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Extension of Fourier algebra homomorphisms to duals of algebras of uniformly continuous functionals

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Abstract

For a locally compact group G, let $\mathcal{X}_G$ be one of the following introverted subspaces of $VN(G)$: $UCB(\widehat{G})$, the C*-algebra of uniformly continuous functionals on $A(G)$; $W(\widehat{G})$, the space of weakly almost periodic functionals on $A(G)$; or $M^*_\rho(G)$, the C*-algebra generated by the left regular representation on the measure algebra of G. We discuss the extension of homomorphisms of (reduced) Fourier–Stieltjes algebras on G and H to cb-norm preserving, weak*–weak*-continuous homomorphisms of $\mathcal{X}^*_G$ into $\mathcal{X}^*_H$, where $(\mathcal{X}_G, \mathcal{X}_H)$ is one of the pairs $(UCB(\widehat{G}), UCB(\widehat{H}))$, $(W(\widehat{G}), W(\widehat{H}))$, or $(M^*_\rho(G), M^*_\rho(H))$. When G is amenable, these extensions are characterized in terms of piecewise affine maps.

Keywords: Fourier algebra; Fourier–Stieltjes algebra; Uniformly continuous functionals; Introverted subspace; Completely bounded maps; Piecewise affine maps

1. Introduction

One of the last century's most striking achievements in abstract harmonic analysis is Paul Cohen's characterization of the homomorphisms from a group algebra $L^1(G)$ into a measure

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algebra $M(H)$, when G and H are locally compact abelian groups. Cohen's solution to the homomorphism problem more precisely (but equivalently) describes the homomorphisms from $A(\widehat{G})$ into $B(\widehat{H})$, the Fourier and Fourier–Stieltjes transforms of $L^1(G)$ and $M(H)$ on the dual groups $\widehat{G}$ and $\widehat{H}$; the description is phrased in terms of continuous piecewise affine maps from $\widehat{H}$ into $\widehat{G}$ [3,28].

Eymard defined versions of $A(G)$, the Fourier algebra of G, and $B(H)$, the Fourier–Stieltjes algebra of H, for any locally compact groups G and H [8] and, naturally, mathematicians have worked to generalize Cohen's theorem to describe the homomorphisms between these algebras. For arbitrary (bounded) homomorphisms $\varphi : A(G) \to B(H)$, Host widely extended Cohen's theorem to the situation in which G contains an abelian subgroup of finite index and H is any locally compact group [14] (also [24], however Lefranc never published his proofs).

Z.-J. Ruan first studied $A(G)$ as an operator space in his influential paper [27], and it has since become clear that it is often essential to consider this operator space structure on $A(G)$. As but one of many examples which justify this statement, the best known generalization of the results of Cohen and Host, due to the first-named author and Nico Spronk, asserts that when G is amenable and H is any locally compact group, every *completely bounded* homomorphism $\varphi : A(G) \to B(H)$ is associated with a piecewise affine continuous map (and conversely) [16,17]. Evidence suggests that this result is best possible.

A problem that remains open, even for abelian groups, asks for a description of all (completely bounded) homomorphisms $\varphi : B(G) \to B(H)$. In [18], assuming that G is amenable, we were able to describe all such homomorphisms that are associated with a continuous, piecewise affine map $\alpha : Y \subset H \to G$. As the dual Banach space of the group C^*-algebra $C^*(G)$, $B(G)$ has a weak*-topology and, when G is amenable, we have also shown that the w^*–w^*-continuous completely bounded homomorphisms $\varphi : B(G) \to B(H)$ are precisely those homomorphisms associated with continuous, piecewise affine, *open* maps [18].

Open maps may be seen as a rarity, so this may be interpreted as a negative result. However, w^*–w^*-continuity is a very attractive property, and the purpose of this paper is to show that for amenable groups, any homomorphism $\varphi : B(G) \to B(H)$ can be extended to a w^*–w^*-continuous homomorphism $\varphi : \mathfrak{X}_G^* \to \mathfrak{X}_H^*$ for a variety of highly manageable and well-studied Banach algebras $\mathfrak{X}_G^*$ and $\mathfrak{X}_H^*$, which respectively contain copies of $B(G)$ and $B(H)$; $\mathfrak{X}_H^*$ can always be chosen to be $B(H)$ itself. For arbitrary locally compact groups, we will consider homomorphisms $\varphi : B_\rho(G) \to B_\rho(H)$ of reduced Fourier–Stieltjes algebras and note that G is known to be amenable exactly when $B_\rho(G) = B(G)$.

More precisely, let $\mathfrak{X}_G$ be any of the following topologically invariant and introverted subspaces of $VN(G)$: $UCB(\widehat{G})$, the C*-algebra of uniformly continuous functionals on $A(G)$; $W(\widehat{G})$, the space of weakly almost periodic functionals on $A(G)$; or $M_\rho^*(G)$, the C*-algebra generated by the left regular representation on the measure algebra of G. Then, as shown by Lau [19] and Lau and Losert [21], $\mathfrak{X}_G^*$ is a Banach algebra containing an isometric copy of $B_\rho(G)$. Let $\varphi : B_\rho(G) \to B_\rho(H)$ be any homomorphism associated with a piecewise affine map α, and let $(\mathcal{X}_G, \mathcal{X}_H)$ be any of the pairs $(UCB(\widehat{G}), UCB(\widehat{H}))$, $(W(\widehat{G}), W(\widehat{H}))$, or $(M_\rho^*(G), M_\rho^*(H))$. In Sections 4 and 5 we prove that φ has a weak*–weak*-continuous homomorphic extension mapping $\mathcal{X}_G^*$ into $\mathcal{X}_H^*$ (or $\mathcal{X}_G^*$ into $B_\rho(H)$) which we explicitly describe in terms of the piecewise affine map α.

All spaces are completely contractive Banach algebras with respect to a natural operator space structure. When G is weakly amenable (respectively amenable) in Sections 3 and 5 we show that the embedding of $B_\rho(G)$ into $\mathfrak{X}_G^*$ is completely bounded (completely isometric) and we prove that our extensions of φ are completely bounded (cb-norm preserving). When G is amenable, we completely characterize the w^*–w^*-continuous, completely bounded homomorphisms $\Phi : \mathfrak{X}_G^* \to \mathfrak{X}_H^*$ which map $A(G)$ into $B_\rho(H)$, in terms of piecewise affine maps. The same is accomplished for w^*–w^*-continuous homomorphisms $\widetilde{\Phi} : \mathfrak{X}_G^* \to B_\rho(H)$ which are completely bounded on $A(G)$.

In Section 6, we describe those piecewise affine maps $\alpha : Y \subset H \to G$ whose associated homomorphisms map into $B_\rho(H)$. It seems natural to wonder which operators $\kappa : C_\rho^*(H) \to C_\rho^*(G)$ between reduced group C^*-algebras, dualize to give homomorphisms $\kappa^* : B_\rho(G) \to B_\rho(H)$ of reduced Fourier–Stieltjes algebras. In Section 7, these operators are characterized in terms of a certain intertwining property.

Isometric isomorphisms of $UCB(\widehat{G})^*$ onto $UCB(\widehat{H})^*$ and dually, of $LUC(G)^*$ into $LUC(H)^*$ where $LUC(G)$ denotes the usual C^*-algebra of left uniformly continuous functions on G, have been studied by Lau, Losert [21], and Ghahramani, Lau, Losert [10]. These authors, and several others, have also studied isometric isomorphisms between a variety of second duals of Banach algebras on locally compact groups. As well as being a natural continuation of Cohen's article, this paper may be seen as complementary to these other works.

2. Preliminaries

Throughout this paper, G and H are locally compact groups with Haar measure dx. The group and measure algebras of G are $L^1(G)$ and $M(G)$. If $F(G)$ is any collection of continuous functions on G, we let $F_c(G)$ denote the set of compactly supported functions in $F(G)$. If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ denotes the algebra of all bounded operators on $\mathcal{H}$.

Let $P(G)$ be the set of all continuous positive definite functions on G; functions in $P(G)$ can be described as coefficient functions $\langle \pi(s)\xi | \xi \rangle$ $(s \in G)$ where $\{\pi, \mathcal{H}\}$ is a (continuous, unitary) representation of G, $\xi \in \mathcal{H}$. The linear span, $B(G)$, of $P(G)$ can be identified with the dual of the group C^*-algebra, $C^*(G)$, the completion of $L^1(G)$ under its largest C^*-norm. With pointwise multiplication and the dual norm, $B(G)$ is a commutative regular semisimple Banach algebra. The Fourier algebra, $A(G)$, is a closed ideal in $B(G)$, also regular and semisimple, which may be defined as the closure of $B_c(G)$ in $B(G)$.

For a representation π of G, A_π is the closed linear span in $B(G)$ of all positive definite coefficient functions associated to π. The w^*-closure of A_π in $B(G)$ is denoted B_π and may be identified with the dual of $C_\pi^* = \pi(C^*(G))$. If $\{\rho_G, L^2(G)\}$ is the left regular representation of G, A_{ρ_G} is the Fourier algebra $A(G)$. We may use the notation $\rho = \rho_G$, and write $B_{\rho_G} = B_\rho(G)$, $C_{\rho_G}^* = C_\rho^*(G)$; $B_\rho(G)$ is a weak*-closed ideal in $B(G)$ called the *reduced Fourier–Stieltjes algebra* of G, and $C_\rho^*(G)$ is called the *reduced group C^*-algebra* of G. We have $C_\rho^*(G)^* = B_\rho(G)$ and $A(G)$ can be identified with the unique predual of $VN(G)$, the von Neumann subalgebra of $B(L^2(G))$ generated by ρ_G. The locally compact group G is amenable if and only if $B_\rho(G) = B(G)$ which is true exactly when $C_\rho^*(G)$ and $C^*(G)$ are $*$-isomorphic [26]. References for the Fourier algebra and other coefficient spaces are [1] and [8].

As a closed ideal in $B(G)$, $A(G)$ is a Banach $B(G)$-bimodule. Therefore $VN(G) = A(G)^*$ is a dual Banach $B(G)$-bimodule via

$$\langle T \cdot \phi, \psi \rangle = \langle \phi \cdot T, \psi \rangle = \langle T, \psi\phi \rangle \quad \big(\psi \in A(G), \phi \in B(G), T \in VN(G)\big).$$

In [11], E. Granirer defined the space of uniformly continuous functionals on $A(G)$, $UCB(\widehat{G})$, to be the (norm-)closure in $VN(G)$ of $A(G) \cdot VN(G)$. Let

$$UCB_c(\widehat{G}) = \big\{T \in VN(G):\ \mathrm{supp}(T) \text{ is compact}\big\},$$

where $a \in \mathrm{supp}(T)$, the support of T, if for each neighbourhood V of a there is some function $\phi \in A(G)$ such that $\mathrm{supp}(\phi) \subset V$ and $\langle T, \phi \rangle \neq 0$ [8, Definition 4.5]. Note that if $\psi \equiv 1$ on a neighbourhood of $\mathrm{supp}(T)$, then $T = \psi \cdot T$, so $UCB_{\mathrm{c}}(\widehat{G}) = A_{\mathrm{c}}(G) \cdot VN(G) \subset UCB(\widehat{G})$ (see [8, Proposition 4.8]). In the footnote to p. 37 of [11] (credited to C. Herz) it is observed that $UCB_{\mathrm{c}}(\widehat{G})$ is a linear subspace of $VN(G)$, and density of $A_{\mathrm{c}}(G)$ in $A(G)$ gives density of $UCB_{\mathrm{c}}(\widehat{G})$ in $UCB(\widehat{G})$. As well, $UCB(\widehat{G})$ is a C^*-algebra [12, Proposition 2]. If G is amenable, $UCB(\widehat{G}) = A(G) \cdot VN(G)$ [11] (and conversely [21]). When G is abelian, $UCB(\widehat{G})$ is (isomorphic to) the usual C^*-algebra of complex-valued uniformly continuous functions on $\widehat{G}$, the dual group of G.

A closed subspace $\mathfrak{X}_G$ of $VN(G)$ is *topologically invariant* if $\mathfrak{X}_G$ is an $A(G)$-submodule of $VN(G)$; $\mathfrak{X}_G$ is then *topologically introverted* if it also satisfies the property that $m \in \mathfrak{X}_G^*$ and $T \in \mathfrak{X}_G$ implies that $m_L(T) \in \mathfrak{X}_G$. Here $m_L(T) \in VN(G) = A(G)^*$ is defined by

$$\big\langle m_L(T), \psi \big\rangle = \langle m, T \cdot \psi \rangle \quad \big(\psi \in A(G)\big).$$

In this case, $\mathfrak{X}_G^*$ is a Banach algebra with respect to its left Arens product

$$\langle n \odot m, T \rangle = \big\langle n, m_L(T) \big\rangle, \quad n, m \in \mathfrak{X}_G^*,\ T \in \mathfrak{X}_G.$$

Topologically invariant and introverted subspaces of $VN(G)$ were first studied by Lau [19] where he showed, among many other things, that $C_\rho^*(G)$, $UCB(\widehat{G})$, $W(\widehat{G})$, and $AP(\widehat{G})$—the latter two spaces are defined in Section 5.2—are topologically invariant and introverted. Duals of introverted subspaces of $VN(G)$ are also studied in [5,15,21–23,25], as well as many other papers.

Any subspace $\mathfrak{X}_G$ of $VN(G)$ is an example of a concrete operator space and its dual is always given the associated canonical dual operator space structure. In fact $A(G)$, $B(G)$, and $UCB(\widehat{G})^*$ are examples of completely contractive Banach algebras; for example see [27] and [25]. Operator space theory has been, and continues to be extremely useful in abstract harmonic analysis [29]. A reference for the general theory of operator spaces is [7].

Let $E = xH_0$ be an open coset in H. A map $\alpha : E \to G$ is called *affine* if there is a homomorphism $\beta : H_0 \to G$ and $g_0 \in G$ such that $\alpha(xh) = g_0\beta(h)$ $(h \in H_0)$. One can show that E is a coset exactly when $EE^{-1}E \subset E$, and that $\alpha : E \to G$ is affine if and only if $\alpha(x_1 x_2^{-1} x_3) = \alpha(x_1)\alpha(x_2)^{-1}\alpha(x_3)$ whenever $x_1, x_2, x_3 \in E$ [16].

Let $\Omega_o(H)$ denote the collection of subsets $Y = E_0 \setminus (\bigcup_1^m E_k)$ of H such that E_0 is an open coset in H and $E_1, \ldots, E_m$ are open subcosets of infinite index in E_0; the smallest open coset containing Y is then E_0 and we use the notation $\mathit{Aff}(Y) = E_0$. If Y is in the ring of sets in H generated by the open cosets in H, then a map $\alpha : Y \subset H \to G$ is called *piecewise affine* if

(i) Y can be written as a disjoint union $Y = \bigcup_{i=1}^n Y_i$, with $Y_i \in \Omega_o(G)$; and
(ii) there is an affine map $\alpha_i : \mathit{Aff}(Y_i) \to G$ such that $\alpha|_{Y_i} = \alpha_i|_{Y_i}$ $(i = 1, \ldots, n)$.

If $\alpha : Y \subset H \to G$ is continuous and piecewise affine, then α induces an associated completely bounded homomorphism $j_\alpha : B(G) \to B(H)$ defined by

$$j_\alpha(\phi) = \begin{cases} \phi \circ \alpha & \text{on } Y, \\ 0 & \text{off } Y. \end{cases}$$

The homomorphism j_α is always completely bounded, and when G is amenable any homomorphism $\varphi : A(G) \to B(H)$ is of the form j_α for some piecewise affine, continuous map α [16,17].

The *multiplier* (*or strict*) *topology* on $B(G)$, τ_M, is the locally convex topology induced by the seminorms p_γ defined by $p_\gamma(\phi) = \|\phi\gamma\|_{A(G)}$ ($\gamma \in A(G), \phi \in B(G)$). When G is amenable, a homomorphism $\varphi : B(G) \to B(H)$ is of the form j_α for some piecewise affine continuous map α if and only if φ is completely bounded and τ_M–w^*-continuous on bounded subsets of $B(G)$, see [18]. The reader is referred to [16–18] for further details.

3. The embedding maps

In this section we provide the definitions and describe some properties of the isometric embedding maps

$$\pi : B_\rho(G) \hookrightarrow UCB(\widehat{G})^* \quad \text{and} \quad \theta : UCB(\widehat{G}) \hookrightarrow B_\rho(G)^*.$$

The maps π and θ were already defined by Lau [19] for amenable groups (where they were denoted by Q and Π, respectively). For any locally compact group, the map π was defined by Lau and Losert in [21], and a version of π was defined for algebras related to the generalized Fourier algebras $A_p(G)$ by Derighetti, Filali, and Sangani Monfared [5].

Define

$$\theta : A(G) \cdot VN(G) \to B_\rho(G)^* : T \to \widehat{T}$$

by

$$\langle \widehat{T}, \phi\rangle = \langle \widehat{\psi \cdot S}, \phi\rangle = \langle S, \phi\psi\rangle$$

when $T = \psi \cdot S$ with $\psi \in A(G)$, $S \in VN(G)$, and $\phi \in B_\rho(G)$. Define

$$\pi : B_\rho(G) \hookrightarrow UCB_c(\widehat{G})^* : \phi \to \widehat{\phi} \quad \text{by } \langle \widehat{\phi}, T\rangle = \langle \phi, \widehat{T}\rangle.$$

We will write $\iota_X : X \hookrightarrow X^{**}$ for the canonical embedding of a normed space into its bidual. Most parts of the following lemma are either implicit in [21, p. 10], or are stated explicitly as in [21, Proposition 4.2]. The fact that π is a homomorphism was established for amenable groups in [19], and in greater generality in [5]. Regarding part (ii) of the lemma, we remark that π is the analogue of the natural embedding of $M(G)$ into $LUC(G)^*$ as defined by Wong [32], and in this case Lau proved that the image of $M(G)$ is precisely the topological centre of $LUC(G)^*$ [20]. For the convenience of the reader we have chosen to include some parts of the proof. When X is a Banach space, we may write $\langle \phi, x\rangle_{X^*-X}$ to stress that ϕ is being regarded as an element of X^*, x an element of X; abbreviations, such as $U^* - U$ for $UCB(\widehat{G})^* - UCB(\widehat{G})$, will often be use

Lemma 3.1. *The following statements hold*:

(i) *θ is well defined.*
(ii) *π is a* (*well-defined*) *isometric algebra homomorphism which extends to $\pi : B_\rho(G) \hookrightarrow UCB(\widehat{G})^*$; π maps $B_\rho(G)$ into the centre of $UCB(\widehat{G})^*$.*
(iii) *For $\phi \in A(G)$, $\pi(\phi) = \iota_A(\phi)|_{UCB(\widehat{G})}$, and $\pi(A(G))$ is w^*-dense in $UCB(\widehat{G})^*$.*
(iv) *For $\phi \in B_\rho(G) = C^*_\rho(G)^*$ and $x \in C^*_\rho(G)$, $\langle \pi(\phi), x\rangle_{U^*-U} = \langle \phi, x\rangle_{B_\rho - C^*_\rho}$, i.e. $\pi(\phi)|_{C^*_\rho(G)} = \phi$.*

Proof. Suppose that $\psi_1, \psi_2 \in A(G)$ and $S_1, S_2 \in VN(G)$ are such that $\psi_1 \cdot S_1 = \psi_2 \cdot S_2$ in $VN(G)$, and let $\phi \in B_\rho(G)$. As noted in [21, p. 10], there is a net (ϕ_i) in $A(G)$ such that for each i, $\|\phi_i\| = \|\phi\|$, and $\phi_i \to \phi$ w^* in $B_\rho(G)$. By [13, Theorem B$_2$], $\|\phi_i\psi_k - \phi\psi_k\|_{A(G)} \to 0$ $(k = 1, 2)$, and therefore

$$\langle \widehat{\psi_1 \cdot S_1}, \phi\rangle = \langle S_1, \phi\psi_1\rangle = \lim_i \langle S_1, \phi_i\psi_1\rangle = \lim_i \langle \psi_1 \cdot S_1, \phi_i\rangle = \lim_i \langle \psi_2 \cdot S_2, \phi_i\rangle = \langle \widehat{\psi_2 \cdot S_2}, \phi\rangle.$$

This establishes statement (i). That π is an isometry (from (ii)), the first part of (iii), and (iv) are respectively parts (c), (a) and (b) of [21, Proposition 4.2]. That $\pi(A(G))$ is w^*-dense in $UCB(\widehat{G})^*$ is thus a consequence of the Hahn–Banach theorem and Goldstine's theorem. To see that π is a homomorphism, let $\phi, \psi \in B_\rho(G)$, $T \in UCB(\widehat{G})$. Then for $\varsigma \in A(G)$,

$$\big\langle \pi(\psi)_L(T), \varsigma\big\rangle = \big\langle \pi(\psi), T \cdot \varsigma\big\rangle = \langle \widehat{T \cdot \varsigma}, \psi\rangle = \langle T, \varsigma\psi\rangle = \langle \psi \cdot T, \varsigma\rangle.$$

Thus, $\pi(\psi)_L(T) = \psi \cdot T$ (recall that $UCB(\widehat{G})$ is a $B(G)$-module). Assuming that $T = \gamma \cdot S$ where $\gamma \in A(G)$ and $S \in VN(G)$,

$$\begin{aligned}\big\langle \pi(\phi) \odot \pi(\psi), T\big\rangle &= \big\langle \pi(\phi), \pi(\psi)_L(T)\big\rangle = \big\langle \pi(\phi), \psi \cdot T\big\rangle = \big\langle \pi(\phi), (\psi\gamma) \cdot S\big\rangle \\ &= \langle S, \phi\psi\gamma\rangle = \big\langle \pi(\phi\psi), \gamma \cdot S\big\rangle = \big\langle \pi(\phi\psi), T\big\rangle.\end{aligned}$$

That π maps into the centre of $UCB(\widehat{G})^*$ is [21, Proposition 4.5(b)]. □

In fact, if $B_\rho(G)$ and $\pi(B_\rho(G)))$ are identified, the direct sum decomposition

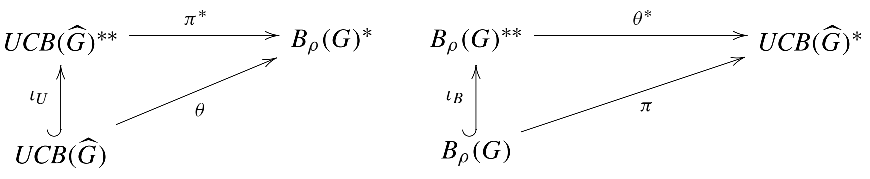
$$UCB(\widehat{G})^* = B_\rho(G) \oplus C_\rho^*(G)^\perp$$

holds (see [23, Lemma 5.2]). The following observation will be useful.

Lemma 3.2. *The map θ extends to an isometry $\theta : UCB(\widehat{G}) \hookrightarrow B_\rho(G)^*$ such that*

(i) *θ extends the canonical embedding $\iota_C : C_\rho^*(G) \hookrightarrow C_\rho^*(G)^{**} = B_\rho(G)^*$, and*
(ii) *for $T \in UCB(\widehat{G})$, $\phi \in A(G)$, $\langle \theta(T), \phi\rangle_{B_\rho^* - B_\rho} = \langle T, \phi\rangle_{VN-A}$, i.e. $\theta(T)|_{A(G)} = T$.*

Moreover, the diagrams commute:



Proof. The validity of the first diagram (on $A(G) \cdot VN(G)$) is a consequence of the definitions of π and θ. It follows from this, and Lemma 3.1(iv), that θ extends ι_C, and because π^* and ι_U are contractive, so too is θ. If $T = \psi \cdot S$ with $\psi \in A(G)$ and $S \in VN(G)$, then for each $\gamma \in A(G)$,

$$\big\langle \theta(T), \gamma\big\rangle = \langle \widehat{\psi \cdot S}, \gamma\rangle = \langle S, \gamma\psi\rangle = \langle \psi \cdot S, \gamma\rangle = \langle T, \gamma\rangle.$$

Hence $\theta(T)|_{A(G)} = T$, as claimed, and consequently $\|\theta(T)\| \geqslant \|T\|$. Therefore, θ is an isometry. That the second diagram commutes follows from the fact that the first one commutes. □

If $\gamma \in A(G)$, then the map $A(G) \to A(G): \psi \mapsto \gamma\psi$ is a completely bounded multiplier of $A(G)$ with cb multiplier norm denoted $\|\gamma\|_{M_{\mathrm{cb}}}$. If $A(G)$ has an approximate identity (e_λ) which is bounded in the cb multiplier norm, then G is called *weakly amenable*. If for each λ, $\|e_\lambda\|_{M_{\mathrm{cb}}} \leqslant 1$, it is convenient for us to call G 1-*weakly amenable*. In particular, the following proposition shows that π and θ are complete isometries when G is amenable.

As in [25], we view $UCB(\widehat{G})$ as an operator subspace of $VN(G)$ (the canonical operator space structure of $UCB(\widehat{G})$ as a C*-algebra) and give $UCB(\widehat{G})^*$ its canonical dual operator space structure (see [7]).

Proposition 3.3. *If G is a* (1-)*weakly amenable locally compact group, then π and θ are completely bounded* (*respectively complete isometries*).

Proof. Let (e_λ) be an approximate identity such that for each λ, $\|e_\lambda\|_{M_{\mathrm{cb}}} \leqslant M$. We begin by showing that for each n, $\|\theta_n\| \leqslant M$, where

$$\theta_n : M_n\big(UCB(\widehat{G})\big) \to M_n\big(B_\rho(G)^*\big) : [T_{i,j}] \to [\widehat{T_{i,j}}].$$

Let $T = [T_{i,j}] \in M_n(UCB_{\mathrm{c}}(\widehat{G}))$. Take $\psi_{i,j} \in A_{\mathrm{c}}(G)$ such that $\psi_{i,j} \equiv 1$ on a neighbourhood of $\mathrm{supp}(T_{i,j})$, so $T_{i,j} = \psi_{i,j} \cdot T_{i,j}$. Let $\phi = [\phi_{k,l}] \in M_n(B_\rho(G))$ with $\|\phi\| \leqslant 1$. Note that each e_λ is a cb multiplier of $B_\rho(G)$ with $\|e_\lambda\|_{M_{\mathrm{cb}}(B_\rho(G))} \leqslant M$ (see, for example [31, Proposition 4.1]), so $\|[e_\lambda \phi_{k,l}]\| \leqslant M$. Hence,

$$\begin{aligned}
\big\|\langle\langle \theta_n(T), \phi\rangle\rangle\big\| &= \big\|\big[\langle \widehat{\psi_{i,j} \cdot T_{i,j}}, \phi_{k,l}\rangle_{B_\rho^* - B_\rho}\big]\big\| \\
&= \big\|\big[\langle T_{i,j}, \phi_{k,l}\psi_{i,j}\rangle_{VN-A}\big]\big\| \\
[7, (2.1.8)] &= \lim_\lambda \big\|\big[\langle T_{i,j}, e_\lambda\phi_{k,l}\psi_{i,j}\rangle_{VN-A}\big]\big\| \\
&= \lim_\lambda \big\|\big[\langle \psi_{i,j} \cdot T_{i,j}, e_\lambda\phi_{k,l}\rangle_{VN-A}\big]\big\| \\
&= \lim_\lambda \big\|\big[\langle T_{i,j}, e_\lambda\phi_{k,l}\rangle_{VN-A}\big]\big\| \\
&\leqslant \sup\big\{\big\|\big[\langle T_{i,j}, \gamma_{k,l}\rangle_{VN-A}\big]\big\| : \gamma = [\gamma_{k,l}] \in M_n\big(A(G)\big), \|\gamma\| \leqslant M\big\} \\
&= M\|T\|_{M_n(UCB(\widehat{G}))}.
\end{aligned}$$

Therefore,

$$\big\|\theta_n(T)\big\| = \sup\big\{\big\|\langle\langle \theta_n(T), \phi\rangle\rangle\big\| : \phi \in M_n\big(B_\rho(G)\big),\ \|\phi\| \leqslant 1\big\} \leqslant M\|T\|_{M_n(UCB(\widehat{G}))},$$

so $\|\theta\|_{\mathrm{cb}} \leqslant M$. It follows that $\|\theta^*\|_{\mathrm{cb}} \leqslant M$ so Lemma 3.2 gives $\|\pi\|_{\mathrm{cb}} \leqslant M$ as well. Also by Lemma 3.2, given $T \in UCB(\widehat{G})$, $\theta(T)|_{A(G)} = T$, so for $T = [T_{i,j}] \in M_n(UCB(\widehat{G}))$, $\|\theta_n(T)\| \geqslant \|T\|_{M_n(UCB(\widehat{G}))}$. Hence, θ is a complete isometry when $M = 1$. As well, for $\phi \in B_\rho(G)$, $\pi(\phi)|_{C_\rho^*(G)} = \phi$, and we can similarly conclude that π is a complete isometry when $M = 1$. □

Recall that τ_M denotes the multiplier topology on $B_\rho(G)$, taken with respect to the closed ideal $A(G)$.

Lemma 3.4. *The embedding $\pi : B_\rho(G) \hookrightarrow UCB(\widehat{G})^*$ is τ_M–w^*-continuous on bounded subsets of $B_\rho(G)$. If G is amenable, then π is τ_M–w^*-continuous on $B_\rho(G) = B(G)$.*

Proof. Let (ϕ_i) be a net in $B_\rho(G)$ such that $\phi_i \to \phi\ \tau_M$. Let $T = \gamma \cdot S$ with $\gamma \in A(G)$ and $S \in VN(G)$. Then $\|\phi_i\gamma - \phi\gamma\|_{A(G)} \to 0$, so

$$\lim\big\langle \pi(\phi_i) - \pi(\phi), T\big\rangle = \lim\langle \widehat{\gamma \cdot S}, \phi_i - \phi\rangle = \lim\langle S, \phi_i\gamma - \phi\gamma\rangle = \langle S, 0\rangle = 0.$$

If G is amenable, then $UCB(\widehat{G}) = A(G) \cdot VN(G)$, so π is τ_M–w^*-continuous. In any case, $A(G) \cdot VN(G)$ is dense in $UCB(\widehat{G})$, so provided that (ϕ_i) is bounded, we can conclude that $\lim\langle \pi(\phi_i) - \pi(\phi), T\rangle = 0$ for $T \in UCB(\widehat{G})$. □

We will need to know when π is w^*–w^*-continuous. The equivalence of (i) and (iii) in the next proposition is [5, Theorem 3.7] in the special case when $p = 2$.

Proposition 3.5. *The following statements are equivalent*:

(i) *G is discrete*;
(ii) *π is w^*–w^*-continuous*;
(iii) *π is surjective.*

Proof. If G is discrete, then [19, Proposition 4.5] gives $UCB(\widehat{G}) = C^*_\rho(G)$ and, by Lemma 3.1, π is the identity map. Hence, statement (ii) holds. Suppose now that π is w^*–w^*-continuous, and let $\pi_* : UCB(\widehat{G}) \to C^*_\rho(G)$ be its predual map. General principles and the first commuting diagram from Lemma 3.2 yield

$$\big\|\pi_*(x)\big\| = \big\|\iota_{C^*_\rho}\big(\pi_*(x)\big)\big\| = \big\|\pi^*\big(\iota_U(x)\big)\big\| = \big\|\theta(x)\big\| = \|x\|,$$

because θ is an isometry. Hence π_* has closed range, and $\operatorname{range}(\pi_*)^\perp = \ker\pi = \{0\}$ because π is injective. Hence, π_* is a bijection, and therefore $\pi = (\pi_*)^*$ is also a bijection. Finally, suppose that π is surjective. Given $m \in UCB(\widehat{G})^*$ such that $m|_{C^*_\rho(G)} = 0$, we can then find $\phi \in B_\rho(G)$ such that $\pi(\phi) = m$. By Lemma 3.1, $\phi = \pi(\phi)|_{C^*_\rho(G)} = m|_{C^*_\rho(G)} = 0$, and so $m = \pi(\phi) = 0$ as well. By the Hahn–Banach separation theorem and [19, Proposition 4.5] we can conclude that $UCB(\widehat{G}) = C^*_\rho(G)$ and therefore G is discrete. □

4. The main extension

Throughout the remainder of the paper, G and H are locally compact groups, and $\alpha : Y \subset H \to G$ is a fixed piecewise affine continuous map. To prove our extension theorems, we will require that j_α, the map associated with α, maps $B_\rho(G)$ into $B_\rho(H)$. That is, *we will always assume that we have*

$$j_\alpha : B_\rho(G) \to B_\rho(H).$$

When H is amenable, $B_\rho(H) = B(H)$ and this holds trivially. In general, we characterize such maps α in Section 6.

If we wish to stress the fact that we are considering j_α as a map on $A(G)$, we will often write $j_A : A(G) \to B_\rho(H)$.

Let $\kappa_\alpha = j_A^* \circ \theta_H$. We have

$$\begin{array}{ccc} B_\rho(H)^* & \xrightarrow{\ j_A^*\ } & VN(G) \\ {\scriptstyle \theta_H} \big\uparrow & \nearrow_{\kappa_\alpha} & \\ UCB(\widehat{H}) & & \end{array}$$

Lemma 4.1. *If $T \in UCB_c(\widehat{H})$ has compact support K, then $\kappa_\alpha(T)$ has compact support with* $\operatorname{supp}(\kappa_\alpha(T)) \subset \alpha(K \cap Y)$. *Hence, $\kappa_\alpha : UCB(\widehat{H}) \to UCB(\widehat{G})$.*

Proof. Let $a \in G \setminus \alpha(K \cap Y)$, and let V be a relatively compact neighbourhood of a such that $\overline{V} \subset G \setminus \alpha(K \cap Y)$. Let $\psi \in A(G)$ be such that $\operatorname{supp}(\psi) \subset V$. To show that $a \notin \operatorname{supp}(\kappa_\alpha(T))$, we only need to show that $\langle \kappa_\alpha(T), \psi \rangle = 0$. The closed set $\alpha^{-1}(\overline{V})$ is disjoint from K, so by regularity of $A(H)$ we can choose $\gamma \in A(H)$ such that $\gamma \equiv 0$ on $\alpha^{-1}(\overline{V})$ and $\gamma \equiv 1$ on a neighbourhood K. Then $T = \gamma \cdot T$ and observe that $(j_A \psi)\gamma \equiv 0$. Hence,

$$\big\langle \kappa_\alpha(T), \psi \big\rangle = \langle \widehat{T}, j_A \psi \rangle = \langle \widehat{\gamma \cdot T}, j_A \psi \rangle = \big\langle T, (j_A \psi)\gamma \big\rangle = \langle T, 0 \rangle = 0,$$

as needed. By continuity of α, $\alpha(K \cap Y)$ is compact, and density of $UCB_c(\widehat{H})$ in $UCB(\widehat{H})$ gives $\kappa_\alpha : UCB(\widehat{H}) \to UCB(\widehat{G})$. $\square$

We now list some useful identities involving κ_α.

Lemma 4.2. *The following identities hold*:

(i) *For $\psi \in A(G)$ and $\gamma \cdot S \in UCB(\widehat{H})$, where $\gamma \in A(H)$ and $S \in VN(H)$,*

$$\big\langle \kappa_\alpha(\gamma \cdot S), \psi \big\rangle = \big\langle S, (j_\alpha \psi)\gamma \big\rangle.$$

(ii) *For $T \in UCB(\widehat{H})$ and $\phi \in B_\rho(G)$, $\phi \cdot \kappa_\alpha(T) = \kappa_\alpha(j_\alpha(\phi) \cdot T)$.*

(iii) *For each $h \in H$,* $\kappa_\alpha(\rho_H(h)) = \begin{cases} \rho_G(\alpha(h)) & \text{if } h \in Y, \\ 0, & \text{otherwise.} \end{cases}$

Proof. Part (i) is immediate from the definition of κ_α. For (ii), suppose that $T = \gamma \cdot S$ where $\gamma \in A(H)$, $S \in VN(H)$, and let $\psi \in A(G)$. Using part (i), we obtain

$$\begin{aligned} \big\langle \phi \cdot \kappa_\alpha(T), \psi \big\rangle &= \big\langle \kappa_\alpha(\gamma \cdot S), \psi\phi \big\rangle = \big\langle S, j_\alpha(\psi\phi)\gamma \big\rangle \\ &= \big\langle S, j_\alpha(\psi)\big(j_\alpha(\phi)\gamma\big) \big\rangle = \big\langle \kappa_\alpha\big(j_\alpha(\phi)\gamma \cdot S\big), \psi \big\rangle \\ &= \big\langle \kappa_\alpha\big(j_\alpha(\phi) \cdot T\big), \psi \big\rangle. \end{aligned}$$

For (iii), let $h \in H$, and choose $\gamma \in A(H)$ such that $\gamma(h) = 1$. Then, $\gamma \cdot \rho_H(h) = \rho_H(h)$, so for $\psi \in A(G)$ we have

$$\langle \kappa_\alpha(\rho_H(h)), \psi \rangle = \langle \kappa_\alpha(\gamma \cdot \rho_H(h)), \psi \rangle = \langle \rho_H(h), j_\alpha(\psi)\gamma \rangle = j_\alpha(\psi)(h).$$

Thus, for $h \in Y$, $\langle \kappa_\alpha(\rho_H(h)), \psi \rangle = \psi(\alpha(h)) = \langle \rho_G(\alpha(h)), \psi \rangle$, and if $h \in H \setminus Y$, then $\langle \kappa_\alpha(\rho_H(h)), \psi \rangle = 0$. □

Lemma 4.3. *The diagram commutes*:

$$\begin{array}{ccc} B_\rho(H)^* & \xrightarrow{\;j_\alpha^*\;} & B_\rho(G)^* \\ \theta_H \big\uparrow & & \big\uparrow \theta_G \\ UCB(\widehat{H}) & \xrightarrow{\;\kappa_\alpha\;} & UCB(\widehat{G}) \end{array}$$

That is, $j_\alpha^* \circ \theta_H = \theta_G \circ \kappa_\alpha$*, meaning that* j_α^* *is an extension of* κ_α.

Proof. Let $T \in UCB_{\mathrm{c}}(\widehat{H})$ with $K = \mathrm{supp}(T)$ compact, and choose $\gamma \in A_{\mathrm{c}}(H)$ such that $\gamma \equiv 1$ on a neighbourhood of K. Then $T = \gamma \cdot T$ and $K \subset \mathrm{supp}(\gamma)$, so by Lemma 4.1 $\mathrm{supp}(\kappa_\alpha(T)) \subset \alpha(\mathrm{supp}(\gamma) \cap Y)$. As $\alpha(\mathrm{supp}(\gamma) \cap Y)$ is compact, we can find $\psi \in A_{\mathrm{c}}(G)$ such that $\psi \equiv 1$ on a neighbourhood of $\alpha(\mathrm{supp}(\gamma) \cap Y)$ and obtain $\kappa_\alpha(T) = \psi \cdot \kappa_\alpha(T)$. Notice that $(j_A\psi)\gamma = \gamma 1_Y$. For $\phi \in B_\rho(G)$, we have

$$\begin{aligned} \langle \theta_G \circ \kappa_\alpha(T), \phi \rangle &= \langle \widehat{\psi \cdot \kappa_\alpha(T)}, \phi \rangle = \langle \kappa_\alpha(\gamma \cdot T), \psi\phi \rangle = \langle T, j_A(\psi\phi)\gamma \rangle \\ &= \langle T, (j_A\psi)(j_\alpha\phi)\gamma \rangle = \langle T, (j_\alpha\phi)\gamma 1_Y \rangle \\ &= \langle T, (j_\alpha\phi)\gamma \rangle = \langle \widehat{\gamma \cdot T}, j_\alpha\phi \rangle \\ &= \langle j_\alpha^*(\theta_H(T)), \phi \rangle, \end{aligned}$$

as needed. □

For the next proof recall from [19] that for $m \in UCB(\widehat{G})^*$, the map

$$n \mapsto n \odot m : UCB(\widehat{G})^* \to UCB(\widehat{G})^* \text{ is } w^*\text{–}w^*\text{-continuous.}$$

Clearly then, if $m \in Z(UCB(\widehat{G})^*)$, the centre of $UCB(\widehat{G})^*$, then

$$n \mapsto m \odot n : UCB(\widehat{G})^* \to UCB(\widehat{G})^* \text{ is also } w^*\text{–}w^*\text{-continuous;}$$

that is $Z(UCB(\widehat{G})^*) \subset Z_t(UCB(\widehat{G})^*)$, the topological centre of $UCB(\widehat{G})^*$. In fact, Lau and Losert have shown that $Z(UCB(\widehat{G})^*) = Z_t(UCB(\widehat{G})^*)$ [21, Theorem 5.8].

The last statement of the next theorem may be compared with Corollary 3.2 of [17] which, in part, states that when G is amenable, $\| j_A \|_{\mathrm{cb}} = \| j_\alpha \|_{\mathrm{cb}}$.

Theorem 4.4. *The dual map* $\kappa_\alpha^* : UCB(\widehat{G})^* \to UCB(\widehat{H})^*$ *is a* w^**–*w^**-continuous homomorphic extension of* $j_\alpha : B_\rho(G) \to B_\rho(H)$*. More precisely, the diagram*

$$\begin{array}{ccc} UCB(\widehat{G})^* & \xrightarrow{\ \kappa_\alpha^*\ } & UCB(\widehat{H})^* \\ {\scriptstyle \pi_G}\uparrow & & \uparrow{\scriptstyle \pi_H} \\ B_\rho(G) & \xrightarrow{\ j_\alpha\ } & B_\rho(H) \end{array}$$

commutes; i.e. $\kappa_\alpha^* \circ \pi_G = \pi_H \circ j_\alpha$*. If* H *is weakly amenable, then* κ_α *is completely bounded, and if* H *is* 1*-weakly amenable, then* $\| j_A \|_{\mathrm{cb}} = \| j_\alpha \|_{\mathrm{cb}} = \| \kappa_\alpha \|_{\mathrm{cb}}$.

Proof. Using Lemmas 3.2 and 4.3, we obtain

$$\begin{aligned} \pi_H \circ j_\alpha &= \theta_H^* \circ \iota_{B_\rho(H)} \circ j_\alpha = \theta_H^* \circ j_\alpha^{**} \circ \iota_{B_\rho(G)} \\ &= \left(j_\alpha^* \circ \theta_H\right)^* \circ \iota_{B_\rho(G)} = (\theta_G \circ \kappa_\alpha)^* \circ \iota_{B_\rho(G)} \\ &= \kappa_\alpha^* \circ \theta_G^* \circ \iota_{B_\rho(G)} \\ &= \kappa_\alpha^* \circ \pi_G. \end{aligned}$$

Letting $m, n \in UCB(\widehat{G})^*$ we now show that $\kappa_\alpha^*(m \odot n) = \kappa_\alpha^*(m) \odot \kappa_\alpha^*(n)$. By Lemma 3.1(iii), we can choose nets (ψ_i) and (γ_l) in $A(G)$ such that $\pi_G(\psi_i) \to m$ and $\pi_G(\gamma_l) \to n w^*$ in $UCB(\widehat{G})^*$. Using the w^*–w^*-continuity of κ_α^*, we obtain

$$\begin{aligned} \kappa_\alpha^*(m \odot n) &= \lim_i \kappa_\alpha^*\big(\pi_G(\psi_i) \odot n\big) \\ (*) &= \lim_i \lim_l \kappa_\alpha^*\big(\pi_G(\psi_i) \odot \pi_G(\gamma_l)\big) \\ &= \lim_i \lim_l \kappa_\alpha^* \circ \pi_G(\psi_i \gamma_l) = \lim_i \lim_l \pi_H \circ j_\alpha(\psi_i \gamma_l) \\ &= \lim_i \lim_l \pi_H(j_\alpha \psi_i) \odot \pi_H(j_\alpha \gamma_l) = \lim_i \lim_l \pi_H(j_\alpha \psi_i) \odot \kappa_\alpha^*(\pi_G \gamma_l) \\ (*) &= \lim_i \pi_H(j_\alpha \psi_i) \odot \kappa_\alpha^*(n) \\ &= \lim_i \kappa_\alpha^*(\pi_G \psi_i) \odot \kappa_\alpha^*(n) \\ &= \kappa_\alpha^*(m) \odot \kappa_\alpha^*(n), \end{aligned}$$

where $(*)$ indicates that we have used Lemma 3.1(ii) and the remarks preceding the statement of the theorem.

By [17, Proposition 3.1], j_A is completely bounded, and when H is weakly amenable θ_H is completely bounded by Proposition 3.3. Hence $\kappa_\alpha = j_A^* \circ \theta_H$ is completely bounded in this case. If H is 1-weakly amenable, then we know that θ_H is a complete isometry, so

$$\|\kappa_\alpha\|_{\mathrm{cb}} = \left\| j_A^* \circ \theta_H \right\|_{\mathrm{cb}} \leqslant \left\| j_A^* \right\|_{\mathrm{cb}} = \| j_A \|_{\mathrm{cb}} \leqslant \| j_\alpha \|_{\mathrm{cb}} = \left\| \kappa_\alpha^* \big|_{B_\rho(G)} \right\|_{\mathrm{cb}} \leqslant \left\| \kappa_\alpha^* \right\|_{\mathrm{cb}} = \|\kappa_\alpha\|_{\mathrm{cb}},$$

proving the second statement of the theorem. □

Let $p_H : UCB(\widehat{H})^* \to C_\rho^*(H)^* = B_\rho(H) : m \mapsto m|_{C_\rho^*(H)}$ be the restriction map. Note that by Lemma 3.1(iv), $p_H \circ \pi_H = id_{B_\rho(H)}$. Corollary 4.5 should be compared with [18, Corollary 5.8] which showed that when G is amenable, j_α is τ_M–w^*-continuous on bounded subsets of $B(G) = B_\rho(G)$. Corollary 4.6 is supplementary to [18, Theorem 5.10].

Corollary 4.5. *The map $j_\alpha : B_\rho(G) \to B_\rho(H)$ factors as $j_\alpha = p_H \circ \kappa_\alpha^* \circ \pi_G$ and is therefore τ_M–w^*-continuous on bounded subsets of $B_\rho(G)$. If G is amenable, then j_α is τ_M–w^*-continuous on $B(G)$.*

Proof. By Theorem 4.4, $\kappa_\alpha^* \circ \pi_G = \pi_H \circ j_\alpha$, so $p_H \circ \kappa_\alpha^* \circ \pi_G = p_H \circ \pi_H \circ j_\alpha = j_\alpha$. As p_H and κ_α^* are w^*–w^*-continuous, the result follows from Lemma 3.4. □

Corollary 4.6. *Let G be amenable, and let ϕ be a mapping of $B(G)$ into $B_\rho(H)$ which is not identically zero. Then ϕ is a τ_M–w^*-continuous completely bounded homomorphism if and only if there is a piecewise affine continuous map $\alpha : Y \subset H \to G$ such that $\phi = j_\alpha$.*

Proof. For the forward implication, note that the w^*-topology on $B_\rho(H)$ agrees with the relative w^* topology on $B_\rho(H)$ inherited from $B(H)$, so $\phi : B(G) \to B(H)$ is τ_M–w^*-continuous on bounded subsets of $B(G)$. By [18, Theorem 5.10], $\phi = j_\alpha$ for some continuous piecewise affine map α. The converse follows from Corollary 4.5. □

5. Mappings between introverted spaces

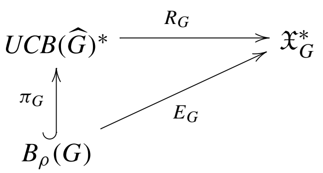
5.1. The general case

Let $\mathfrak{X}_G$ be a closed subspace of $UCB(\widehat{G})$ which is topologically invariant and introverted in $VN(G)$. Define

$$I_G : \mathfrak{X}_G \hookrightarrow UCB(\widehat{G}) : T \to T,$$

$$R_G = I_G^* : UCB(\widehat{G})^* \to \mathfrak{X}_G^* : m \mapsto m|_{\mathfrak{X}_G},$$

$$E_G = R_G \circ \pi_G : B_\rho(G) \to \mathfrak{X}_G^*,$$



Remark 5.1. Observe that R_G is a w^*–w^*-continuous, completely contractive, surjective algebra homomorphism. By Lemmas 3.1 and 3.4 we hence obtain:

(i) E_G is a contractive algebra homomorphism mapping $B_\rho(G)$ into the centre of $\mathfrak{X}_G^*$; $E_G(A(G))$ is w^*-dense in $\mathfrak{X}_G^*$; E_G is τ_M–w^*-continuous on bounded subsets of $B_\rho(G)$, and τ_M–w^*-continuous on $B_\rho(G)$ when G is amenable.

(ii) If G is weakly amenable, then E_G is completely bounded.

(iii) If $\mathfrak{X}_G$ contains $C_\rho^*(G)$, then E_G is an isometric algebra isomorphism such that

$$E_G(\phi)\big|_{C_\rho^*(G)} = \phi, \quad \phi \in B_\rho(G).$$

When $\mathfrak{X}_G$ contains $C_\rho^*(G)$ and G is 1-weakly amenable, E_G is a complete isometry.

Proposition 5.2. *Let $\mathfrak{X}_G$ and $\mathfrak{X}_H$ be topologically invariant and introverted subspaces of $UCB(\widehat{G})$ and $UCB(\widehat{H})$, respectively. Suppose that $\kappa_\alpha(\mathfrak{X}_H) \subset \mathfrak{X}_G$ and let*

$$\bar{\kappa}_\alpha : \mathfrak{X}_H \to \mathfrak{X}_G : T \mapsto \kappa_\alpha(T).$$

Then

(i) *$\bar{\kappa}_\alpha^* : \mathfrak{X}_G^* \to \mathfrak{X}_H^*$ is a w^*–w^*-continuous algebra homomorphism such that the diagram*

$$\begin{array}{ccc} \mathfrak{X}_G^* & \xrightarrow{\ \bar{\kappa}_\alpha^*\ } & \mathfrak{X}_H^* \\ {\scriptstyle E_G}\uparrow & & \uparrow{\scriptstyle E_H} \\ B_\rho(G) & \xrightarrow{\ j_\alpha\ } & B_\rho(H) \end{array}$$

commutes. If H is (1-)weakly amenable, then $\bar{\kappa}_\alpha$ is completely bounded (and $\|\bar{\kappa}_\alpha\|_{\mathrm{cb}} = \|\bar{\kappa}_\alpha^*\|_{\mathrm{cb}} = \|j_\alpha\|_{\mathrm{cb}}$).

(ii) *Therefore, if $C_\rho^*(G) \subset \mathfrak{X}_G$ and $C_\rho^*(H) \subset \mathfrak{X}_H$, then $\bar{\kappa}_\alpha^*$ is a w^*–w^*-continuous extension of j_α.*

Proof. Obviously, $\kappa_\alpha \circ I_H = I_G \circ \bar{\kappa}_\alpha$ and therefore, $R_H \circ \kappa_\alpha^* = \bar{\kappa}_\alpha^* \circ R_G$. We have seen that κ_α^* and R_H are algebra homomorphisms, and R_G is a surjective homomorphism, from which it easily follows that $\bar{\kappa}_\alpha^*$ is an algebra homomorphism. As well, by Theorem 4.4 we have $\kappa_\alpha^* \circ \pi_G = \pi_H \circ j_\alpha$, so

$$\bar{\kappa}_\alpha^* \circ E_G = \bar{\kappa}_\alpha^* \circ R_G \circ \pi_G = R_H \circ \kappa_\alpha^* \circ \pi_G = R_H \circ \pi_H \circ j_\alpha = E_H \circ j_\alpha.$$

The remaining parts of the proposition follow from Theorem 4.4 and Remark 5.1. □

We will say that $\mathfrak{X}_H$ is *$B_\rho(H)$-invariant* if it is a $B_\rho(H)$-submodule of $VN(H)$. Let $\mathfrak{X}_{H,c} = \{T \in \mathfrak{X}_H\colon \operatorname{supp}(T) \text{ is compact}\}$. The next proposition shows that topological invariance often implies $B_\rho(H)$-invariance and that κ_α preserves topological introversion of $B_\rho(H)$-invariant subspaces of $UCB(\widehat{H})$.

Proposition 5.3. *Let $\mathfrak{X}_H$ be a topologically invariant closed subspace of $UCB(\widehat{H})$, and let $\mathfrak{X}_G = \overline{\kappa_\alpha(\mathfrak{X}_H)}$.*

(i) *If $\mathfrak{X}_H$ is $B_\rho(H)$-invariant (and topologically introverted) in $VN(H)$, then $\mathfrak{X}_G$ is $B_\rho(G)$-invariant (and topologically introverted) in $VN(G)$.*

(ii) *If $\mathfrak{X}_{H,c}$ is dense in $\mathfrak{X}_H$, then $\mathfrak{X}_H$ is $B_\rho(H)$-invariant and $\mathfrak{X}_{G,c}$ is dense in $\mathfrak{X}_G$. If H is amenable, then $\mathfrak{X}_{H,c}$ is dense in $\mathfrak{X}_H$.*

Proof. (i) Suppose that $\mathfrak{X}_H$ is $B_\rho(H)$ invariant and let $\kappa_\alpha(T) \in \mathfrak{X}_G$, where $T \in \mathfrak{X}_H$. For $\phi \in B_\rho(G)$, Lemma 4.2(ii) gives $\phi \cdot \kappa_\alpha(T) = \kappa_\alpha(j_\alpha(\phi) \cdot T)$ which belongs to $\mathfrak{X}_G$. Hence, $\mathfrak{X}_G$ is $B_\rho(G)$-invariant. Assume as well that $\mathfrak{X}_H$ is topologically introverted. Letting $m \in \mathfrak{X}_G^*$ and $S \in \mathfrak{X}_G$, we must show that $m_L(S) \in \mathfrak{X}_G$. Let $n = \bar{\kappa}_\alpha^*(m)$ where $\bar{\kappa}_\alpha : \mathfrak{X}_H \to \mathfrak{X}_G : T \mapsto \kappa_\alpha(T)$ and assume that $S = \bar{\kappa}_\alpha(T)$, $T \in \mathfrak{X}_H$. Then for $\psi \in A(G)$,

$$\langle m_L(S), \psi \rangle = \langle m, \psi \cdot \bar{\kappa}_\alpha(T) \rangle = \langle m, \bar{\kappa}_\alpha(j_\alpha(\psi) \cdot T) \rangle = \langle n, j_\alpha(\psi) \cdot T \rangle$$

where we have used Lemma 4.2(ii). Let $\tilde{n} \in UCB(\widehat{G})^*$ be an extension of n and choose a sequence $(\gamma_i \cdot T_i)$ in $UCB(\widehat{H})$ with each $\gamma_i \in A(H)$, $T_i \in UCB(\widehat{H})$ and $\|\gamma_i \cdot T_i - T\| \to 0$. Then

$$\begin{aligned} \langle m_L(S), \psi \rangle &= \lim_i \langle \tilde{n}, j_\alpha(\psi) \cdot (\gamma_i \cdot T_i) \rangle = \lim_i \langle \tilde{n}_L(T_i), j_\alpha(\psi)\gamma_i \rangle \\ \text{(see Lemma 4.2(i))} &= \lim_i \langle \kappa_\alpha(\gamma_i \cdot \tilde{n}_L(T_i)), \psi \rangle \\ &= \lim_i \langle \kappa_\alpha(\tilde{n}_L(\gamma_i \cdot T_i)), \psi \rangle = \langle \kappa_\alpha(\tilde{n}_L(T)), \psi \rangle \\ &= \langle \kappa_\alpha(n_L(T)), \psi \rangle. \end{aligned}$$

Hence, $m_L(S) = \kappa_\alpha(n_L(T))$ which belongs to $\mathfrak{X}_G$, because $\mathfrak{X}_H$ is topologically introverted.

(ii) Let $T \in \mathfrak{X}_{H,c}$, $\phi \in B_\rho(H)$. Taking $\gamma \in A(H)$ such that $\gamma \equiv 1$ on a neighbourhood of $\operatorname{supp}(T)$, $\phi \cdot T = \phi \cdot (\gamma \cdot T) = (\phi\gamma) \cdot T \in \mathfrak{X}_{H,c}$ because $\mathfrak{X}_H$ it topologically invariant. Therefore, if $\mathfrak{X}_{H,c}$ is dense in $\mathfrak{X}_H$, then $\mathfrak{X}_H$ is $B_\rho(H)$-invariant. In this case $\mathfrak{X}_{G,c}$ is dense in $\mathfrak{X}_G$ by Lemma 4.1. When H is amenable, $A(H)$ has a bounded approximate identity (e_i) such that each e_i has compact support. For any $T \in \mathfrak{X}_H$, $e_i \cdot T \in \mathfrak{X}_{H,c}$ and it is easy to see that $\|e_i \cdot T - T\| \to 0$. □

Remark 5.4. If we assume that $\mathfrak{X}_H$ is $B_\rho(H)$-invariant and topologically introverted in $UCB(\widehat{H})$ and $\mathfrak{X}_G = \overline{\kappa_\alpha(\mathfrak{X}_H)}$, then $\bar{\kappa}_\alpha^* : \mathfrak{X}_G^* \to \mathfrak{X}_H^*$ is a homomorphism by Propositions 5.2 and 5.3. In order to properly say that this extends $j_\alpha : B_\rho(G) \to B_\rho(H)$, we need $C_\rho^*(H) \subset \mathfrak{X}_H$ and $C_\rho^*(G) \subset \mathfrak{X}_G$. If this is not the case, we can replace $\mathfrak{X}_H$ by $\mathcal{Z}_H = \mathfrak{X}_H + C_\rho^*(H) = \overline{\{S + T\colon S \in \mathfrak{X}_H\ T \in C_\rho^*(H)\}}$ (the smallest $B_\rho(H)$-invariant and topologically introverted subspace of $UCB(\widehat{H})$ containing both $\mathfrak{X}_H$ and $C_\rho^*(H)$) and $\mathfrak{X}_G$ by $\kappa_\alpha(\mathcal{Z}_H) + C_\rho^*(G)$.

5.2. *Special cases*

Following R. Smith and N. Spronk [30], we will denote the operator norm closure of $\rho_G(M(G))$ in $B(L^2(G))$ by $M_\rho^*(G)$. An operator $T \in VN(G)$ is *weakly almost periodic* (*almost periodic*), written $T \in W(\widehat{G})$ ($T \in AP(\widehat{G})$), if its orbit in $VN(G)$, $\{\phi \cdot T\colon \phi \in A(G),\ \|\phi\| \leqslant 1\}$ is relatively weakly compact (compact). Lau [19] has shown that $W(\widehat{G})$ and $AP(\widehat{G})$ are topologically invariant and introverted in $VN(G)$. Moreover, $AP(\widehat{G}) \subset W(\widehat{G})$ and when G is amenable, $W(\widehat{G}) \subset UCB(\widehat{G})$ [11, Proposition 1]; $UCB(\widehat{G}) \subset W(\widehat{G})$ if and only if G is discrete [12].

If $\mathcal{X}$ and $\mathcal{Y}$ are topologically invariant and introverted subspaces of $VN(G)$, then it is easy to see that the same is true of $\mathcal{X} \cap \mathcal{Y}$. In particular, the spaces

$$AP_{\mathrm{u}}(\widehat{G}) = AP(\widehat{G}) \cap UCB(\widehat{G}) \quad \text{and} \quad W_{\mathrm{u}}(\widehat{G}) = W(\widehat{G}) \cap UCB(\widehat{G})$$

are topologically invariant and introverted subspaces of $UCB(\widehat{G})$. When G is amenable, $AP_{\mathrm{u}}(\widehat{G}) = AP(\widehat{G})$ and $W_{\mathrm{u}}(\widehat{G}) = W(\widehat{G})$ by the aforementioned result of Granirer.

Theorem 2.8 of [6] shows that when G is compact, $M^*_\rho(G) \subset W(\widehat{G})$, and in [6, Section 8] the authors state that the containment holds for any locally compact group. In this subsection we will give a different proof of this general statement. We then show that if $(\mathfrak{X}_G, \mathfrak{X}_H)$ is one of the pairs $(M^*_\rho(G), M^*_\rho(H))$, $(W_{\mathrm{u}}(\widehat{G}), W_{\mathrm{u}}(\widehat{H}))$, or $(AP_{\mathrm{u}}(\widehat{G}), AP_{\mathrm{u}}(\widehat{H}))$, then κ_α maps $\mathfrak{X}_H$ into $\mathfrak{X}_G$ (and Proposition 5.2 applies).

We first observe that $M^*_\rho(G) \subset VN(G)$, $A(G) \subset C_0(G)$ and

$$\langle \rho_G(\mu), \gamma \rangle_{VN-A} = \langle \mu, \gamma \rangle_{M-C_0} \quad \big(\mu \in M(G),\ \gamma \in A(G)\big). \tag{5.1}$$

Indeed, if $\xi, \eta \in L^2(G)$ and γ is the coefficient function $\gamma(\cdot) = \langle \rho_G(\cdot)\xi | \eta \rangle$, then

$$\langle \rho_G(\mu), \gamma \rangle_{VN-A} = \langle \rho_G(\mu)\xi | \eta \rangle = \int_G \langle \rho_G(s)\xi | \eta \rangle \, d\mu(s) = \int_G \gamma(s)\, d\mu(s) = \langle \mu, \gamma \rangle_{M-C_0}.$$

Proposition 5.5. *The C^*-algebra $M^*_\rho(G)$ is a topologically invariant and topologically introverted subspace of $VN(G)$. Moreover, $M^*_\rho(G) \subset W_{\mathrm{u}}(\widehat{G})$.*

Proof. Let $\mu \in M(G)$, $\psi \in A(G)$. Then for $\gamma \in A(G)$,

$$\langle \psi \cdot \rho_G(\mu), \gamma \rangle_{VN-A} = \langle \rho_G(\mu), \gamma\psi \rangle_{VN-A} = \int_G \gamma(s)\psi(s)\, d\mu(s) = \langle \rho_G(\psi \cdot \mu), \gamma \rangle_{VN-A},$$

so $\psi \cdot \rho_G(\mu) = \rho_G(\psi \cdot \mu) \in M^*_\rho(G)$. This establishes topological invariance and shows that $M^*_\rho(G) \subset UCB(\widehat{G})$. Let (ϕ_i) be a net in $P_1(G) \cap A(G)$ and suppose that $T \in VN(G)$ is such that $\phi_i \cdot \rho_G(\mu) = \rho_G(\phi_i \cdot \mu) \to T \sigma(VN(G), A(G))$. Here, $P_1(G) = \{\phi \in P(G)\colon \|\phi\| = 1\}$. By Lemma 5.1 of [19], to establish topological introversion of $M^*_\rho(G)$ it suffices to show that $T \in M^*_\rho(G)$. As $\|\phi_i \cdot \mu\|_{M(G)} \leqslant \|\phi_i\|_\infty \|\mu\|_{M(G)} = \|\mu\|_{M(G)}$, by passing to a subnet we may assume that $\phi_i \cdot \mu \to \nu\, \sigma(M(G), C_0(G))$. For any $\gamma \in A(G)$,

$$\begin{aligned}\langle T, \gamma \rangle_{VN-A} &= \lim_i \langle \rho_G(\phi_i \cdot \mu), \gamma \rangle_{VN-A} = \lim_i \langle \phi_i \cdot \mu, \gamma \rangle_{M-C_0} = \langle \nu, \gamma \rangle_{M-C_0} \\ &= \langle \rho_G(\nu), \gamma \rangle_{VN-A},\end{aligned}$$

so $T = \rho_G(\nu) \in M^*_\rho(G)$.

By [19, Theorem 5.6], to show that $M^*_\rho(G) \subset W(\widehat{G})$, it suffices to show that multiplication in $M^*_\rho(G)$ commutes. For this, let $m, n \in M^*_\rho(G)^*$, let $\widetilde{m}, \widetilde{n} \in VN(G)^*$ be extensions of m and n, and take bounded nets (ϕ_i), (ψ_j) in $A(G)$ such that $\phi_i \to \widetilde{m}$, $\psi_j \to \widetilde{n}\ \sigma(VN(G)^*, VN(G))$. Then (ϕ_i), (ψ_j) are also bounded in $C_0(G)$ so there exist $m', n' \in M(G)^*$ such that, by passing to subnets if necessary, $\phi_i \to m'$, $\psi_j \to n' \sigma(M(G)^*, M(G))$. For $\mu \in M(G)$ we have

$$\begin{aligned}\langle m \odot n, \rho_G(\mu) \rangle &= \langle \widetilde{m} \odot \widetilde{n}, \rho_G(\mu) \rangle = \lim_i \lim_j \langle \rho_G(\mu), \phi_i \psi_j \rangle \\ &= \lim_i \lim_j \langle \mu, \phi_i \psi_j \rangle_{M-C_0}\end{aligned}$$

$$
\begin{aligned}
(*) &= \lim_j \lim_i \langle \mu, \phi_i \psi_j \rangle_{M-C_0} \\
&= \lim_j \lim_i \langle \mu, \psi_j \phi_i \rangle_{M-C_0} \\
&= \big\langle n \odot m, \rho_G(\mu) \big\rangle,
\end{aligned}
$$

where at line $(*)$ we have used the fact the C*-algebra $C_0(G)$ is Arens regular (see [4, Eq. (2.6.28) and Corollary 3.2.37]. □

We should point out that any topologically invariant subspace of $W(\widehat{G})$ is automatically topologically introverted by [22, Lemma 1.2]. We were unable to obtain a proof of Proposition 5.5 which made use of this fact.

Proposition 5.6. *The operator κ_α maps $M^*_\rho(H)$ into $M^*_\rho(G)$.*

Proof. First note that $j_\alpha : A(G) \to B_\rho(H)$ is contractive with respect to the uniform norms on $A(G)$ and $B_\rho(H)$, so we can extend j_α to a contractive mapping $\tau_\alpha : C_0(G) \to LUC(H)$ also described by

$$
\tau_\alpha(f) = \begin{cases} f \circ \alpha & \text{on } Y, \\ 0 & \text{off } Y \end{cases} \qquad \big(f \in C_0(G)\big).
$$

(We remark that the same formula defines a contractive homomorphism of $LUC(G)$ into $LUC(H)$.) Let $I_H : M(H) \to LUC(H)^*$ be the isometric embedding given by

$$
\langle I_H \mu, f \rangle = \int_H f \, d\mu \quad \big(f \in LUC(H),\ \mu \in M(H)\big)
$$

(see e.g. [10]) and define $\sigma_\alpha : M(H) \to M^*_\rho(G)$ so that the diagram

$$
\begin{array}{ccc}
LUC(H)^* & \xrightarrow{\ \tau_\alpha^*\ } & M(G) \\
\uparrow{\scriptstyle I_H} & & \downarrow{\scriptstyle \rho_G} \\
M(H) & \xrightarrow{\ \sigma_\alpha\ } & M^*_\rho(G)
\end{array}
$$

commutes. For $\mu \in M(H)$ we *claim* that $\sigma_\alpha(\mu) = \kappa_\alpha(\rho_H(\mu))$. Assuming that μ has compact support K, $\rho_H(\mu)$ also has compact support K [8, Remarque 4.7]. Take $\gamma \in A(H)$ such that $\gamma \equiv 1$ on a neighbourhood of K so that $\rho_H(\mu) = \gamma \cdot \rho_H(\mu)$. For $\psi \in A(G)$ we have

$$
\begin{aligned}
\big\langle \sigma_\alpha(\mu), \psi \big\rangle_{VN-A} &= \big\langle \rho_G\big(\tau_\alpha^*\big(I_H(\mu)\big)\big), \psi \big\rangle_{VN-A} \\
\text{by (5.1)} &= \big\langle \tau_\alpha^*\big(I_H(\mu)\big), \psi \big\rangle_{M-C_0} \\
&= \int_H \tau_\alpha \psi \, d\mu = \int_H (j_\alpha \psi) \gamma \, d\mu
\end{aligned}
$$

$$= \langle \rho_H(\mu), (j_\alpha\psi)\gamma \rangle_{VN-A} = \langle \kappa_\alpha(\gamma \cdot \rho_H(\mu)), \psi \rangle_{VN-A}$$
$$= \langle \kappa_\alpha(\rho_H(\mu)), \psi \rangle_{VN-A},$$

giving the claim. Thus, κ_α maps $\rho_H(M(H))$ into $M_\rho^*(G)$ and therefore κ_α maps $M_\rho^*(H)$ into $M_\rho^*(G)$. □

Remark 5.7. The map $\kappa_\alpha^* : M_\rho^*(G)^* \to M_\rho^*(H)^*$ is of interest to us (see Theorem 5.9) and can be described as follows.

Let $m \in M_\rho^*(G)^*$ and take $(\phi_i)_i$ to be a net in $A(G)$ such that $\pi_G(\phi_i) \to m w^*$ in $M_\rho^*(G)^*$. Then for each $\mu \in M(H)$,

$$\langle \kappa_\alpha^*(m), \rho_H(\mu) \rangle = \lim_i \int_H j_\alpha\phi_i \, d\mu = \lim_i \int_Y \phi_i \circ \alpha \, d\mu$$

where $\alpha : Y \subset H \to G$.

Indeed, w^*–w^*-continuity of κ_α^* and the calculation found in the proof of Proposition 5.6 give

$$\langle \kappa_\alpha^*(m), \rho_H(\mu) \rangle = \lim_i \langle \kappa_\alpha^*(\pi_G(\phi_i)), \rho_H(\mu) \rangle = \lim_i \langle \pi_G(\phi_i), \kappa_\alpha(\rho_H(\mu)) \rangle$$
$$= \lim_i \langle \kappa_\alpha(\rho_H(\mu)), \phi_i \rangle_{VN-A} = \lim_i \int_H j_\alpha\phi_i \, d\mu.$$

Proposition 5.8. *The operator κ_α maps $W_u(\widehat{H})$ into $W_u(\widehat{G})$ and $AP_u(\widehat{H})$ into $AP_u(\widehat{G})$.*

Proof. It is obvious that $W_u(\widehat{H})$ and $AP_u(\widehat{H})$ are $B_\rho(H)$-invariant, so we know from Proposition 5.3 that $\mathcal{X}_W = \overline{\kappa_\alpha(W_u(\widehat{H}))}$ and $\mathcal{X}_A = \overline{\kappa_\alpha(AP_u(\widehat{H}))}$) are topologically invariant and introverted in $VN(G)$. By a theorem of Lau [19, Theorem 5.6], to prove that $\mathcal{X}_W$ is contained in $W(\widehat{G})$ it suffices to show that multiplication in $\mathcal{X}_W^*$ is separately w^*-continuous on bounded sets.

Let $m \in \mathcal{X}_W^*$. Suppose that (n_i) is a bounded net in $\mathcal{X}_W^*$, $n \in \mathcal{X}_W^*$, and $n_i \to n \ w^*$ in $\mathcal{X}_W^*$. Let $S = \kappa_\alpha(T) \in \mathcal{X}_W$ where $T \in W_u(\widehat{H})$. As $(\kappa_\alpha^*(n_i))$ is bounded in $W_u(\widehat{H})^*$, and $\kappa_\alpha^* : \mathcal{X}_W^* \to W_u(\widehat{H})^*$ is a w^*–w^*-continuous homomorphism, Ref. [19, Theorem 5.6] gives

$$\lim_i \langle m \odot n_i, S \rangle = \lim_i \langle m \odot n_i, \kappa_\alpha(T) \rangle = \lim_i \langle \kappa_\alpha^*(m) \odot \kappa_\alpha^*(n_i), S \rangle$$
$$= \langle \kappa_\alpha^*(m) \odot \kappa_\alpha^*(n), S \rangle = \langle m \odot n, T \rangle.$$

Continuity in the other variable is trivial. Hence, $\mathcal{X}_W$ is contained in $W(\widehat{G})$. One can similarly use [19, Theorem 5.8] to show that multiplication in $\mathcal{X}_A^*$ is jointly w^*-continuous on bounded subsets of $\mathcal{X}_A^*$; Ref. [19, Theorem 5.8] then gives $\mathcal{X}_A \subset AP(\widehat{G})$. □

5.3. Summary

The next two theorems are immediate consequences of Propositions 5.2, 5.6, and 5.8. The uniqueness statements follow from w^*-density of $E_G(A(G))$ in $\mathfrak{X}_G^*$, (see Remark 5.1(i) and Lemma 3.1(iii)).

Theorem 5.9. *Let $(\mathfrak{X}_H, \mathfrak{X}_G)$ be one of the pairs $(UCB(\widehat{H}), UCB(\widehat{G}))$, $(M_\rho^*(H), M_\rho^*(G))$, or $(W_{\mathrm{u}}(\widehat{H}), W_{\mathrm{u}}(\widehat{G}))$. Then κ_α maps $\mathfrak{X}_H$ into $\mathfrak{X}_G$, the diagram*

$$\begin{array}{ccc} \mathfrak{X}_G^* & \xrightarrow{\ \kappa_\alpha^*\ } & \mathfrak{X}_H^* \\ {\scriptstyle E_G}\uparrow & & \uparrow{\scriptstyle E_H} \\ B_\rho(G) & \xrightarrow{\ j_\alpha\ } & B_\rho(H) \end{array}$$

commutes, and $\kappa_\alpha^* : \mathfrak{X}_G^* \to \mathfrak{X}_H^*$ is the (unique) w^*–w^*-continuous, homomorphic extension of $j_\alpha : B_\rho(G) \to B_\rho(H)$. If H is (1-)weakly amenable, then κ_α is completely bounded (and $\|\kappa_\alpha\|_{\mathrm{cb}} = \|\kappa_\alpha^*\|_{\mathrm{cb}} = \|j_\alpha\|_{\mathrm{cb}}$).

If $\mathfrak{X}_H$ contains $C_\rho^*(H)$, let $P_H : \mathfrak{X}_H^* \to B_\rho(H) = C_\rho^*(H)^* : m \mapsto m|_{C_\rho^*(H)}$. Observe that P_H is w^*–w^*-continuous and, by Remark 5.1(iii), $P_H \circ E_H = id_{B_\rho(H)}$.

Theorem 5.10. *Let $(\mathfrak{X}_H, \mathfrak{X}_G)$ be one of the pairs $(UCB(\widehat{H}), UCB(\widehat{G}))$, $(M_\rho^*(H), M_\rho^*(G))$, or $(W_{\mathrm{u}}(\widehat{H}), W_{\mathrm{u}}(\widehat{G}))$, and let $\widetilde{\kappa}_\alpha^* = P_H \circ \kappa_\alpha^*$. Then the diagram*

$$\begin{array}{ccc} \mathfrak{X}_G^* & \xrightarrow{\ \kappa_\alpha^*\ } & \mathfrak{X}_H^* \\ {\scriptstyle E_G}\uparrow & \searrow^{\widetilde{\kappa}_\alpha^*} & \downarrow{\scriptstyle P_H} \\ B_\rho(G) & \xrightarrow{\ j_\alpha\ } & B_\rho(H) \end{array}$$

commutes, and $\widetilde{\kappa}_\alpha^* : \mathfrak{X}_G^* \to B_\rho(H)$ is the (unique) w^*–w^*-continuous, homomorphic extension of $j_\alpha : B_\rho(G) \to B_\rho(H)$. If H is (1-)weakly amenable, then κ_α is completely bounded (and $\|\widetilde{\kappa}_\alpha^*\|_{\mathrm{cb}} = \|j_\alpha\|_{\mathrm{cb}}$).

Corollary 5.11. *Let G be an amenable locally compact group, $(\mathfrak{X}_H, \mathfrak{X}_G)$ be one of the pairs $(UCB(\widehat{H}), UCB(\widehat{G}))$, $(M_\rho^*(H), M_\rho^*(G))$, or $(W_{\mathrm{u}}(\widehat{H}), W(\widehat{G}))$.*

(i) *Then every completely bounded homomorphism $\varphi : A(G) \to B_\rho(H)$ extends (uniquely) to a τ_M–w^*-continuous homomorphism $\widetilde{\varphi} : B(G) \to B_\rho(H)$. This further extends (uniquely) to w^*–w^*-continuous homomorphisms*

$$\Phi : \mathfrak{X}_G^* \to \mathfrak{X}_H^* \quad \text{and} \quad \widetilde{\Phi} : X_G^* \to B_\rho(H).$$

Moreover, there is a piecewise affine, continuous map $\alpha : Y \subset H \to G$ *such that*

$$\varphi = j_\alpha, \qquad \widetilde{\varphi} = j_\alpha, \qquad \Phi = \kappa_\alpha^*, \qquad \widetilde{\Phi} = \widetilde{\kappa}_\alpha^*.$$

If H *is* (1-)*weakly amenable, then all extensions are completely bounded* (*and cb-norm preserving*).

(ii) *Conversely, let* $\Phi : \mathfrak{X}_G^* \to \mathfrak{X}_H^*$ *be a* w^*–w^**-continuous homomorphism which maps* $A(G)$ *into* $B_\rho(H)$ *and is completely bounded on* $A(G)$*. Then there is a piecewise affine, continuous map* $\alpha : Y \subset H \to G$ *such that* $\Phi = \kappa_\alpha^*$*; if* H *is weakly amenable,* Φ *is completely bounded on* $\mathfrak{X}_G^*$.

(iii) *Let* $\widetilde{\Phi} : \mathfrak{X}_G^* \to B_\rho(H)$ *be a* w^*–w^**-continuous homomorphism which is completely bounded on* $A(G)$*. Then there is a piecewise affine, continuous map* $\alpha : Y \subset H \to G$ *such that* $\widetilde{\Phi} = \widetilde{\kappa}_\alpha^*$*; if* H *is weakly amenable,* $\widetilde{\Phi}$ *is completely bounded on* $\mathfrak{X}_G^*$.

Proof. By [17, Theorem 3.7], there is a continuous piecewise affine map $\alpha : Y \subset H \to G$ such that $\varphi = j_\alpha$; this extends to $j_\alpha : B(G) \to B(H)$. As $A(G)$ has a contractive bounded approximate identity, the unit ball of $A(G)$ is τ_M-dense in the unit ball of $B(G)$ (for example, see the proof of [18, Theorem 5.6]). By [18, Theorem 5.10], j_α is τ_M–w^*-continuous on bounded subsets of $B(G)$ and $B_\rho(H)$ is w^*-closed in $B(H)$, so j_α maps $B(G)$ into $B_\rho(H)$; this gives $\widetilde{\varphi} = j_\alpha : B(G) \to B_\rho(H)$. The remaining statements follow from Corollary 4.5 and Theorems 5.9 and 5.10. □

We know from Theorems 5.10 and 5.9 that $\widetilde{\kappa}_\alpha^* : UCB(\widehat{G})^* \to B_\rho(H)$ and $\kappa_\alpha^* : UCB(\widehat{G})^* \to UCB(\widehat{H})^*$ are the unique w^*–w^*-continuous extensions of j_α between the specified spaces. We also have the very simply described homomorphisms $j_\alpha \circ P_G : UCB(\widehat{G})^* \to B_\rho(H)$ and $\pi_H \circ j_\alpha \circ P_G : UCB(\widehat{G})^* \to UCB(\widehat{H})^*$, however as the following example shows, these maps do not necessarily agree with $\widetilde{\kappa}_\alpha^*$ and κ_α^* and are therefore not necessarily w^*–w^*-continuous.

Example 5.12. The following diagrams do *not* necessarily commute:

(i)
$$\begin{array}{ccc} UCB(\widehat{G})^* & & \\ \downarrow P_G & \searrow^{\widetilde{\kappa}_\alpha^*} & \\ B_\rho(G) & \xrightarrow{\ j_\alpha\ } & B_\rho(H) \end{array}$$

(ii)
$$\begin{array}{ccc} UCB(\widehat{G})^* & \xrightarrow{\ \kappa_\alpha^*\ } & UCB(\widehat{H})^* \\ \downarrow P_G & & \uparrow \pi_H \\ B_\rho(G) & \xrightarrow{\ j_\alpha\ } & B_\rho(H) \end{array}$$

For (i), take G to be a non-discrete amenable locally compact group. Then $\rho_G(e_G) \notin C_\rho^*(G)$ so we can choose $m \in UCB(\widehat{G})^*$ such that $m|_{C_\rho^*(G)} = 0$ and $m(\rho_G(e_G)) = 1$. Let $\alpha : G \to G : g \mapsto e_G$, so $j_\alpha : B_\rho(G) \to B_\rho(G) : \phi \mapsto \phi(e_G)1_G$. Then $j_\alpha \circ P_G(m) = 0$, however $\widetilde{\kappa}_\alpha^*(m) = P_G \circ \kappa_\alpha^*(m) \neq 0$. To see this, let $f \in L^1(G)$ be such that $\int_G f = 1$ and take (ϕ_i) to be a net in $A(G)$ such that $\pi_G(\phi_i) \to m\, w^*$ in $UCB(\widehat{G})^*$. Then $\rho_G(f) \in C_\rho^*(G)$ and w^*–w^*-continuity of $\widetilde{\kappa}_\alpha^*$ gives

$$\begin{aligned} \langle \widetilde{\kappa}_\alpha^*(m), \rho_G(f) \rangle &= \lim_i \langle P_G \circ \kappa_\alpha^* \circ \pi_G(\phi_i), \rho_G(f) \rangle = \lim_i \langle j_\alpha(\phi_i), \rho_G(f) \rangle_{B_\rho - C_\rho^*} \\ &= \lim_i \phi_i(e_G) \int_G f = \lim_i \langle \pi_G(\phi_i), \rho_G(e_G) \rangle = \langle m, \rho_G(e_G) \rangle = 1. \end{aligned}$$

To see that diagram (i) does not necessarily commute, take G to be any non-discrete group and let $\alpha : G \to G$ be the identity isomorphism. Then $j_\alpha = id_{B_\rho(G)} : B_\rho(G) \to B_\rho(G)$ has w^*–w^*-continuous homomorphic extension $id_{UCB(\widehat{G})^*} : UCB(\widehat{G})^* \to UCB(\widehat{G})^*$. As κ_α^* is the unique map with this property, $\kappa_\alpha^* = id_{UCB(\widehat{G})^*}$ which is surjective. However, we know from Proposition 3.5 that π_G is not surjective, so $\kappa_\alpha^* \neq \pi_G \circ j_\alpha \circ P_G$.

6. Homomorphisms of reduced Fourier–Stieltjes algebras

Throughout this paper we have assumed that the induced homomorphism j_α maps $B_\rho(G)$ into $B_\rho(H)$. In this section we characterize, in terms of α, when this is the case.

Recall first that if π and σ are unitary representations of G, then π is *weakly contained* in σ ($\pi \preccurlyeq \sigma$) if, given any $\epsilon > 0$, any compact subset K of G, and any positive definite function ϕ associated with π, there is a finite sum, ψ, of positive definite functions associated with σ such that $|\phi(x) - \psi(x)| < \epsilon$ $(x \in K)$. We will use the fact that $\pi \preccurlyeq \sigma$ if and only if $B_\pi \subset B_\sigma$ [1, Proposition 3.1]. For much more about weak containment see [2] and [9]. As well, we recall that the idempotents in $B(G)$ are precisely the characteristic functions 1_Z of sets Z in the open coset ring of G [14]. For an element $x \in G$, $l_x, r_x : B(G) \to B(G)$ denote the left and right translation operators.

In the following theorem, $\alpha : Y \subset H \to G$ is a continuous piecewise affine map, with Y written as a disjoint union $Y = \bigcup_{i=1}^n Y_i$, $Y_i \in \Omega_o(G)$, and $\alpha_i : E_i \to G$ an affine map on $E_i = \mathit{Aff}(Y_i)$ such that $\alpha|_{Y_i} = \alpha_i|_{Y_i}$ $(i = 1, \dots, n)$. For each i, H_i is the subgroup $H_i = E_i^{-1}E_i$ of H and $\beta_i : H_i \to G$ is the homomorphism given by $\beta_i(h) = \alpha(y_i)^{-1}\alpha(y_i h)$ $(h \in H_i, y_i \in Y_i)$. By taking $\pi = \omega_G$ and $\pi = \rho_G$, the theorem respectively describes when j_α maps $B(G)$ and $B_\rho(G)$ into $B_\rho(H)$.

Theorem 6.1. *Let π be a continuous unitary representation of G. Then j_α maps B_π into $B_\rho(H)$ if and only if for each $i = 1, \dots, n$, the representation $\pi \circ \beta_i$ of H_i is weakly contained in ρ_{H_i}, the left regular representation of H_i. In particular, if each H_i is amenable, then j_α maps $B(G)$ into $B_\rho(H)$.*

Proof. First note that if H_i is amenable, then for any continuous unitary representation σ_i of H_i, $\sigma_i \preccurlyeq \rho_{H_i}$ by the weak containment property of amenable groups [2, Appendix G]. Hence, the second statement follows from the first statement of the theorem.

Suppose that for each i, $\pi \circ \beta_i \preccurlyeq \rho_{H_i}$. By [8, Theorem 2.20] and [1, Proposition 3.1], $j_{\beta_i}(B_\pi) \subset B_{\pi\circ\beta_i} \subset B_{\rho_{H_i}} = B_\rho(H_i)$. The expansion map $s_i : B(H_i) \to B(H) : \phi \mapsto \phi^\circ$ is w^*–w^*-continuous (see [18, Lemma 2.2]) and by [8, Proposition 3.21], s_i maps $A(H_i)$ into $A(H)$; hence s_i maps $B_\rho(H_i)$ into $B_\rho(H)$. It follows that $j_{\alpha_i} = l_{y_i^{-1}} \circ s_i \circ j_{\beta_i} \circ l_{\alpha(y_i)}$ maps B_π (which is translation invariant) into $B_\rho(H)$ and therefore, because $B_\rho(H)$ is an ideal in $B(H)$, $j_\alpha = \sum_{i=1}^n \chi_i \circ j_{\alpha_i}$ maps B_π into $B_\rho(H)$ as well; here $\chi_i\phi = \phi 1_{Y_i}$ $(\phi \in B_\rho(H))$.

Conversely, suppose that j_α maps B_π into $B_\rho(H)$. Then for each i, $\chi_i \circ j_\alpha = j_{\alpha_i'}$ maps B_π into $B_\rho(H)$ where $\alpha_i' = \alpha|_{Y_i} = \alpha_i|_{Y_i}$. Suppose for now that we can show that j_{α_i} also maps B_π into $B_\rho(H)$. The restriction map r_i of $B(H)$ into $B(H_i)$ is w^*–w^*-continuous and maps $A(H)$ into $A(H_i)$ so it also maps $B_\rho(H)$ into $B_\rho(H_i)$ (see, for example [18, Lemma 2.2] and [8, Proposition 3.21]). Therefore $j_{\beta_i} = r_i \circ l_{y_i} \circ j_{\alpha_i} \circ l_{\alpha(y_i)^{-1}}$ maps B_π into $B_\rho(H_i)$, and hence maps A_π into $B_\rho(H_i)$. That is, $j_{\beta_i}(A_\pi) = A_{\pi\circ\beta_i} \subset B_\rho(H_i)$, using [1, Proposition 2.10], and therefore $B_{\pi\circ\beta_i}$, the weak*-closure of $A_{\pi\circ\beta_i}$, is also contained in $B_\rho(H_i)$. Hence, $\pi\circ\beta_i \preccurlyeq \rho_{H_i}$, $i=1,\dots,n$.

We now fix i, let $Z=Y_i$, $E=\mathit{Aff}(Y_i)$, $\gamma=\alpha_i$, and $\gamma'=\alpha_i'=\gamma|_Z$. Assuming that $j_{\gamma'}$ maps B_π into $B_\rho(H)$, we now complete the proof by showing that j_γ also maps B_π into $B_\rho(H)$. By [16, Lemma 4.5], we can choose a finite subset $F=\{p_1,\dots,p_m\}$ of $E^{-1}E$ such that $E=ZF$. Let

$$Z_1=Zp_1,\qquad Z_k=Zp_k\setminus\bigcup_{l=1}^{k-1}Zp_l\quad \text{for } k=2,3,\dots,m.$$

Then each Z_k is in the open coset ring of H and E is the disjoint union $E=\bigcup_{k=1}^m Z_k$. Therefore,

$$j_\gamma\phi=\sum_{k=1}^m (j_\gamma\phi)\cdot 1_{Z_k}\quad(\phi\in B_\pi). \tag{6.1}$$

Let $x_k, z_k\in E$ be such that $p_k=x_k^{-1}z_k$, $q_k=\gamma(x_k)^{-1}\gamma(z_k)$, and let $z\in Z_k$. Then $z=yp_k$ for some $y\in Z\subset E$, so

$$j_\gamma\phi(z)=\phi\big(\gamma\big(yx_k^{-1}z_k\big)\big)=\phi\big(\gamma'(y)\gamma(x_k)^{-1}\gamma(z_k)\big)=j_{\gamma'}(r_{q_k}\phi)(y)=r_{p_k^{-1}}\big(j_{\gamma'}(r_{q_k}\phi)\big)(z).$$

We are assuming that $j_{\gamma'}$ maps B_π into $B_\rho(H)$, so $j_\gamma\phi\cdot 1_{Z_k}=r_{p_k^{-1}}(j_{\gamma'}(r_{q_k}\phi))\cdot 1_{Z_k}\in B_\rho(H)$, for $\phi\in B_\pi$. That j_γ maps B_π into $B_\rho(H)$ now follows from (6.1). □

7. Weak*–weak*-continuous homomorphisms $\mathfrak{X}_G^*\to\mathfrak{X}_H^*$

In this final section, $\mathfrak{X}_G$ and $\mathfrak{X}_H$ are, respectively, subspaces of $UCB(\widehat{G})$ and $UCB(\widehat{H})$ which are topologically invariant and introverted in $VN(G)$ and $VN(H)$. It seems natural to wonder when a map $\kappa:\mathfrak{X}_H\to\mathfrak{X}_G$ dualizes to give a homomorphism $j:\mathfrak{X}_G^*\to\mathfrak{X}_H^*$. In this section we describe such maps as those which intertwine various module actions.

We let $\mathfrak{A}_G=\mathfrak{X}_G^*$, $\mathfrak{A}_H=\mathfrak{X}_H^*$. The dual $\mathfrak{A}_G$-module action on $\mathfrak{A}_G^*$ is denoted by

$$\langle m.M,n\rangle=\langle M,n\odot m\rangle\quad\text{and}\quad\langle M.m,n\rangle=\langle M,m\odot n\rangle\quad\big(M\in\mathfrak{A}_G^*,\ m,n\in\mathfrak{A}_G\big).$$

As before $\mathfrak{X}_G$ is also a Banach $A(G)$-submodule of $VN(G)$, and is a $B_\rho(G)$-submodule of $VN(G)$ when $\mathfrak{X}_G$ is $B_\rho(G)$-invariant (see Section 5.1); we continue to denote these module operations by $\phi\cdot T=T\cdot\phi$ for $\phi\in B_\rho(G)$ and $T\in\mathfrak{X}_G$. Defining $E_G=R_G\circ\pi_G:B_\rho(G)\to\mathfrak{X}_G^*$ as in Section 5.1 we will slightly abuse notation and write $\widehat{\phi}=E_G\phi(=\widehat{\phi}|_{\mathfrak{X}_G})$. The map $\phi\mapsto\widehat{\phi}$ is an isometric embedding (completely isometric if G is 1-weakly amenable) when $C_\rho^*(G)\subset\mathfrak{X}_G$; see Remark 5.1.

Lemma 7.1. *For $\phi\in A(G)$ and $T\in\mathfrak{X}_G\hookrightarrow\mathfrak{X}_G^{**}=\mathfrak{A}_G^*$, $\widehat{\phi}.T=\phi\cdot T=T\cdot\phi=T.\widehat{\phi}$. When $\mathfrak{X}_G$ is $B_\rho(G)$-invariant the statement holds for $\phi\in B_\rho(G)$.*

Proof. Assume that $\mathfrak{X}_G$ is $B_\rho(G)$-invariant, let $T\in\mathfrak{X}_G$, $\phi\in B_\rho(G)$. For $m\in\mathfrak{A}_G$,

$$\langle\widehat{\phi}.T,m\rangle=\langle T,m\odot\widehat{\phi}\rangle=\big\langle m,\widehat{\phi}_L(T)\big\rangle=\langle m,\phi\cdot T\rangle=\langle\phi\cdot T,m\rangle$$

as needed. □

Let

$$\kappa : \mathfrak{X}_H \to \mathfrak{X}_G, \qquad j = \kappa^* : \mathfrak{A}_G \to \mathfrak{A}_H.$$

The following proposition describes when j is a homomorphism.

Proposition 7.2. *The map $j : \mathfrak{A}_G \to \mathfrak{A}_H$ is a homomorphism if and only if*

$$\kappa(T) \cdot \phi = \kappa^{**}\big(T.j(\widehat{\phi})\big) \quad \big(T \in \mathfrak{X}_H,\ \phi \in A(G)\big). \tag{7.1}$$

Proof. Suppose that κ satisfies (7.1) and let $m, n \in \mathfrak{A}_G$. Then for $T \in \mathfrak{X}_H$ and $\phi \in A(G)$,

$$\begin{aligned}
\big\langle n_L\big(\kappa(T)\big), \phi\big\rangle_{VN-A} &= \big\langle n, \kappa(T) \cdot \phi\big\rangle_{\mathfrak{X}_G^*-\mathfrak{X}_G} = \big\langle \kappa(T) \cdot \phi, n\big\rangle_{\mathfrak{A}_G^*-\mathfrak{A}_G} \\
&= \big\langle \kappa^{**}\big(T.j(\widehat{\phi})\big), n\big\rangle_{\mathfrak{A}_G^*-\mathfrak{A}_G} = \big\langle T.j(\widehat{\phi}), j(n)\big\rangle_{\mathfrak{A}_H^*-\mathfrak{A}_H} \\
&= \big\langle j(\widehat{\phi}) \odot j(n), T\big\rangle_{\mathfrak{X}_H^*-\mathfrak{X}_H} = \big\langle \widehat{\phi}, \kappa\big(j(n)_L(T)\big)\big\rangle_{\mathfrak{X}_G^*-\mathfrak{X}_G} \\
&= \big\langle \kappa\big(j(n)_L(T)\big), \phi\big\rangle_{VN-A}.
\end{aligned}$$

Therefore,

$$\big\langle j(m \odot n), T\big\rangle = \big\langle m \odot n, \kappa(T)\big\rangle = \big\langle m, n_L\big(\kappa(T)\big)\big\rangle = \big\langle m, \kappa\big(j(n)_L(T)\big)\big\rangle = \big\langle j(m) \odot j(n), T\big\rangle.$$

Conversely, suppose that j is a homomorphism. Then

$$\big\langle m, \kappa\big(j(n)_L(T)\big)\big\rangle = \big\langle j(m) \odot j(n), T\big\rangle = \big\langle j(m \odot n), T\big\rangle = \big\langle m, n_L\big(\kappa(T)\big)\big\rangle,$$

so $n_L(\kappa(T)) = \kappa(j(n)_L(T))$ $(n \in \mathfrak{A}_G)$. Hence,

$$\begin{aligned}
\big\langle \kappa(T) \cdot \phi, n\big\rangle_{\mathfrak{A}_G^*-\mathfrak{A}_G} &= \big\langle n_L\big(\kappa(T)\big), \phi\big\rangle_{VN-A} = \big\langle \kappa\big(j(n)_L(T)\big), \phi\big\rangle_{VN-A} \\
&= \big\langle \kappa^{**}\big(T.j(\widehat{\phi})\big), n\big\rangle_{\mathfrak{A}_G^*-\mathfrak{A}_G},
\end{aligned}$$

where we have used parts of the first calculation found in this proof. □

Further to the hypotheses of the last proposition, suppose that $\mathfrak{X}_H$ is $B_\rho(H)$-invariant and that $j(A(G)) \subset B_\rho(H)$ (identifying $B_\rho(H)$ with $E_H(B_\rho(H))$). Under these hypotheses, Lemma 7.1 and Proposition 7.2 give the following corollary.

Corollary 7.3. *$j = \kappa^* : \mathfrak{A}_G \to \mathfrak{A}_H$ is a homomorphism if and only if*

$$\kappa(T) \cdot \phi = \kappa\big(T \cdot j(\phi)\big) \quad \big(T \in \mathfrak{X}_H,\ \phi \in A(G)\big).$$

For $\phi \in B_\rho(G)$ and $f \in L^1(G)$, $\phi \cdot \rho_G(f) = \rho_G(\phi f)$, so $C_\rho^*(G)$ is $B_\rho(G)$-invariant. Also note that the module action agrees with the dual action of $B_\rho(G)$ on $C_\rho^*(G) \hookrightarrow C_\rho^*(G)^{**} = B_\rho(G)^*$, and that $E_G : B_\rho(G) \hookrightarrow C_\rho^*(G)^* = B_\rho(G)$ is the identity map in this case. The next corollary says that the dual map $j = \kappa^*$ of $\kappa : C_\rho^*(H) \to C_\rho^*(G)$ is a homomorphism if and only if, for each $\phi \in A(G)$, κ is intertwining with respect to the module actions by $j(\phi)$ and ϕ.

Corollary 7.4. *Let* $\kappa : C^*_\rho(H) \to C^*_\rho(G)$, $j = \kappa^* : B_\rho(G) \to B_\rho(H)$, *and consider the following statements*:

(i) *j is a homomorphism*;
(ii) *for any $T \in C^*_\rho(H)$ and $\phi \in A(G)$, $\kappa(T) \cdot \phi = \kappa(T \cdot j(\phi))$*;
(iii) *there is a continuous piecewise affine open map $\alpha : Y \subset H \to G$ such that*

$$\langle \phi, \kappa(\rho_H(f)) \rangle = \int_Y \phi(\alpha(s)) f(s)\, ds \quad (\phi \in B_\rho(G),\ f \in L^1(H)).$$

Then (i) ⇔ (ii) *and* (i) ⇔ (ii) ⇔ (iii) *when G is amenable and κ is completely bounded.*

Proof. The first statement is immediate from Corollary 7.3; the second statement is a consequence of [18, Theorem 5.11]. □

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