{j=0}^{v-1} (n/a_n)^{j} |x - x_{kn}|^j
\leq C(a_n/n) \sum_{x=1}^{\ell} \left| x \right|, \quad \text{say.} \tag{3.7}
\]

where

\[
\sum_{x=1}^{\ell} \left| x \right| = \sum_{s=1}^{\ell} \left(1 + |x|\right)^{-(d + \eta + m/6)} W^\prime(x) \sum_{k=1}^{n} |P_s^{(s)}(x_kn) h_{sk\theta}(x)|
\times \sum_{j=0}^{v-1} (n/a_n)^{j} |x - x_{kn}|^j.
\]

Now, since \(P_s(x)\) is a polynomial defined by \(f\) and \(\varepsilon\), we have

\[
(1 + |x|)^{s + \eta + m/6} W^\prime(x) |P_s^{(s)}(x)|
\leq C(s, \varepsilon, f), \quad x \in R, \quad s = 1, 2, ..., \ell,
\]

where \(C(s, \varepsilon, f)\) is a positive constant independent of \(n\). Therefore, by Lemma 3.2 we have

\[
\sum_{x=1}^{\ell} \left| x \right| \leq C(s, \varepsilon, f) \log(1 + n), \tag{3.8}
\]
where \( C(s, e, f) \) is independent of \( n \), and may depend on \( s, e \), and \( f \). Consequently, by (3.7) and (3.8) we obtain (3.6), therefore (1.10) was shown.

4. PROOF OF THEOREM 5

In the rest of the paper we investigate the mean convergence of the Hermite–Fejér interpolation polynomial \( L_n(v; x) \). Since for the Lagrange case we have Theorem 3, the order \( v \) is assumed \( v = 2, 3, 4, \ldots \). In this section we obtain a direct theorem, then the following are assumed. Let \( 1 < p < \infty, x > 0, A \in \mathbb{R} \), and let the conditions (1.11) or (1.12) or (1.13) be satisfied. A real valued continuous function \( f \in C(\mathbb{R}) \) satisfies (1.14).

Lemma 4.1 [6, Lemma 2.7]. Let \( 0 < \beta < 2 \), then, for \( x \in \mathbb{R} \)
\[
W(x) \sum_{|x_k| > \beta a_n} (1 + |x_k|)^{-\alpha} W^{-1}(x_k) / A_k(x) \leq C a_n^{-\alpha} \times \begin{cases} 1, & |x| \leq \beta a_n/2, \\ \left( a_n^{1/2} W(x) p_n(x) \right)^{v^{-1} + 1} + \log(1 + n), & \beta a_n/2 < |x| \leq 2a_n, \\ a_n/|x|, & 2a_n < |x|. \end{cases}
\]

Let us define
\[
\tilde{h}_{kn}(x) = \sum_{j=0}^{v-1} |e_j(x - x_k)|. \tag{4.1}
\]

Lemma 4.2. Let \( 0 < \beta < 2 \). Then, for \( x \in \mathbb{R} \),
\[
\sum (x) = W(x) \sum_{|x_k| > \beta a_n} (1 + |x_k|)^{-\alpha} W^{-1}(x_k) / A_k(x), \tag{4.3}
\]

Proof. First, we set
\[
\sum (x) = \left\{ W(x) \sum_{|x_k| > \beta a_n} (1 + |x_k|)^{-\alpha} W^{-1}(x_k) / A_k(x) \right\} A_4(x). \tag{4.4}
\]
where
\[ A_k(x) = |W^{-1}(x_{kn}) W(x) f_{kn}(x)|^{r-1} \sum_{j=0}^{r-1} |a_j(x_{kn} - x_k)| (1 + |x_k|)^{-(r-1)m/6}. \]

Then, we show that
\[ A_k(x) = C \times \begin{cases} 1, & |x| \leq \beta a_n/2 \quad \text{or} \quad 2a_n < |x|, \\ (|a_n^{1/2} W(x) p_n(x)|^{r-1} + 1), & \beta a_n/2 < |x| \leq 2a_n. \end{cases} \]

We note (2.12). For \(|x - x_{kn}| < \delta a_n/n\), we use (2.11).
\[ A_k(x) \leq C \left| W^{-1}(x_{kn}) W(x) f_{kn}(x) \right|^{r-1} \]
\[ \times (1 + |x_{kn}|)^{-(r-1)m/6} \sum_{j=0}^{r-1} (n/a_n)^j |x - x_{kn}|^j \]
\[ \leq C (1 + |x_{kn}|)^{-(r-1)m/6} \sum_{j=0}^{r-1} (n/a_n)^j/2 \leq C. \] (4.5)

Let \(|x| \leq \beta a_n/2 \) or \(2a_n < |x|\), and \(|x - x_{kn}| \geq \delta a_n/n\). Then, by (2.5) and (2.7),
\[ A_k(x) \leq C \left| a_n^{1/2} W(x) p_n(x) / [(x - x_{kn})/a_n]^{1/4} \right|^{r-1} (1 + |x_{kn}|)^{-(r-1)m/6} \]
\[ \times \left[ \max(n^{-2/3}, 1 - |x_{kn}/a_n|) \right]^{1/4} \sum_{j=0}^{r-1} (n/a_n)^j/2 \]
\[ \leq C \left[ \max(n^{-2/3}, 1 - |x_{kn}/a_n|) \right]^{-(r-1)/4} (1 + |x_{kn}|)^{-(r-1)m/6} \]
\[ \leq C. \] (4.6)

If \(\beta a_n/2 < |x| \leq 2a_n\) and \(|x - x_{kn}| \geq \delta a_n/n\), then we have
\[ A_k(x) \leq C \left| a_n^{1/2} W(x) p_n(x) / [(x - x_{kn})/a_n]^{1/4} \right|^{r-1} \]
\[ \times \left[ \max(n^{-2/3}, 1 - |x_{kn}/a_n|) \right]^{1/4} \sum_{j=0}^{r-1} (n/a_n)^j/2 \]
\[ \leq C \left| a_n^{1/2} W(x) p_n(x) \right|^{r-1} \left[ \max(n^{-2/3}, 1 - |x_{kn}/a_n|) \right]^{-(r-1)/4} \]
\[ \times (1 + |x_{kn}|)^{-(r-1)m/6} \]
\[ \leq C \left| a_n^{1/2} W(x) p_n(x) \right|^{r-1}. \] (4.7)

Therefore, by (4.5), (4.6), and (4.7) we obtain (4.4), consequently (4.3).
Applying Lemma 4.1 to (4.3) we obtain (4.2).

**Lemma 4.3** (cf. [6, Lemma 3.1]). We set $0 < \beta < 2$, and we let $n = 1, 2, 3, \ldots$. If $f_a(x) = 0$ for $|x| < \beta a_n$, furthermore,
\[
|W'(x)f_a(x)| \leq n(1 + |x|)^{-(p - 1)m/6}, \quad x \in \mathbb{R},
\]
then we have
\[
\limsup_{n \to \infty} \|(1 + |x|)^{-d} W'(x) L_n(v, f_a; x)\|_{L_p(\mathbb{R})} \leq C e. \tag{4.8}
\]

**Proof.** By Lemma 4.2
\[
|W'(x)L_n(v, f_a; x)|
\begin{align*}
\leq C n a_n^{-8} & \times \sum_{|x_k| \geq \beta a_n} (1 + |x_k|)^{-(p - 1)m/6} W^*(x_k) h_n(v; x) \\
& \leq C n a_n^{-8} \times \begin{cases} 1, & |x| \leq \beta a_n/2, \\
\left( |a_k|^{1/2} W(x) p_n(x) \right)^{r - 1} + 1 \left( |a_k|^{1/2} W(x) p_n(x) \right) + \log(1 + n) \right), & \beta a_n/2 < |x| \leq 2a_n, \\
\left( a_n/|x| \right), & 2a_n < |x|. \end{cases} \tag{4.9}
\end{align*}
\]
We repeat the same method as the proof of [6, Lemma 3.1] below. From (4.9),
\[
\tau_n^{(1)} = \|(1 + |x|)^{-d} W'(x) L_n(v, f_a; x)\|_{L_p(\mathbb{R})} \leq \begin{cases} 1, & Ap > 1, \\
\left( \log(1 + n) \right)^{1/p}, & Ap = 1 \\
\left( a_n^{-p} \right)^{1/p - d}, & Ap < 1. \end{cases}
\]
Here, we see that all conditions of (1.11), (1.12), and (1.13) imply
\[
1/p - (x + A) \leq 1/p - (\langle x \rangle + A) < 0. \tag{4.10}
\]
Therefore,
\[
\tau_n^{(1)} \leq C e.
\]
Next, we estimate
\[
\tau_n^{(2)} = \|(1 + |x|)^{-d} W'(x) L_n(v, f_a; x)\|_{L_p(\beta a_n/2 < |x| \leq 2a_n)}.
\]
Using Lemma 4.2, we have, again
\[
\tau_n^{(2)} \leq Cn^{-\alpha} [a_n^{(p-\delta-\delta_0)} \|W(x) p_n(x)\|_{L_p(\beta_n/2 < |x| < 2\alpha_n)}^p + a_n^{1/2 - \delta} \|W(x) p_n(x)\|_{L_p(\beta_n/2 < |x| < 2\alpha_n)} + \{\log(1 + n)\} a_n^{(p-1)/2 - \delta} \|W(x) p_n(x)\|_{L_p(\beta_n/2 < |x| < 2\alpha_n)}^{-1} + \{\log(1 + n)\} a_n^{p-\delta}].
\]

Since, by (2.2) and (2.9),
\[
\|W(x) p_n(x)\|_{L_p(\mathbb{R})} \sim a_n^{1/p - 1/2} \times \left\{ \begin{array}{ll}
1, & p < 4/v, \\
\{\log(1 + n)\}^{1/4}, & p = 4/v, \\
a_n^{(m_0)(1 - 4/p)}, & p > 4/v,
\end{array} \right.
\]
we have
\[
\tau_n^{(2)} \leq Cn^{-\alpha} [a_n^{1/p - (x + \delta)} \times \left\{ \begin{array}{ll}
1, & 1 < p < 4/v, \\
\{\log(1 + n)\}^{1/4}, & p = 4/v, \\
a_n^{(m_0)(1 - 4/p)}, & p > 4/v,
\end{array} \right.
+ \{\log(1 + n)\} \times \left\{ \begin{array}{ll}
1, & 1 < p < 4/(v - 1), \\
\{\log(1 + n)\}^{(p-1)/4}, & p = 4/(v - 1), \\
a_n^{(m_0)(v - 1 - 4/p)}, & p > 4/(v - 1),
\end{array} \right.
+ \{\log(1 + n)\}].
\]
Therefore, by our assumption (1.11) or (1.12), or (1.13),
\[
\tau_n^{(2)} \leq Ce.
\]
Finally, from (4.2),
\[
\tau_n^{(3)} = \|[1 + |x|]^{-d} W'(x) L_d(v, f_n; x)\|_{L_p(\mathbb{R})} \geq 2\alpha_n}\leq Cn^{-\alpha} \|[x]^{-1} (1 + |x|)^{-d} \|_{L_p(\mathbb{R})} \geq 2\alpha_n}.
\]
Therefore, by (4.10),
\[ \tau_n^{(3)} \leq C \alpha_n^{1/p - (x + \delta)} \leq C \varepsilon. \]

Consequently, we obtained (4.8), that is, the proof of Lemma 4.3 is complete.

**Lemma 4.4** (cf. [6, Lemma 3.2]). Let \( \varepsilon > 0, 0 < \beta < 1 \). We assume that \( \Psi_n \in C(R), n = 1, 2, 3, \ldots \) are the functions satisfying
\[ \Psi_n(x) = 0, \quad |x| > \beta \alpha_n, \]
and
\[ |W^n(x) \Psi_n(x)| \leq \varepsilon |1 + |x||^{-(x + (r - 1) \alpha_n)}, \quad x \in R. \]

Then,
\[ \limsup_{n \to \infty} \| (1 + |x|)^{-\delta} W^n(x) \mathcal{L}_n(x, \Psi_n; x) \mathcal{L}_n(|x| > 2 \beta \alpha_n) \|_p \leq C \varepsilon, \]
where \( C \) is independent of \( \varepsilon, n, \) and \( \Psi_n \).

**Proof.** We see that
\[ |W^n(x) \mathcal{L}_n(r, \Psi_n; x)| \]
\[ \leq \varepsilon \sum_{|x|^{\alpha_n} \leq \beta \alpha_n} |W^{-1}(x_{\alpha_n}) W(x) \mathcal{L}_n(x)| (1 + |x_{\alpha_n}|)^{-\alpha A_k(x)}, \]
where \( A_k(x) \) is given by (4.3). Then, by (4.5), (4.6), and (4.7),
\[ A_k(x) \leq C (a_n^{1/2} W(x) p_d(x))^{r-1} + 1. \]
Since \( |x| > 2 \beta \alpha_n \) and \( |x_{\alpha_n}| \leq \beta \alpha_n \), we obtain \( |x_{\alpha_n} - x| \sim |x| \). Hence, by (2.10),
\[ |W^n(x) \mathcal{L}_n(r, \Psi_n; x)| \]
\[ \leq C \varepsilon (a_n^{1/2} W(x) p_d(x))^{r-1} + 1 \]
\[ \times \sum_{|x_{\alpha_n}| \leq \beta \alpha_n} |W^{-1}(x_{\alpha_n}) W(x) \mathcal{L}_n(x)| (1 + |x_{\alpha_n}|)^{-\alpha A_k(x)} \]
\[ \leq C \varepsilon (a_n^{1/2} W(x) p_d(x))^{r-1} + (a_n^{1/2} W(x) p_d(x)) |x|^{-1} \]
\[ \times \left( a_n/n \right) \sum_{|x_{\alpha_n}| \leq \beta \alpha_n} (1 + |x_{\alpha_n}|)^{-\alpha A_k(x)}. \]
\[
\begin{align*}
&\leq C\varepsilon (|a_n^{-1/2} W(x) p_n(x)|^{+} + |a_n^{-1/2} W(x) p_n(x)|) |x|^{-1} \\
&\quad \times \sum_{|x_k| \leq \beta_n} (1 + |x_k|)^{-\alpha} (x_{k-1,n} - x_{k+1,n}) \quad \text{(by (2.4))}
&\leq C\varepsilon (|a_n^{-1/2} W(x) p_n(x)|^{+} + |a_n^{-1/2} W(x) p_n(x)|) |x|^{-1} \\
&\quad \times \int_{2\beta_n}^{\infty} (1 + |t|)^{-\alpha} dt
&\leq C\varepsilon (|a_n^{-1/2} W(x) p_n(x)|^{+} + |a_n^{-1/2} W(x) p_n(x)|) |x|^{-1} a_n^{1-\varepsilon\alpha}(\log n)^*,
\end{align*}
\]

where
\[
(\log n)^* = \begin{cases} 
\log(1+n), & \alpha = 1, \\
1, & \text{otherwise.}
\end{cases}
\]

Therefore, by (2.9),
\[
(1 + |x|)^{-\alpha} W(x) L_{p}(v, \mathcal{W}_n; x, x)|x_0| > 2\beta_n
\]
\[
\leq C\varepsilon a_n^{1-\varepsilon\alpha}(\log n)^* a_n^{-(d+1)} \\
\times \left( |a_n^{-1/2} W(x) p_n(x)|^{+} + |a_n^{-1/2} W(x) p_n(x)| \right) \quad \text{(by } d+1 > 0\text{)}
\]
\[
\leq C\varepsilon a_n^{1-\varepsilon\alpha}(\log n)^* \\
\times \left\{ \begin{array}{ll}
1, & 1 < p < 4/v, \\
\log(1+n)^{+\delta}, & p = 4/v, \\
\frac{1}{n^{(1/v)(4/v - 4/p)}}, & p > 4/v,
\end{array} \right.
\]
\[
\left\{ \begin{array}{ll}
1, & 1 < p < 4, \\
\log(1+n)^{1+\delta}, & p = 4, \\
\frac{n^{(1/v)(1-4/p)}}{n^{(1/v)(4/v - 4/p)}}, & p > 4,
\end{array} \right.
\]
\[
\leq C\varepsilon ( by (1.11) or (1.12) or (1.13)).
\]

\text{KASUGA AND SAKEI}
Lemma 4.5 (cf. [6, Lemma 3.4]). Let \( \varepsilon > 0, 0 < \beta < 1/2 \), and assume that \( \mathcal{P}_n(x) \in C(R) \), \( n = 1, 2, 3, \ldots \), are the functions satisfying
\[
\mathcal{P}_n(x) = 0, \quad |x| \geq \beta a_n,
\]
and
\[
|W'(x) \mathcal{P}_n(x)| < \varepsilon (1 + |x|)^{-\{(n + 1)m/6\}}, \quad x \in R, \quad n \geq 1.
\]
Then,
\[
\limsup_{n \to \infty} \left\| (1 + |x|)^{-d} W'(x) L_n(v, \mathcal{P}_n; x) \right\|_{L_p(|x| < 2\beta a_n)} \leq C \varepsilon.
\]

Proof. By definition
\[
|W'(x) L_n(v, \mathcal{P}_n; x)|
\]
\[
\leq \varepsilon \sum_{|x_k| < \beta a_n} \left| (1 + |x_k|)^{-n} W'(x_k) L_n(x) A_k(x) \right|
\]
\[
\leq C \varepsilon \sum_{|x_k| < \beta a_n} \left| (1 + |x_k|)^{-n} W'(x_k) W(x) \ell_{\alpha}(x) \right|
\]
where \( A_k(x) \) is defined by (4.3), and then, \( A_k(x) \leq C, x \leq 2\beta a_n \). We use the expression (2.20). By (2.7), (2.11), and (2.5),
\[
|W'(x) L_n(v, \mathcal{P}_n; x)|
\]
\[
\leq C \varepsilon \sum_{|x_k| < \beta a_n} \left| (1 + |x_k|)^{-n} |d_n^{1/2} W(x) p_n(x)/\ell(k, x)| \right|
\]
\[
\leq C \varepsilon \sum_{|x_k| < \beta a_n} \left| (1 + |x_k|)^{-n} |1/\ell(k, x)| \right|
\]
Therefore, we have
\[
|W'(x) L_n(v, \mathcal{P}_n; x)| \leq C \varepsilon \left\{ \log(1 + n) \right\}.
\]
By (4.11),
\[
\left\| (1 + |x|)^{-d} W'(x) L_n(v, \mathcal{P}_n; x) \right\|_{L_p(|x| < 2\beta a_n)}
\]
\[
\leq C \varepsilon \left\{ \log(1 + n) \right\} \left\| (1 + |x|)^{-d} \right\|_{L_p(|x| < 2\beta a_n)}
\]
\[
\leq C \varepsilon \left\{ \log(1 + n) \right\} \alpha_n^{1/\beta - d} \quad \text{(by (1.11), (1.12), and (1.13))}
\]
\[
\leq C \varepsilon.
\]
Consequently, we see that the proof of Lemma 4.5 is complete. \( \Box \)
Remark 4.6. In the above consideration of Section 4 we can replace \( h_{kn}(x) \) in (4.1) by \( h_{kn}(x) = |l_{kn}(x)| \sum_{j=0}^{m} (n/a_n)^j |x - x_{kn}| \) (defined in (3.3)).

Proof of Theorem 5. By (1.14) there exists a polynomial \( P_s(x) \) such that

\[
[(1 + |x|)^{(r - 1)m/6} W^q(x) \{ f(x) - P_s(x) \}] < \varepsilon, \quad x \in \mathbb{R}
\]

(cf. [2, p. 180]). Since (by (2.17)),

\[
L_n(v - 1, v, P_s; x) = P_s(x) \quad \text{and} \quad h_{kn}(v; x) = h_{kn}(v; x), \quad x \in \mathbb{R},
\]

we have

\[
(1 + |x|)^{-d} W^q(x) [L_n(v, f; x) - f(x)]
\]

\[
= (1 + |x|)^{-d} W^q(x) \left[ L_n(v, f - P_s; x) + \{ P_s(x) - f(x) \} \right]
\]

\[
+ \sum_{k=1}^{n} \sum_{s=1}^{n} \sum_{j=1}^{m} \left( p_s(x) h_{kn}(x) \right]
\]

\[
= \sum_{k=1}^{n} \sum_{s=1}^{n} \sum_{j=1}^{m} (x). \]

Let \( \chi[-a_n/4, a_n/4] \) denote the characteristic function of \([-a_n/4, a_n/4]\) and write

\[
f - p_s = (f - p_s) \chi[-a_n/4, a_n/4] + (f - p_s)(1 - \chi[-a_n/4, a_n/4])
\]

\[
= \Psi_n + f_n.
\]

Applying Lemma 4.3, 4.4, and 4.5 to \( f_n \) or \( \Psi_n \), we obtain

\[
\left\| \sum_{i} (x) \right\|_{L_p(\mathbb{R})} \leq C\varepsilon.
\]

Since, by (4.10) we see that \(-p \{ A + \alpha + (v - 1)m/6 \} < -p(\alpha + \alpha) < -1, \)

we also have

\[
\left\| \sum_{i} (x) \right\|_{L_p(\mathbb{R})} \leq C\varepsilon \left| (1 + |x|)^{-d + \alpha + (v - 1)m/6} \right|_{L_p(\mathbb{R})} \leq C\varepsilon.
\]
Finally, we estimate $\sum_3 (x)$. We see that

$$\left| (1 + |x|)^{-d} W^\alpha(x) \sum_{k=1}^n \sum_{s=1}^r P_{\varepsilon}^{(s)}(x_{kn}) h_{kn}(x) \right|$$

$$\leq C \sum_{s=1}^r (1 + |x|)^{-d} W^\alpha(x) \sum_{k=1}^n |P_{\varepsilon}^{(s)}(x_{kn}) /\ell_{kn}(x)|$$

$$\times \sum_{j=0}^{r-1} (n/a_n)^j |x - x_{kn}|^j$$

$$\leq C \sum_{s=1}^r (a_n/n)^s (1 + |x|)^{-d} W^\alpha(x) \sum_{k=1}^n |P_{\varepsilon}^{(s)}(x_{kn}) /\ell_{kn}(x)|$$

$$\times \sum_{j=0}^{r-1} (n/a_n)^j |x - x_{kn}|^j$$

$$= C(a_n/n) \sum_3^{r'} (x),$$

where

$$\sum_3^{r'} (x) = \sum_{s=1}^r (1 + |x|)^{-d} W^\alpha(x) \sum_{k=1}^n |P_{\varepsilon}^{(s)}(x_{kn}) /\ell_{kn}(x)|$$

$$\times \sum_{j=0}^{r-1} (n/a_n)^j |x - x_{kn}|^j.$$
(we replace $e$ to $M(s, v, f)$ in each lemma). Consequently, we see that the proof of Theorem 5 is complete.

5. PROOF OF THEOREM 6

In this section we let $v = 3, 5, 7, \ldots$, and we will obtain an inverse theorem. We need the following lemmas.

**Lemma 5.1** [6, Lemma 2.5]. Let $\xi \in \mathbb{R}$. There exists $C > 0$ such that for $\lambda \geq 2$, there exist polynomials $P_\lambda^s$ of degree $\leq C\lambda \log \lambda$ satisfying

$$P_\lambda^s(t) \sim (1 + t^2)^\xi,$$

uniformly for $-\lambda \leq t \leq \lambda$.

**Lemma 5.2** [6, Lemma 3.5]. Let $0 < \sigma < 1$, $0 < \theta < 1 - \sigma$, and $1 < p < \infty$. Then, there exists $C$ such that for $n \geq 1$ and $P$ of degree at most $\theta n$, we have

$$\|P\|_{L_p[-a_n, a_n]} \leq C a_n^{1/2} \sum_{j=n-1}^n \|p_j WP\|_{L_p[-a_n, a_n]}.$$

The following proposition is important itself, and to prove Theorem 6 we use it as one of the lemmas. We use the number $q_n = (2\pi/m)^{1/m}$ instead of $a_n$, defined in Section 2 (see (2.2)). Let $\beta = (1/2)\left\{\pi^{1/2}/F(m/2)/F(m + 1/2)\right\}^{1/m}$ be Freud's constant, and let $\varepsilon = m(m/2)^{m-1/2}(m^{m-1} - 1)^{1/m}$.

In [3], we showed that the proposition held for $x_{kn} \in [\theta, \Theta]$, where $\theta$ and $\Theta$ are positive constants. We omit the proof of Proposition 5.3, because we can show it by careful repeating the same line of the consideration as one in [3].

**Proposition 5.3** (cf. [3, Lemma 14]). For $j = 0, 1, 2, \ldots$, there exists a polynomial $\Psi_j(x)$ of degree $j$ such that $(-1)^j \Psi_j(-v) > 0$ for $v = 1, 2, 3, \ldots$, and the following relation holds: Let $0 < \varepsilon < 1$. Then, we have an expression

$$e_{2n,k} = (-1)^j \{1/(2\pi)!\} \Psi_j(-v) v^{2\varepsilon q_n^{2m-1}}\delta + \eta_{\varepsilon q_n}(v, s), \quad (5.1)$$

where $\eta_{\varepsilon q_n}(v, s)$ satisfies

$$|\eta_{\varepsilon q_n}(v, s)| \leq C \delta, \quad (5.2)$$

for $k$ with $|x_{kn}| \leq \varepsilon q_n$ and $s = 0, 1, \ldots, v$. Here, the positive constant $C$ is independent of $n, k$, and $\varepsilon$, and may depend on $v, s$, and $m$; $\varepsilon$ is the largest integer not exceeding $(v - 1)/2$. 


Proof of Theorem 6. Let \( r = 3, 5, 7, \ldots \). We repeat the line of [6, proof of the necessary conditions of Theorem 1.3]. Let \( \zeta(x) \) be an even continuous function that is decreasing in \([0, \infty)\), with
\[
\zeta(x) \geq \left\{ \log(2 + |x|) \right\}^{-1/2r} \quad (x \in R), \quad \lim_{x \to \infty} \zeta(x) = 0.
\]

Let us define two spaces: \( X \) consists of all continuous functions satisfying
\[
\|f\|_X = \|(1 + |x|)^{n+(r-1)m} W^{r}(x) f(x) \zeta^{-1}(x)\|_{C(R)} < \infty,
\]
and \( Y \) consists of all measurable functions satisfying
\[
\|f\|_Y = \|(1 + |x|)^{-d} W^{r}(x) f(x)\|_{L_d(R)} < \infty.
\]

For each \( f \in X \), (1.14) is satisfied, so our hypothesis ensures that
\[
\lim_{n \to \infty} \|L_n(v, f) - f\|_Y = 0.
\]

Since \( X \) is a Banach space, by the uniform boundedness principle, there exists \( C > 0 \) such that for \( n = 1, 2, 3, \ldots \), and every \( f \in X \),
\[
\|L_n(v, f) - f\|_Y \leq C \|f\|_X.
\]

Noting \( L_n(v, f; x) = f(0), \ x \in R \), we have for every \( f \in C(R) \) with \( f(0) = 0 \) that
\[
\|f\|_Y \leq C \|f\|_X,
\]
consequently, we obtain
\[
\|L_n(v, f)\|_Y \leq C \|f\|_X, \quad \text{(5.3)}
\]
that is,
\[
\|(1 + |x|)^{-d} W^{r}(x) L_n(v, f; x)\|_{L_d(R)} \leq C \|\zeta^{-1}(x)(1 + |x|)^{n+(r-1)m} W^{r}(x) f(x)\|_{C(R)}. \quad \text{(5.4)}
\]

Let \( 0 < \varepsilon \) be small enough, and let us consider the function \( g_n \in C(R) \) such that \( g_n(x) = 0 \) in \([0, \infty) \cup (-\infty, -\varepsilon \Delta_n)\);
\[
\|g_n\|_X = \|\zeta^{-1}(x)(1 + |x|)^{n+(r-1)m} W^{r}(x) g_n(x)\|_{C(R)} = 1; \quad \text{(5.5)}
\]

and for \(-\varepsilon a_n \leq x_{kn} < 0,\)
\[
\zeta^{-1}(x_{kn})(1 + |x_{kn}|)^{\nu + (\nu - 1)\varepsilon - \nu/6} W(\varepsilon, x_{kn}) g_n(x_{kn}) \mathrm{sign}\{p_n'(x_{kn})\} = 1.
\]

Then, for \(x \geq 1,\) we have
\[
|L_n(\nu, g_n; x)| = \sum_{x_{kn} \in \{-\varepsilon a_n, 0\}} g_n(x_{kn}) \left[p_n(x)/\{(x - x_{kn}) p_n'(x_{kn})\}\right]^{r-1} \times \sum_{j=0}^{r-1} e_j(x - x_{kn}).
\]

(5.6)

Here, we show that for \(r \geq 3\) and so \(n\) large enough,
\[
(-1)^{(r-1)/2} \frac{1}{\nu} |x - x_{kn}|^{r-1} \sum_{j=0}^{r-1} e_j(x - x_{kn})^j \geq C(n/a_n)^{r-1}. \tag{5.7}
\]

In fact, using the expression (2.20) we see that for \(x \geq 1\) and \(x_{kn} \in [-\varepsilon a_n, 0),\)
\[
|t(k, x)| \delta a_n/n \geq x = t(k) \delta a_n/n \geq 1,
\]

where \(t(x)\) is a positive number. Therefore, we have
\[
|t(k, x)| \geq t(x) \geq (1/\delta)(n/a_n). \tag{5.8}
\]

By (5.1) and (5.2), there exists a positive constant \(C(\nu)\) such that
\[
(-1)^{(r-1)/2} e_{r-1,k} \geq C(\nu)(n/a_n)^{r-1}. \tag{5.9}
\]

From (5.8) and (5.9),
\[
(-1)^{(r-1)/2} \frac{1}{\nu} |x - x_{kn}|^{r-1} \sum_{j=0}^{r-1} e_j(x - x_{kn})^j
\]
\[
= (-1)^{(r-1)/2} \left\{ e_{r-1,k} + \sum_{j=0}^{r-2} e_j(x - x_{kn})^{j-r+1} \right\}
\]
\[
\geq C(n/a_n)^{r-1} - C \sum_{j=0}^{r-2} (n/a_n)^j \{ |t(k, x)| \delta \}^{j-r+1} (n/a_n)^{r-1-j}
\]
\[
= (n/a_n)^{r-1} \left[ C(\nu) - C \sum_{j=0}^{r-2} \{|t(k, x)| \delta \}^{j-r+1} \right]
\]
\[
\geq (n/a_n)^{r-1} \left[ C(\nu) - C(\nu/n) \right]
\]
\[
\geq C(n/a_n)^{r-1}.
\]

Therefore, we obtain (5.7).
Let $1 \leq x \leq 2a_n$. Applying (5.7) to (5.6), we have

$$|L_n(v, g_n; x)| \geq C \left| \sum_{x_{kn} \in [-a_n, 0]} \varphi_d(x_{kn}) \left[ p_d(x) / p'_d(x_{kn}) \right]^r (x - x_{kn})^{-1} (n/a_n)^{r-1} \right|$$

$$\geq C(a_n/n) \left| a_n^{1/2} p_d(x) \right|^r \sum_{x_{kn} \in [-a_n, 0]} (1 + |x_{kn}|)^{-\left\{ x + (r-1)/6 \right\}}$$

$$\times \zeta(x_{kn}) (x - x_{kn})^{-1}$$

$$\geq C\zeta(a_n) \left| a_n^{1/2} p_d(x) \right|^r \sum_{x_{kn} \in [-a_n, 0]} (1 + |x_{kn}|)^{-\left\{ x + (r-1)/6 \right\}}$$

$$\times (x - x_{kn})^{-1} (x_{kn-1} - x_{kn+1})$$

(by (2.4))

$$\geq C\zeta(a_n) \left| a_n^{1/2} p_d(x) \right|^r \int_{-a_n/2}^{a_n/2} \left[ (1 + t)^{-\left\{ x + (r-1)/6 \right\}} / (x + t) \right] dt$$

$$\geq C\zeta(a_n) \left| a_n^{1/2} p_d(x) \right|^r \int_{-a_n/2}^{a_n/2} (1 + t)^{-\left\{ x + (r-1)/6 \right\}} dt$$

$$\geq C\zeta(a_n) \left| a_n^{1/2} p_d(x) \right|^r \{ x^\ast \}$$

where

$$\{ x^\ast \} = \begin{cases} 1, & x + (r-1)/6 \geq 1, \\ \log(1 + \min(a_n/2, x)), & x + (r-1)/6 = 1, \\ \left( \min(a_n/2, x) \right)^{1 - \left\{ x + (r-1)/6 \right\}}, & x + (r-1)/6 < 1, \\ \log(1 + x), & x + (r-1)/6 < 1, \\ 1, & \text{otherwise.} \end{cases}$$

The last inequality is obtained by considering $1 \leq x \leq a_n/2$ and $a_n/2 < x \leq 2a_n$ separately. Since by (5.3) we see that

$$\|L_n(v, g_n)\|_x \leq C \left\| g_n \right\|_x \leq C,$$

we have

$$C \geq \left\| (1 + |x|)^{-d} W^r(x) L_n(v, g_n; x) \right\|_{L_p(1, 2a_n)}$$

$$\geq C \left\{ \log(1 + n) \right\}^{-\left\{ 1/2p \right\}} \left\| (1 + |x|)^{-d} \zeta(x + (r-1)/6) \right\|_{L_p(1, 2a_n)}$$

$$\times \left\| a_n^{1/2} W(x) p_d(x) \right\|_p(p(1, 2a_n))$$

(see the definition $\zeta(x)$). (5.11)
Since by (2.5) we have
\[
\| (1 + |x|)^{-(d + \log (1 + x))} a_n^{1/2} W(x) p_a(x) \|_{L_p([-1, 1])} \leq C, \tag{5.11}
\]
(5.11) implies that
\[
C \geq \left[ \log(1 + n) \right]^{- (1/2p^2)} \left[ \| (1 + |x|)^{-(d + \log (1 + x))} \right] \times \| a_n^{1/2} W(x) p_a(x) \| L_p([-2a_n, 2a_n]) - C.
\]
Therefore,
\[
C \left[ \log(1 + n) \right]^{1/2p^2} \geq a_n^{1/2} \left[ (1 + |x|)^{-(d + \log (1 + x))} \right] \times \| W(x) p_a(x) \|_{L_p([-2a_n, 2a_n])} - C,
\]
that is,
\[
C \left[ \log(1 + n) \right]^{1/2p^2} \geq a_n^{1/2} \left[ (1 + |x|)^{-(d + \log (1 + x))} \right] \times \| W(x) p_a(x) \|_{L_p([-2a_n, 2a_n])} - C. \tag{5.12}
\]
Now, let \( P_{2n}^* \) be the polynomial of Lemma 5.1 of degree \( o(n \log n) = o(n) \) such that for \( |x| \leq 2a_n \),
\[
P_{2n}^*(x) \sim (1 + x^2)^{-(d + \log (1 + x))} \times \left( (1 + |x|)^{-(d + \log (1 + x))} \right) \times \| W(x) p_a(x) \|_{L_p([-2a_n, 2a_n])} - C.
\]
We obtain from (5.12) that
\[
C \left[ \log(1 + n) \right]^{1/2p^2} \geq a_n^{1/2} \sum_{j = n - 1}^{n} \| W(x) p_a(x) P_{2n}^*(x) \|_{L_p([-2a_n, 2a_n])} - C.
\]
In Lemma 5.2 setting \( \sigma = 1/2 \) and \( \theta = 1/4 \), we have
\[
C \left[ \log(1 + n) \right]^{1/2p^2} \geq C \left\| P_{2n}^*(x) \right\|_{L_p([-a_n, a_n])} - C,
\]
\[
\geq C \left\| (1 + |x|)^{-(d + \log (1 + x))} \right\|_{L_p([-a_n, a_n])} - C
\]
\[
\geq C \times \left\{ \log(1 + n) \right\}^{1/2p^2} - C,
\]
\[
A = \left( 1/p \right) - \left( \log (1 + x) \right),
\]
\[
A = \left( 1/p \right) - \left( \log (1 + x) \right),
\]
\[
1 - C, \quad A = (1/p) - \left( \log (1 + x) \right).
\]

However, for these inequalities can occur only the last one, that is, \( A > (1/p) - \ll x + ( v - 1 ) \ m/6 \rr \). Therefore, we obtain the necessary conditions for \( 1 < p \leq 4/v \) (but \( v < 4 \)).

Next, we consider the case of \( p > 4/v \). We return to (5.10), that is,

\[
|L_n(v, g_n; x)| \geq C\zeta(a_n) [a_n^{1/2} p_n(x)]^* x^{-\ll x + ( v - 1 ) \ m/6 \rr} \log(x)^* \tag{5.13}
\]

First, by (2.5), (2.6), and (2.8) we see that for \( 0 < \kappa < 1/2 \) small enough,

\[
\| W(x) p_n(x) \|_{L_p(x)} \sim \| W(x) p_n(x) \|_{L_p(x)} \tag{5.14}
\]

Therefore, by (5.4), (5.5), (5.13), (5.14), and (2.9), we have

\[
C \geq \| (1 + |x|)^{-d} W'(x) L_n(v, g_n; x) \|_{L_p(x)} \geq C\zeta(a_n) a_n^{1/2} a_n^{-(d + \ll x + ( v - 1 ) \ m/6 \rr)} (\log n)^* \tag{5.15}
\]

Consequently, if \( \alpha + (v - 1) m/6 = 1 \), then we see that

\[
1/p - (A + \ll x + (v - 1) \ m/6 \rr) + (m/6)(v - 4/p) < 0
\]

(recall the definition of \((\log n)^*\), therefore we have (1.17). If \( \alpha + (v - 1) m/6 \neq 1 \), then (5.15) implies that

\[
C \| \log(1 + n) \|^{1/(2p)} \geq a_n^{1/p - (d + \ll x + ( v - 1 ) \ m/6 \rr) + (m/6)(v - 4/p)} (\log n)^* \tag{5.15}
\]

Therefore, we have

\[
1/p - (A + \ll x + (v - 1) \ m/6 \rr) + (m/6)(v - 4/p) \leq 0.
\]

Thus, we have (1.18). Consequently, the theorem follows.

**Proof of Corollary 7.** Let \( v = 3, 5, 7, \ldots \), and let \( x \geq 1 \). Furthermore, we assume that \((m/6)(v - 4/p) > 1\) for \( v > 3 \), or if \( v = 3 \), then \( p > 4/3 \). Then, the condition (1.13) is equivalent to the condition (1.18).
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