DISCRETE MATHEMATICS

On growth of Lie algebras, generalized partitions, and analytic functions ☆

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Abstract

In this paper we discuss some recent results on two different types of growth of Lie algebras that lead to some combinatorial problems. First, we study the growth of finitely generated Lie algebras (Sections 1–4). This problem leads to a study of generalized partitions. Recently the author has suggested a series of q-dimensions of algebras Dim^q, $q \in \mathbb{N}$ which includes, as first terms, dimensions of vector spaces, Gelfand–Kirillov dimensions, and superdimensions. These dimensions enabled us to describe the change of a growth in transition from a Lie algebra to its universal enveloping algebra. In fact, this is a result on some generalized partitions. In this paper we give some results on asymptotics for those generalized partitions. As a main application, we obtain an asymptotical result for the growth of free polynilpotent finitely generated Lie algebras. As a corollary, we specify the asymptotic growth of lower central series ranks for free polynilpotent finitely generated groups. We essentially use Hilbert–Poincaré series and some facts on growth of complex functions which are analytic in the unit circle. By growth of such functions we mean their growth when the variable tends to 1. Also we discuss for all levels $q = 2, 3, \ldots$ what numbers $\alpha > 0$ can be a q-dimension of some Lie (associative) algebra. Second, we discuss a 'codimension growth' for varieties of Lie algebras (Sections 5 and 6). It is useful to consider some exponential generating functions called complexity functions. Those functions are entire functions of a complex variable provided the varieties of Lie algebras are nontrivial. We compute the complexity functions for some varieties. The growth of a complexity function for an arbitrary polynilpotent variety is evaluated. Here we need to study the connection between the growth of a fast increasing entire function and the behavior of its Taylor coefficients. As a result we obtain a result for the asymptotics of the codimension growth of a polynilpotent variety of Lie algebras. Also we obtain an upper bound for a growth of an arbitrary nontrivial variety of Lie algebras. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

Dans ce travail nous traitons quelques résultats récents portant sur deux types différents de croissance pour les algèbres de Lie, qui nous conduisent à des problèmes combinatoires.

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Dans un premier temps, nous étudions la croissance des algèbres de Lie finiment engendrées (Sections 1–4). Ce probème nous amène à étudier les partitions généralisées. Récemment l'auteur avait proposé une série de q-dimensions des algèbres Dim^q, $q \in \mathbb{N}$, qui contient comme premiers termes des dimensions d'espaces vectoriels, les dimensions de Gelfand–Kirillov et les superdimensions. Ces dimensions nous ont permis de décrire le changement de croissance quand on passe d'une algèbre de Lie à son algèbre enveloppante universelle. Dans le travail présent nous proposons une asymtotique plus précise pour les partitions généralisées. Comme application majeure nous obtenons une asymptotique pour la croissance des algèbres de Lie polynilpotentes libres et finiment engendrées. Comme corollaire nous spécifions une croissance asymptotique des rangs de la série contrale descendante pour les groupes polynilpotents libres et finiment engendrés. On utilise éssentiellement les séries de Hilbert–Poincaré et quelques énoncés sur la croissance des fonctions analytiques dans le disque unité. La croissance d'une telle fonction s'entend comme croissance quand la variable tend vers 1. En plus, nous discutons pour tous les niveaux $q = 2, 3, \ldots$ quels sont les nombres $\alpha > 0$ qui peuvent apparaître comme q-dimension d'une algèbre de Lie (associative). Dans un deuxième temps, nous étudions la "croissance en codimension" pour les variétés d'algèbres de Lie (Sections 5 and 6). Il convient de regarder quelques fonctions génératrices exponentielles, ici appellés fonctions de complexité. Ces fonctions sont des fonctions entières d'une variable complexe pourvu que la variété ne soit pas triviale. On calcule la fonction de complexité de plusieurs variétés et on évalue la croissance de la fonction de complexité pour une variété polynilpotente quelconque. Pour cela, on a besoin d'étudier la relation entre la croissance d'une fonction entière rapidement croissante et le comportement de ses coéfficients de Taylor. On en déduit une asymptotique pour la croissance en codimension d'une variété polynilpotente d'algèbres de Lie. En plus on obtient une borne supérieure pour la croissance d'une variété nontriviale d'algèbres de Lie quelconque.

1. Growth of finitely generated Lie algebras and series of dimensions

Let A be a Lie (associative) algebra over a field K, generated by a finite set X. Denote by $A^{(X,n)}$ subspace spanned by all monomials in X of length not exceeding n. Denote

$$\gamma_A(n) = \gamma_A(X,n) = \dim_K A^{(X,n)}, \qquad \lambda_A(n) = \gamma_A(n) - \gamma_A(n-1),$$

where $\dim_K$ stands for the dimension of a vector space over K. If A is an associative algebra with unit then we consider that this unit belongs to $A^{(X,n)}$, $n \geqslant 0$, and $\gamma_A(0) = \lambda_A(0) = 1$. On functions $f : \mathbb{N} \to \mathbb{R}^+$, where $\mathbb{R}^+ = \{\alpha \in \mathbb{R} \mid \alpha > 0\}$, we consider a partial order $f(n) \overset{\mathrm{a}}{\leqslant} g(n)$ iff there exists $N > 0$, such that $f(n) \leqslant g(n)$, $n \geqslant N$.

The growth less than any exponent is called subexponential. If it is also greater than any polynomial growth, then it is called intermediate. For study of such growths the following series of dimensions has been suggested [20,22]. Denote by iteration

$$\ln^{(1)} n = \ln n; \quad \ln^{(q+1)} n = \ln(\ln^{(q)} n), \; q = 1, 2, \ldots$$

Consider a series of functions $\Phi_\alpha^q(n)$, $q=1,2,3,\dots$ of a natural argument with the parameter $\alpha\in\mathbb{R}^+$:

$$\begin{aligned}
&\Phi_\alpha^1(n)=\alpha,\\
&\Phi_\alpha^2(n)=n^\alpha,\\
&\Phi_\alpha^3(n)=\exp(n^{\alpha/(\alpha+1)}),\\
&\Phi_\alpha^q(n)=\exp\left(\frac{n}{(\ln^{(q-3)}n)^{1/\alpha}}\right);\quad q=4,5,\dots .
\end{aligned}\tag{1}$$

Suppose that $f(n)$ is any positive valued function of a natural argument. We define the (*upper*) *dimension of level* q, $q=1,2,3,\dots$ and the *lower dimension of level* q by

$$\mathrm{Dim}^q f(n)=\inf\{\alpha\in\mathbb{R}\mid f(n)\overset{\mathrm{a}}{\leq}\Phi_\alpha^q(n)\},$$

$$\underline{\mathrm{Dim}}^q f(n)=\sup\{\alpha\in\mathbb{R}\mid f(n)\overset{\mathrm{a}}{\geqslant}\Phi_\alpha^q(n)\}.$$

Suppose that A is a finitely generated algebra with $\gamma_A(n)$ as above. We define the *q-dimension* (*lower q-dimension*), $q=1,2,3,\dots$ of A by

$$\mathrm{Dim}^q A=\mathrm{Dim}^q\gamma_A(n),\quad \underline{\mathrm{Dim}}^q A=\underline{\mathrm{Dim}}^q\gamma_A(n).$$

q-dimensions do not depend on a generating set X. Remark that 1-dimension coincides with the dimension of a vector space A over K. Dimensions of level 2 are exactly the upper and lower Gelfand–Kirillov dimensions [8]. Dimensions of level 3 correspond to the superdimensions of [5] up to normalization (see [22]). Dimensions of levels $q=4,5,\dots$ correspond to growths which are subexponential but are greater than any function $\exp(n^\beta)$, $\beta<1$. Such growths were not known and has not been studied until [20].

Theorem 1.1 (Petrogradsky [20,22]). *Let L be a finitely generated Lie algebra with* $\mathrm{Dim}^q L=\alpha>0$, $q=1,2,\dots$. *Also for* $q\geqslant 3$ *suppose that* $\underline{\mathrm{Dim}}^q L=\alpha$ *and for* $q=2$ *suppose that* $\underline{\mathrm{Dim}}^2\lambda_L(n)=\alpha-1$, $\alpha\geqslant 1$. *Then*

$$\underline{\mathrm{Dim}}^{q+1}U(L)=\mathrm{Dim}^{q+1}U(L)=\alpha.$$

In fact, this is a result on generalized partitions. If we have a sequence $\{b_n\in\mathbb{N}_0\mid n=1,2,\dots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, then we can obtain another sequence $\{a_n\in\mathbb{N}_0\mid n=0,1,2,\dots\}$:

$$\prod_{n=1}^{\infty}\frac{1}{(1-t^n)^{b_n}}=\sum_{n=0}^{\infty}a_nt^n.\tag{2}$$

If $b_n=1$, $n\in\mathbb{N}$, then $a_n=\rho(n)$ is the number of partitions of n. In a general case we obtain *generalized partitions*. We can illustrate their meaning by the following. a_n is a number of the Young diagrams with rows of length m marked by b_m colors, the order of colors being nonessential; $m\in\mathbb{N}$. In other words, a_n is the number of partitions

$$n=\lambda_1+\lambda_1+\cdots+\lambda_s,\quad \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant l_s,\quad \lambda_i\in\mathbb{N};$$

where all numbers λ_i equal to m are painted into b_m colors, $m=1,\dots,n$; where the order of equal numbers of different colors in our partitions we consider nonessential.

An importance for us of this relation (2) is explained by the following fact. If we take a finitely generated Lie algebra L and consider its universal enveloping algebra as generated by the same set, then two sequences $b_n = \lambda_L(n)$, $n = 1,2,\dots$ and $a_n = \lambda_{U(L)}(n)$, $n = 0,1,2,\dots$ do satisfy (2) [31,33].

Let $f(n)$ be a function of a natural argument, we will compare it with our etalon functions (e.g. (1)). Suppose that $\tau_\gamma(n)$ is a function of a natural argument, which is continuous and increasing with respect to the parameter γ. Let the number $c \in \mathbb{R}$ be fixed. Then we introduce a notation:

$$f(n) \sim^c \tau_c(n) \Leftrightarrow \inf\{\gamma \mid f(n) \overset{\mathrm{a}}{\leqslant} \tau_\gamma(n)\} = c.$$

Now, Theorem 1.1 and some other results of [22] can be reformulated in other way.

Theorem 1.2. *Suppose that two sequences* $\{b_n \in \mathbb{N}_0 \mid n = 1,2,\dots\}$ *and* $\{a_n \in \mathbb{N}_0 \mid n = 0,1,2,\dots\}$ *satisfy* (2).
1. *If* $b_n = \Phi^2_{\alpha-1+\mathrm{o}(1)}(n) = n^{\alpha-1+\mathrm{o}(1)}$, *then* $a_n = \Phi^3_{\alpha+\mathrm{o}(1)}(n) = \exp(n^{\alpha/(\alpha+1)+\mathrm{o}(1)})$.
2. *If* $b_n = \Phi^q_{\alpha+\mathrm{o}(1)}(n)$, $q \geqslant 3$, *then* $a_n = \Phi^{q+1}_{\alpha+\mathrm{o}(1)}(n)$.
3. *If* $b_n \sim^\alpha \Phi^q_\alpha(n)$, $q \geqslant 3$, *then* $a_n \sim^\alpha \Phi^{q+1}_\alpha(n)$.

In other words, we move upwards on the staircase (1).

If L is a Lie algebra then by iteration the lower central series is defined $L^1 = L$, $L^{i+1} = [L, L^i]$, $i = 1,2,\dots$. Now L is called *nilpotent* of class s iff $L^{s+1} = \{0\}$, $L^s \neq \{0\}$. All Lie algebras nilpotent of class s form the variety $\boldsymbol{N}_s$. Recall that L is *polynilpotent* with tuple $(s_q,\dots,s_2,s_1)$ iff there exists a chain of ideals

$$0 = L_{q+1} \subset L_q \subset \cdots \subset L_2 \subset L_1 = L$$

with $L_i/L_{i+1} \in \boldsymbol{N}_{s_i}$. All polynilpotent Lie algebras with fixed tuple form a variety denoted by $\boldsymbol{N}_{s_q}\dots\boldsymbol{N}_{s_2}\boldsymbol{N}_{s_1}$. If $\boldsymbol{M}$ is a variety of Lie algebras then by $F(\boldsymbol{M},k)$ we denote its free algebra of rank k (this is an algebra generated by k elements $x_1,\dots,x_k$ and such that for all $H \in \boldsymbol{M}$ and any $y_1,\dots,y_k \in H$ there exists a homomorphism $\phi : F \to H$ with $\phi(x_i) = y_i$, $i = 1,\dots,k$). In the case $s_q = \cdots = s_1 = 1$ one has the variety $\boldsymbol{A}^q$ of *solvable* Lie algebras of length q. For the theory of *varieties* see monograph [2].

Theorem 1.3 (Petrogradsky [22]). *Let* $L = F(\boldsymbol{N}_{s_q}\dots\boldsymbol{N}_{s_2}\boldsymbol{N}_{s_1},k)$, $q \geqslant 2$ *be the free polynilpotent Lie algebra of rank k,* $k \geqslant 2$. *Then*

$$\underline{\mathrm{Dim}}^q L = \mathrm{Dim}^q L = s_2 \dim_K F(\boldsymbol{N}_{s_1},k).$$

Recall that for a graded algebra $A = \bigoplus_{n=0}^\infty A_n$ a Hilbert–Poincaré series is defined as $\mathscr{H}(A,t) = \sum_{n=0}^\infty \dim_K A_n\, t^n$. It is interesting when a Hilbert–Poincaré series is the rational function.

Corollary 1.1 (Petrogradsky [22]). *The Hilbert–Poincaré series for* $L = F(\boldsymbol{N}_{s_q}\dots\boldsymbol{N}_{s_1},k)$ *is rational iff* $q \leqslant 2$.

We would like to mention one interesting result. Let W_n be the Witt algebra and $\mathbf{W}_n = \mathrm{var}(W_n)$ be the variety defined by all identical relations of this algebra. Asymptotic formulae from [18] in terms of q-dimensions imply the following theorem.

Theorem 1.4 (Molev [18]). *Suppose that* $L = F(\mathbf{W}_n, k)$, $k \geqslant n+1$. *Then* $\underline{\mathrm{Dim}}^3 L = \mathrm{Dim}^3 L = n$.

The following hypothesis is an analogue of Theorem 5.3.

Conjecture 1. Suppose that L is a finitely generated Lie algebra satisfying nontrivial identity of degree d. Then (1) $\mathrm{Dim}^q L < \infty$ for some $q \in \mathbb{N}$. (2) We can take $q = d$.

2. Generalized partitions and growth of polynilpotent Lie algebras and groups

Let us fix some notations. By $\zeta(x)$ and $\Gamma(x)$ we denote the zeta and gamma functions; $\psi_k(n)$ is the dimension of elements of degree n in a free Lie algebra with k generators.

Now we obtain more precise asymptotic than those in Theorem 1.2.

Theorem 2.1 (Petrogradsky [26]). *Suppose that two sequences* $\{b_n \in \mathbb{N}_0 \mid n = 1, 2, \ldots\}$ *and* $\{a_n \in \mathbb{N}_0 \mid n = 0, 1, 2, \ldots\}$ *satisfy* (2). *Suppose that* b_n *has one asymptotic, then* a_n *has other asymptotic as below*:

1. $b_n = (\sigma + \mathrm{o}(1))\, n^{\alpha - 1}$, $\alpha \geqslant 1$; $\quad a_n = \exp((\theta + \mathrm{o}(1))\, n^{\alpha/(\alpha+1)})$;

2a. $b_n \sim^{\sigma} \exp(\sigma\, n^{\alpha/(\alpha+1)})$; $\quad a_n \sim^{\kappa} \exp\left(\kappa \dfrac{n}{(\ln n)^{1/\alpha}}\right)$;

2b. $b_n = \exp((\sigma + \mathrm{o}(1))\, n^{\alpha/(\alpha+1)})$; $\quad a_n = \exp\left((\kappa + \mathrm{o}(1)) \dfrac{n}{(\ln n)^{1/\alpha}}\right)$;

3a. $b_n \sim^{\sigma} \exp\left(\sigma \dfrac{n}{(\ln^{(s)} n)^{1/\alpha}}\right)$, $s \geqslant 1$; $\quad a_n \sim^{\sigma} \exp\left(\sigma \dfrac{n}{(\ln^{(s+1)} n)^{1/\alpha}}\right)$;

3b. $b_n = \exp\left((\sigma + \mathrm{o}(1)) \dfrac{n}{(\ln^{(s)} n)^{1/\alpha}}\right)$, $s \geqslant 1$;

$$a_n = \exp\left((\sigma + \mathrm{o}(1)) \frac{n}{(\ln^{(s+1)} n)^{1/\alpha}}\right);$$

where the constants are $\theta = (1 + 1/\alpha)(\sigma\zeta(\alpha+1)\Gamma(\alpha+1))^{1/(\alpha+1)}$, $\kappa = \alpha\,(\sigma/(\alpha+1))^{1+1/\alpha}$.

Remark. Assertion 1, as well as assertion 1 of Theorem 1.2, are versions of known asymptotics [1,4]; but in [1] exact asymptotics for a_n are given, provided that on b_n stronger assumptions are imposed. The virtue of the present approach is that our arguments are elementary. To the best of our knowledge, assertions 2a, 2b, 3a, 3b were not known.

Let us consider an application of this result to so called *r-fold partitions of n* [10] [1] A double partition (2-fold partition) of n is a representation of n as a double sum

$$\begin{aligned} n &= \alpha_1 + \cdots + \alpha_k, \\ \alpha_1 &= \beta_{11} + \cdots + \beta_{1m_1}, \\ &\vdots \\ \alpha_k &= \beta_{k1} + \cdots + \beta_{km_k}; \end{aligned}$$

where α_i, β_{ij} are positive integers; and the order of $\alpha_1, \ldots, \alpha_k$ and $\beta_{i1}, \ldots, \beta_{im_i}$, $i = 1, \ldots, k$ is disregarded. By iteration of this process, i.e. decomposing β_{ij} further we can obtain r-fold partitions, let us denote this number by $\rho(r,n)$. We have $\rho(1,n) = \rho(n)$ the number of ordinary partitions, and, by definition, we consider that $\rho(0,n) = 1$, $n \in \mathbb{N}$. The following recursive relation holds [10]

$$1 + \sum_{n=1}^{\infty} \rho(r,n) t^n = \prod_{m=1}^{\infty} \frac{1}{(1 - t^m)^{\rho(r-1,m)}}, \quad r \geqslant 1.$$

By applying above theorem we obtain an asymptotic

Corollary 2.1.

$$\rho(r,n) = \exp\left(\left(\frac{\pi^2}{6} + o(1) \right) \frac{n}{\ln^{(r-1)} n} \right), \quad r \geqslant 2.$$

For $r = 2$ much more precise asymptotic is given in [10], but for $r \geqslant 3$ this result is new.

Theorem 2.1 enables us to obtain asymptotics for growth of free polynilpotent Lie algebras and groups (Table 1).

Theorem 2.2 (Petrogradsky [23,26]). *Let $\boldsymbol{V} = \boldsymbol{N}_{s_q} \cdots \boldsymbol{N}_{s_1}$, $q \geqslant 2$ be a polynilpotent variety of Lie algebras. Suppose that $L = F(\boldsymbol{V}, k)$ is its free algebra of rank k, freely generated* by $X = \{x_1, \ldots, x_k\}$. *Then there exists an infinitesimal such that*

$$\gamma_L(X,n) = \begin{cases} \dfrac{A + o(1)}{N!} n^N, & q = 2, \\ \exp\left((C + o(1))\, n^{N/(N+1)} \right), & q = 3, \\ \exp\left((B^{1/N} + o(1)) \dfrac{n}{(\ln^{(q-3)} n)^{1/N}} \right), & q \geqslant 4; \end{cases}$$

[1] The author is grateful to V. Liskovets for this reference.

Table 1
Asymptotics for polynilpotent varieties of Lie algebras

	Affine algebras of rank k		Codimension growth	
Variety $\boldsymbol{M}$	Growth function $\gamma_{F(\boldsymbol{M},k)}(n)$	Generating function $\mathscr{H}_k(\boldsymbol{M},z)$	Growth function $c_n(\boldsymbol{M})$	Complexity function $\mathscr{C}(\boldsymbol{M},z)$
Polynilpotent varieties				
$\boldsymbol{N}_a$	$\dim F(\boldsymbol{N}_a,k)$	$\sum_{i=1}^{a}\psi_k(i)z^i$	0	$=\sum_{i=1}^{a}\frac{z^i}{i}$
$\boldsymbol{N}_b\boldsymbol{N}_a$	$\frac{A}{N!}n^N$	$\frac{A}{(1-z)^N}$	$b^{n/a}(n!)^{\frac{a-1}{a}}$	$\exp(\frac{b}{a}z^a)$
	$N=b\dim F(\boldsymbol{N}_a,k)$	$A=\frac{1}{b}\left(\frac{k-1}{\prod_{q=2}^{a}q^{\psi_k(q)}}\right)^b$		
$\boldsymbol{N}_c\boldsymbol{N}_b\boldsymbol{N}_a$	$\exp(Cn^{\frac{N}{N+1}})$	$\exp\left(\frac{B}{(1-z)^N}\right)$	$\frac{n!\cdot\theta^n}{(\ln n)^{n/a}}$	$\exp^{(2)}(\frac{b}{a}z^a)$
	$N=b\dim F(\boldsymbol{N}_a,k)$	$B=cA\zeta(N+1)$	$\theta=(\frac{b}{a})^{1/a}$	
	$C=(1+\frac{1}{N})(BN)^{\frac{1}{1+N}}$			
$\boldsymbol{N}_{s_q}\cdots\boldsymbol{N}_{s_1}$	$\exp\left(\frac{n\cdot B^{1/N}}{(\ln^{(q-3)}n)^{1/N}}\right)$	$\exp^{(q-2)}\left(\frac{B}{(1-z)^N}\right)$	$\frac{n!\cdot\theta^n}{(\ln^{(q-2)}n)^{n/s_1}}$	$\exp^{(q-1)}(\frac{s_2}{s_1}z^{s_1})$
$q\geqslant 4$	$N=s_2\dim F(\boldsymbol{N}_{s_1},k)$	$B=s_3A\zeta(N+1)$	$\theta=(\frac{s_2}{s_1})^{1/s_1}$	

where the constants are as follows:

$$N=s_2\dim_K F(\boldsymbol{N}_{s_1},k),\ A=\frac{1}{s_2}\left(\frac{k-1}{\prod_{q=2}^{s_1}q^{\psi_k(q)}}\right)^{s_2},\qquad B=s_3A\zeta(N+1),$$
$$C=\left(1+\frac{1}{N}\right)(BN)^{1/(1+N)}.$$

Remark. In Theorem 1.3 q-dimensions do not depend on the choice of the generating set. But here the constants are valid only for the standard generating set X.

Suppose that we have a group G, denote by $\gamma_n(G)$, $n=1,2,\ldots$ terms of the lower central series. Then we can construct in a standard way a Lie algebra [13]:

$$L_K(G)=\bigoplus_{n=1}^{\infty}(\gamma_n(G)/\gamma_{n+1}(G))\otimes_{\mathbb{Z}}K.$$

If G was a free polynilpotent group, then $L_K(G)$ is the free polynilpotent Lie algebra of the same rank and with the same tuple [30]. This allows us to derive the following.

Corollary 2.2 (Petrogradsky [26]). *Let $G=G(\boldsymbol{N}_{s_q}\ldots\boldsymbol{N}_{s_1},k)$, $q\geqslant 2$ be the free polynilpotent group of rank k. Let $\gamma_n(G)$, $n=1,2,\ldots,$ be the terms of the lower central series.*

1. *Suppose that* $b_n = \operatorname{rank} \gamma_n(G)/\gamma_{n+1}(G)$. *Then*
$$b_n = \begin{cases} \dfrac{A+\mathrm{o}(1)}{(N-1)!}\, n^{N-1}, & q=2; \\ \exp((C+\mathrm{o}(1))\, n^{N/N+1}), & q=3; \\ \exp\left((B^{1/N}+\mathrm{o}(1))\, \dfrac{n}{(\ln^{(q-3)} n)^{1/N}}\right), & q\geqslant 4; \end{cases}$$
where the constants N, A, B, C *are the same as in Theorem* 2.2.
2. *Suppose that* K *is the field of any characteristic,* Δ *is the augmentation ideal in* $K[G]$, *and* $a_n = \dim_K \Delta^n/\Delta^{n+1}$, $n = 0, 1, 2, \ldots$. *Then* $\{a_n \mid n = 0, 1, 2, \ldots\}$ *do not depend on* K *and*
$$a_n = \begin{cases} \exp\left((C+\mathrm{o}(1))\, n^{N/(N+1)}\right), & q=2; \\ \exp\left((B^{1/N}+\mathrm{o}(1))\, \dfrac{n}{(\ln^{(q-3)} n)^{1/N}}\right), & q\geqslant 3; \end{cases}$$
where N, A, B, C *are computed as in Theorem* 2.2 (*in case* $q=2$ *we take* $s_3 = 1$).

M.I. Kargapolov raised the problem 2.18 in [11] to describe the lower central series ranks for a free polynilpotent finitely generated group. Exact recursive formulae were given in [6]. But those formulae do not give any idea about the character of the growth of these ranks. We suggest another answer to this problem by specifying the asymptotic behavior of these ranks. Also, Theorem 2.2 and Corollary 2.2 may be viewed as an analogue of the Witt formula for free Lie algebras and groups. Let us recall this formula. Suppose that $L = \bigoplus_{n=1}^{\infty} L_n$ is a free Lie algebra of rank k and G is a free group of rank k, then the lower central series factors $\gamma_n(G)/\gamma_{n+1}(G)$ are free abelian groups and their ranks are equal to
$$\operatorname{rank}\ \gamma_n(G)/\gamma_{n+1}(G) = \dim_K L_n = \psi_k(n) = \frac{1}{n}\sum_{a|n} k^a \mu\left(\frac{n}{a}\right) \approx \frac{k^n}{n}.$$

In [26] we study two kinds of p-central series for polynilpotent groups. These series are important instruments in the recent positive solution of the Restricted Burnside problem by Zelmanov [35]. First, we consider *lower exponent* p-*central series*
$$G = G_1 \supset G_2 \supset \cdots; \qquad G_1 = G, \quad G_{i+1} = (G_i, G)\cdot G_i^p,\ i\in\mathbb{N};$$
$$\tilde{L}(G) = \bigoplus_{i=1}^{\infty} G_i/G_{i+1}; \qquad \tilde{b}_n = \dim_{\mathbf{F}_p} G_n/G_{n+1}.$$
the associated Lie ring $\tilde{L}(G)$ turns out a Lie algebra over the field $\mathbb{F}_p$ of residues modulo p. The asymptotic of $\tilde{b}_n$ differs to that of $b_n = \operatorname{rank}\gamma_n(G)/\gamma_{n+1}(G)$ in the above corollary only for $q=2$, namely the degree of a polynomial increases by one.

Second, we consider *lower* p-*central series* due to Brauer, Jennings and Zassenhaus [13]:
$$G = D_{1,p} \supset D_{2,p} \supset \cdots, \qquad D_{n,p} = D_{n,p}(G) = \prod_{m p^j \geqslant n} (\gamma_m(G))^{p^j}, \quad n\in\mathbb{N}.$$

The associated Lie ring is a Lie p-algebra over $\mathbb{F}_p$ (see, e.g. [19]); also let as consider corresponding Hilbert–Poincaré series:

$$L_{\mathbb{F}_p}(G) = \bigoplus_{n=1}^{\infty} D_{n,p}/D_{n+1,p},$$

$$\mathscr{H}(L_{\mathbb{F}_p}(G), t) = \sum_{n=1}^{\infty} d_n t^n, \quad d_n = \dim_{\mathbb{F}_p}(D_{n,p}/D_{n+1,p}), \quad n \in \mathbb{N}.$$

Here also minor changes in asymptotics happen only for $q=2$. In this case we have an interesting example of polynomially growing coefficients d_n whereas the respective Hilbert–Poincaré function $\mathscr{H}(L_{\mathbb{F}_p}(G), t)$ is not rational.

3. Hilbert–Poincaré series for some Lie algebras

An important step in our considerations is a study of a behaviour of a Hilbert–Poincaré function $\mathscr{H}(F(\boldsymbol{V}, k), t)$ while $t \to 1-0$, where $\boldsymbol{V}$ is a free polynilpotent variety.

Suppose that coefficients of one series $\psi(t) = \sum_{n=0}^{\infty} a_n t^n$ are obtained from coefficients of another series $\phi(t) = \sum_{n=1}^{\infty} b_n t^n$ by relation (2); in this case we write $\psi(t) = \mathscr{E}(\phi(t))$.

Lemma 3.1. $\mathscr{E}(\phi(t)) = \exp(\sum_{m=1}^{\infty} (1/m)\phi(t^m))$.

In terms of $\mathscr{E}$ we can find the Hilbert–Poincaré series of a free solvable Lie algebra.

Lemma 3.2 (Petrogradsky [26]). *Let $L = F(\boldsymbol{A}^q, k)$ be the free solvable Lie algebra of rank k. Define recursively functions $h_0(t) = kt$, $h_{i+1}(t) = 1 + (h_i(t) - 1)\mathscr{E}(h_i(t))$, $i = 1, 2, \ldots$. Then*

$$\mathscr{H}(L, t) = h_0(t) + \cdots + h_{q-1}(t).$$

For some cases we can compute Hilbert–Poincaré series explicitly.

Lemma 3.3 (Petrogradsky [22]). *Let $L = F(\boldsymbol{N}_c\boldsymbol{A}, k)$. Denote by $\mu(n)$ the Möbius function. Then*

$$\mathscr{H}(L, t) = kt + \sum_{m=1}^{c} \frac{1}{m} \sum_{a|m} \mu\left(\frac{m}{a}\right)\left(1 + \frac{kt^{m/a} - 1}{(1 - t^{m/a})^k}\right)^a,$$

In case of an arbitrary polynilpotent variety $\boldsymbol{V}$ we can only evaluate the growth of $\mathscr{H}(F(\boldsymbol{V}, k), t)$ while $t \to 1-0$.

Theorem 3.1 (Petrogradsky [26]). *Let $\boldsymbol{V}=\boldsymbol{N}_{s_q}\cdots\boldsymbol{N}_{s_1}$, $q\geqslant 2$ be the polynilpotent variety. Suppose that $L=F(\boldsymbol{V},k)$ is its free algebra of rank k. Then*

$$\lim_{t\to 1-0}(1-t)^N\mathscr{H}(L,t)=A,\quad q=2,$$

$$\lim_{t\to 1-0}(1-t)^N\ln^{(q-2)}(\mathscr{H}(L,t))=s_3\zeta(N+1)A,\quad q\geqslant 3;$$

where N, A are the same as in Theorem 2.2.

Suppose that $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is a series that converges for all $z\in\mathbb{C}$, $|z|<1$. Denote $\mathsf{M}_f(r)=\max_{|z|=r}|f(z)|$, for all $0\leqslant r<1$. We prove some facts, similar to Theorem 6.4 below, on connection between a growth of coefficients a_n and a growth of the function $\mathsf{M}_f(r)$, while $r\to 1-0$; where $f(t)$ behaves as the Hilbert–Poincaré series in Theorem 3.1. This is one of steps in proofs of Theorems 2.2 and 2.1.

4. Lie and associative algebras with nonintegeral q-dimensions

By Theorem 1.3, q-dimensions for free polynilpotent Lie algebras are integers. For Gelfand–Kirillov dimension and superdimension of associative algebras the following facts are known.

Theorem 4.1 (Borho and Kraft [5]). *For any $\sigma\in[2,+\infty)$ there exists a two-generated associative algebra A with* $\underline{\mathrm{Dim}}^2 A=\mathrm{Dim}^2 A=\sigma$.

Also the Gelfand–Kirillov dimension may equal 1 and 0 (latter are finite-dimensional algebras over the field); and there is a trivial gap: $\mathrm{Dim}^2 A\notin(0,1)$. It is more interesting that there is another gap:

Theorem 4.2 (Bergman [12]). *Gelfand–Kirillov dimension of an associative algebra does not belong to the interval* $(1,2)$.

Theorem 4.3 (Borho and Kraft [5]). *For any $\sigma\in(0,1]$ there exists a two-generated associative algebra A with* $\mathrm{Dim}^3 A=\sigma$.

So, the Gelfand–Kirillov dimension of associative algebras can take the values $0,1,[2,\infty]$. It is interesting that for Jordan algebras we have the same picture [14,15].

For q-dimensions of Lie algebras we have the following.

Theorem 4.4 (Petrogradsky [24]). *For levels $q=2,3\dots$ and any $\sigma\in[1,+\infty)$ there exists a two-generated Lie algebra $H\in\boldsymbol{A}^{q-2}\boldsymbol{N}_s\boldsymbol{A}$, $s=\lceil\sigma\rceil$ with* $\mathrm{Dim}^q L=\underline{\mathrm{Dim}}^q L=\sigma$; *where $\lceil\alpha\rceil$ for $\alpha\in\mathbb{R}$ denotes the least integer greater or equal than α.*

Remark. In the case $q=2$ it is interesting that, unlike associative algebras, Lie algebras have no gap $(1,2)$. As for higher levels $q=3,4,\ldots$, we have an open problem whether for all $\sigma\in(0,1)$ there exist Lie algebras with $\mathrm{Dim}^q L=\underline{\mathrm{Dim}}^q L=\sigma$.

Corollary 4.1 (Petrogradsky [24]). *For levels $q=1,2,3,\ldots$ there exist finitely generated associative algebras A with the following nonzero finite q-dimensions*:

$$\underline{\mathrm{DIM}}^q A=\mathrm{Dim}^q A\in\begin{cases}\mathbb{N}, & q=1,\\ 1,[2,+\infty) & q=2,\\ (0,+\infty) & q=3,\\ [1,+\infty) & q=4,5,\ldots.\end{cases}$$

Remark. For high levels $q=4,5,\ldots$ we also have intervals $\sigma\in(0,1)$ for further study.

Lemma 4.1 (Petrogradsky [24]). *Let $L\in\boldsymbol{A}^2$ be an arbitrary finitely generated metabelian Lie algebra. Then* $\mathrm{Dim}^2 L=\underline{\mathrm{Dim}}^2 L$ *is an integer*.

5. Asymptotics for codimension growth

Let $\boldsymbol{V}$ be a variety of Lie algebras and $F(\boldsymbol{V},X)$ be its free algebra freely generated by a countable set $X=\{x_i\mid i\in\mathbb{N}\}$. Let $P_n(\boldsymbol{V})\subset F(\boldsymbol{V},X)$ be a subspace of all multilinear elements in $\{x_1,\ldots,x_n\}$ and consider its dimension $c_n(\boldsymbol{M})=\dim_K P_n(\boldsymbol{V})$. It is not difficult to see that for another distinct elements $x_{i_1},\ldots,x_{i_n}\in X$ we obtain the same number. The sequence $c_n(\boldsymbol{V})$, $n=1,2,\ldots$ is called *codimension growth* and is an important characteristic of a variety. In case of associative algebras the basic fact is as follows:

Theorem 5.1 (Regev [28]). *Let $\boldsymbol{V}$ be a variety of associative algebras with a nontrivial identical relation of degree d. Then $c_n(\boldsymbol{V})\leqslant C^n$, $n\in\mathbb{N}$; where $C=(d-1)^2$.*

Codimension growths of Lie varieties not exceeding exponential growth have been studied extensively, see review [16]. Unlike associative case, the growth of a rather small variety $\boldsymbol{AN}_2$ is overexponential [34]. So, we have a vast area of overexponential growths for Lie algebras, which stretches between the exponent and the factorial. In order to explore this area we consider a series of functions of a natural argument $\Psi_\alpha^q(n)$, $q=2,3,\ldots$ with a real parameter α:

$$\Psi_\alpha^q(n)=\begin{cases}(n!)^{(\alpha-1)/\alpha}, & \alpha>1,\ q=2;\\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/\alpha}}, & \alpha>0,\ q=3,4\ldots\end{cases}\tag{3}$$

More fine scale is formed by a series of functions with two real parameters α, β:

$$\Psi^q_{\alpha,\beta}(n) = \begin{cases} (n!)^{(\alpha-1)/\alpha}\beta^{n/\alpha}, \ \alpha \geqslant 1, \ \beta > 0; \ \ q=2; \\ \dfrac{n! \cdot (\beta/\alpha)^{n/\alpha}}{(\ln^{(q-2)} n)^{n/\alpha}}, \ \ \alpha > 0, \ \beta > 0; \ q = 3,4\ldots \end{cases} \tag{4}$$

The following is the more precise asymptotic than that in [21].

Theorem 5.2 (Petrogradsky [25]). *Let $V = N_{s_q} \cdots N_{s_1}$, $q \geqslant 2$ be a polynilpotent variety of Lie algebras. Then*

$$c_n(V) = \Psi^q_{s_1,s_2+o(1)}(n) = \begin{cases} (n!)^{(s_1-1)/s_1}(s_2+o(1))^{n/s_1}; & q=2; \\ \dfrac{n!}{(\ln^{(q-2)} n)^{n/s_1}} \left(\dfrac{s_2+o(1)}{s_1}\right)^{n/s_1}; & q=3,4\ldots \end{cases}$$

In [21] the result was obtained by direct evaluations of some sums of coefficients. But in Theorem 5.2 another important instrument is the *complexity function* of a variety (see below). By the following result we see that the scale, suggested above, is rather complete:

Theorem 5.3 (Petrogradsky [25]). *Let V be a variety of Lie algebras satisfying some nontrivial identity of degree $m > 3$. Then there exists infinitesimal* (*depending only on m*), *such that*

$$c_n(V) \leqslant \frac{n!}{(\ln^{(m-3)} n)^n}(1+o(1))^n.$$

Remark. For a codimension growth not exceeding exponent it is typical to use the Young diagrams [16]. But even for $V = AN_2$ by Theorem 5.2 we have $c_n(V) = \sqrt{n!}(1+o(1))^n$. It is easy to see that for an irreducible representation π of the symmetric group S_n one has $\dim_K \pi < \sqrt{n!}$; also the number of irreducible representations is rather small:

$$\rho(n) \approx \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}).$$

So, from the dimension viewpoint, S_n-module $P_n(V)$ may contain all irreducible S_n-modules. Thus, for Lie algebras we need other technique, than the Young diagrams.

6. Complexity functions for some varieties

Recall that for a free Lie algebra one has $c_n(F) = (n-1)!$. For an arbitrary variety V the *complexity function* is defined as follows:

$$\mathbb{C}(V,z) = \sum_{n=1}^{\infty} \frac{c_n(V)}{n!} z^n, \quad z \in \mathbb{C}.$$

Complexity functions were introduced by Yu.P. Razmyslow in a wider context of varieties of Lie pairs [29]. An upper bound for the codimension growth of an arbitrary variety of Lie algebras is formulated in a nice way:

Theorem 6.1 (Razmyslow [29]). *Suppose that $\boldsymbol{V}$ is a nontrivial variety of Lie algebras. Then $\mathscr{C}(\boldsymbol{V},z)$ is an entire function of a complex variable.*

For some varieties it is possible to compute the complexity function explicitly.

Lemma 6.1 (Petrogradsky [21]). *For the variety of solvable Lie algebras one has $\mathscr{C}(\boldsymbol{A}^q,z)=\sigma_0(z)+\cdots+\sigma_{q-1}(z)$, where $\sigma_0(z)=z$, $\sigma_{i+1}=1+(\sigma_i(z)-1)\exp(\sigma_i(z))$, $i=0,1,2,\ldots$.*

Lemma 6.2 (Petrogradsky [21]). $\mathscr{C}(\boldsymbol{N}_c\boldsymbol{A},z)=z+\sum_{i=1}^{c}(1/i)(1+\exp(z)(z-1))^i$.

We write $f(n)\approx g(n)$ iff $\lim_{n\to\infty}f(n)/g(n)=1$. Complexity functions are useful for computations of the codimension growth. For example.

Corollary 6.1. $c_n(\boldsymbol{N}_c\boldsymbol{A})\approx c^{n-c-1}n^c$.

An important tool in our study is the following fact [9,21,25,29].

Lemma 6.3. *Suppose that $L=F(\boldsymbol{V},X)$ is a free algebra for some variety of Lie algebras. Then for its universal enveloping algebra it is possible to define a complexity function, moreover $\mathscr{C}(U(L),z)=\exp(\mathscr{C}(L,z))$.*

It is interesting to find an analogue of Lemma 6.1 for polynilpotent varieties. In this case we can only prove the following.

Theorem 6.2 (Petrogradsky [25]). *Let $\boldsymbol{V}=\boldsymbol{N}_{s_q}\cdots\boldsymbol{N}_{s_1}$, $q\geqslant 2$ be a polynilpotent variety and $f(z)=\mathscr{C}(\boldsymbol{V},z)$. Then*

$$\lim_{r\to\infty}\frac{\ln^{(q-1)}\mathsf{M}_f(r)}{r^{s_1}}=\frac{s_2}{s_1}.$$

To make use of this fact we need some facts about entire functions. Let $f(z)$ be an entire function. Denote $\mathsf{M}_f(r)=\max_{|z|=r}|f(z)|$. Then the *order* and *type* (for the known order) are defined as [7]:

$$\operatorname{ord} f=\overline{\lim_{r\to\infty}}\frac{\ln\ln\mathsf{M}_f(r)}{\ln r};\qquad \operatorname{typ} f=\overline{\lim_{r\to\infty}}\frac{\ln\mathsf{M}_f(r)}{r^{\rho}}.$$

The following is a classical result.

Theorem 6.3 (Evgrafov [7, Section 3.2.3]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ and type σ. Then*

$$\overline{\lim_{n\to\infty}}\, n^{1/\rho} \sqrt[n]{|a_n|} = (\sigma e \rho)^{1/\rho}.$$

By analogy we can define the order and type of a level q, $q \in \mathbb{N}$, as

$$\operatorname{ord}_q f = \overline{\lim_{r\to\infty}} \frac{\ln^{(q+1)} \mathsf{M}_f(r)}{\ln r}; \qquad \operatorname{typ}_q f = \overline{\lim_{r\to\infty}} \frac{\ln^{(q)} \mathsf{M}_f(r)}{r^{\rho}}.$$

In these notations Theorem 6.2 implies

Corollary 6.2. *Let $V = N_{s_q} \cdots N_{s_1}$, $q \geqslant 2$, be a polynilpotent Lie variety and $f(z) = \mathscr{C}(V, z)$ be its complexity function. Then* $\operatorname{ord}_{q-1} f(z) = s_1$, $\operatorname{typ}_{q-1} f(z) = s_2/s_1$.

In particular, we see that for the polynilpotent variety with $q > 2$ the complexity function has an infinite ordinary order. To study such functions we prove an analogue of Theorem 6.3.

Theorem 6.4 (Petrogradsky [25]). *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function. Then for any fixed numbers $q \in \mathbb{N}$, $q \geqslant 3$; $\lambda > 0$ one has*

$$\overline{\lim_{r\to\infty}} \frac{\ln^{(q-1)} \mathsf{M}_f(r)}{r^{\lambda}} = \overline{\lim_{n\to\infty}} \ln^{(q-2)} n\, |a_n|^{\lambda/n}.$$

We use this result along with Theorem 6.2 to prove Theorem 5.2.

Further study of complexity functions of Lie varieties see in [27].

References

[1] G.E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, MA, 1976.

[2] Yu.A. Bahturin, Identical Relations in Lie Algebras, VNV Science Press, Utrext, 1987.

[4] A.E. Berezniy, Discrete subexponential groups, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 123 (1983) 155–166; (Engl. transl.) J. Soviet Math. 28 (1985) 570–579.

[5] W. Borho, H. Kraft, Über die Gelfand–Kirillov-Dimension, Math. Ann. 220 (1) (1976) 1–24.

[6] G.P. Egorychev, Integral representation and the computation of combinatorial sums, Transl. Math. Monogr., Vol. 59, Amer. Math. Soc., Providence, RI, 1984.

[7] M.A. Evgrafov, Asymptotic Estimates and Entire Functions, Gordon and Breach, New York, 1961.

[8] I.M. Gelfand, A.A. Kirillov, Sur les corps lies aux algèbres enveloppantes des algèbres de Lie, Inst. Hautes Études Sci. Publ. Math. 31 (1966) 509–523.

[9] I.P. Goulden, D.M. Jackson, Combinatorial Enumeration, Wiley, New York, 1983.

[10] R. Kaneiwa, An asymptotic formula for Cayley's double partition function $p(2; n)$, Tokyo. J. Math. 2 (1) (1979) 137–158.

[11] Kourovskaya tetrad, Unsolved problems in group theory, Nauka, Novosibirsk, 1967.

[12] G.R. Krause, T.H. Lenagan, Growth of algebras and Gelfand–Kirillov dimension, Pitman, Boston, 1985.

[13] M. Lazard, Sur les groupes nilpotents et les anneaux de Lie, Ann. Sci. École Norm. Sup. 71 (1954) 101–190.

[14] C. Martinez, E. Zelmanov, Jordan algebras of Gel'fand–Kirillov dimension one, J. Algebra 180 (1) (1996) 211–238.

[15] C. Martinez, Gel'fand–Kirillov dimension in Jordan algebras, Trans. Amer. Math. Soc. 348 (1) (1996) 119–126.
[16] S.P. Mishchenko, Growth of varieties of Lie algebras, Usp. Mat. Nauk 45 (6) (1990) 25–45; (Engl. transl.) Russian Math. Surveys 45 (6) (1990) 27–52.
[18] A.I. Molev, On growth of some algebras of vector fields, Usp. Mat. Nauk 41 (2) (1986) 205–206; (Engl. transl.) Russian Math. Surveys 41 (2) (1986) 213–214.
[19] I.B.S. Passi, Group Rings and Their Augmentation Ideals, Lecture Notes in Mathematics, Vol. 715, Springer, Berlin/Heidelberg/New York, 1979.
[20] V.M. Petrogradsky, On some types of intermediate growth in Lie algebras, Usp. Mat. Nauk 48 (5) (1993) 181–182; (Engl. transl.) Russian Math. Surveys 48 (5) (1993) 181–182.
[21] V.M. Petrogradsky, On types of overexponential growth of identities in Lie PI-algebras (Russian), Fundamentalnaya i Prikladnaya Matematika 1 (4) (1995) 989–1007.
[22] V.M. Petrogradsky, Intermediate growth in Lie algebras and their enveloping algebras, J. Algebra 179 (1996) 459–482.
[23] V.M. Petrogradsky, Growth in polynilpotent Lie algebras, Thesises of Ring Theory Conference, Miskols, Hungary, July 1996, pp. 46–47.
[24] V.M. Petrogradsky, On Lie Algebras with nonintegral q-dimensions, Proc. Amer. Math. Soc. 125 (3) (1997) 649–656.
[25] V.M. Petrogradsky, Growth of polynilpotent varieties of Lie Algebras and fast increasing entire functions, Mat. Sb. 188 (1997) (6) 119–138; Engl. transl. Russian Acad. Sci. Sb. Math. 188 (1997) (6) 913–931.
[26] V.M. Petrogradsky, Growth of finitely generated polynilpotent Lie algebras and groups, generalized partitions, and functions analytic in the unit circle, Internat. J. Algebra Comput. 9 (2) (1999) 179–212.
[27] V.M. Petrogradsky, Exponential generating functions and complexity of Lie varieties (1999).
[28] A. Regev, Existence of polynomial identities in $A \otimes B$, Bull. Amer. Math. Soc. 77 (6) (1971) 1067–1069.
[29] Yu.P. Razmyslow, Identities of Algebras and their Representations, Transl. Math. Monogr. Vol. 138, Amer. Math. Soc., Providence, RI, 1994.
[30] A.L. Shmel'kin, Free polynilpotent groups, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1) (1964) 91–122; (English transl.) Amer. Math. Soc. Transl. Ser. 2 (55) (1966) 270–304.
[31] M.K. Smith, Universal enveloping algebras with subexponential but not polynomially bounded growth, Proc. Amer. Math. Soc. 60 (1976) 22–24.
[33] V.A. Ufnarovskiy, On growth of algebras, Vestnik Moskov. Univ. Ser. 1 Mat. Mekh. (4) (1978) 59–65; (Engl. transl.) Moscow Univ. Math. Bull. 33 (4) (1978) 47–52.
[34] I.B. Volichenko, On variety $\boldsymbol{AN}_2$ over field of zero characteristic, Dokl. Akad. Nauk Belarusi XXV (12) 1063–1066.
[35] E.I. Zelmanov, On the restricted Burnside problem, Proceedings of the International Congress of Mathematicians, Vols. I and II (Kyoto, 1990) 395–402, Math. Soc. Japan, Tokyo, 1991.