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Uniform $W^{1,p}$ estimates for systems of linear elasticity in a periodic medium

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Abstract

Let $\{\mathcal{L}_\varepsilon\}$ be a family of elliptic systems of linear elasticity with rapidly oscillating periodic coefficients. We obtain the uniform $W^{1,p}$ estimate $\|\nabla u_\varepsilon\|_p \leqslant C\|f\|_p$ in a Lipschitz domain Ω in $\mathbb{R}^n$ for solutions to the Dirichlet problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f)$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$, where $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n} + \delta$ and $C, \delta > 0$ are constants independent of $\varepsilon > 0$. The ranges are sharp for $n = 2$ or 3.

Keywords: System of elasticity; Homogenization; Lipschitz domain

1. Introduction

The primary purpose of this paper is to study uniform $W^{1,p}$ estimates for a family of elliptic systems of linear elasticity with rapidly oscillating coefficients. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$. Consider the Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where

$$\mathcal{L}_\varepsilon = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right] = -\operatorname{div}\left[A\left(\frac{x}{\varepsilon}\right)\nabla\right], \quad \varepsilon > 0. \tag{1.2}$$

We will assume that the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ is real and satisfies

$$a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y) \quad \text{for } 1 \leqslant i, j, \alpha, \beta \leqslant n \text{ and } y \in \mathbb{R}^n, \tag{1.3}$$

$$\mu|\xi|^2 \leqslant a_{ij}^{\alpha\beta}(y)\xi_i^\alpha \xi_j^\beta \leqslant \frac{1}{\mu}|\xi|^2 \quad \text{for } y \in \mathbb{R}^n, \tag{1.4}$$

where μ is a positive constant and $\xi = (\xi_i^\alpha)$ is any $n \times n$ *symmetric* matrix with real entries, and the periodicity condition

$$A(y+z) = A(y) \quad \text{for } y \in \mathbb{R}^n \text{ and } z \in \mathbb{Z}^n. \tag{1.5}$$

We say $A \in \mathcal{M}(\mu, \lambda, \tau)$ if it satisfies (1.3), (1.4), (1.5) and the smoothness condition

$$\left|A(x) - A(y)\right| \leqslant \tau|x-y|^\lambda \quad \text{for some } \lambda \in (0,1) \text{ and } \tau \geqslant 0. \tag{1.6}$$

The following is the main result of the paper.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla)$ with $A \in \mathcal{M}(\mu, \lambda, \tau)$. Then for any $f \in L^p(\Omega)$ with $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n} + \delta$, there exists a unique $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f)$ in Ω. Moreover, the solution u_ε satisfies*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leqslant C_p \|f\|_{L^p(\Omega)}, \tag{1.7}$$

and constants $\delta > 0$ and $C_p > 0$ are independent of ε.

We will also consider a family of general second-order elliptic systems $\{-\operatorname{div}(A(x/\varepsilon)\nabla)\}$, where $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \leqslant i, j \leqslant n$ and $1 \leqslant \alpha, \beta \leqslant m$. We say $A \in \Lambda(\mu, \lambda, \tau)$ if it satisfies (1.5)–(1.6) and the ellipticity condition (1.4) for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{nm}$. The symmetry condition $A = A^*$, i.e., $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$, is also needed in the following theorem.

Theorem 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla)$ with $A \in \Lambda(\mu, \lambda, \tau)$ and $A = A^*$. Then for any $f \in L^p(\Omega)$ with $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n} + \delta$, there exists a unique $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f)$ in Ω. Moreover, the solution u_ε satisfies* (1.7) *and constants $\delta > 0$ and $C_p > 0$ are independent of ε.*

Elliptic equations and systems with rapidly oscillating coefficients arise in the theory of homogenization (see e.g. [4,18]). It is well known that as $\varepsilon \to 0$, the solution u_ε of (1.1) converges to u_0 weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$, where $u_0 \in W_0^{1,2}(\Omega)$ is the solution of the homogenized elliptic system. Uniform regularity estimates of u_ε are an important tool in the study of various convergence problems for $\mathcal{L}_\varepsilon$. We remark that if Ω is $C^{1,\alpha}$ and $A \in \Lambda(\mu, \lambda, \tau)$, the uniform $W^{1,p}$ estimate (1.7) was established in [2,3] for any $1 < p < \infty$, without the symmetry condition $A = A^*$ (see [15] for its extension to the Neumann boundary condition with the symmetry condition). It was pointed out in [3] that the same approach also gives estimate (1.7) for $1 < p < \infty$ if Ω is $C^{1,\alpha}$ and $A \in \mathcal{M}(\mu, \lambda, \tau)$. We also mention that if Ω is Lipschitz and $m = 1$, the $W^{1,p}$ estimate (1.7) was obtained in [20] for $(4/3) - \delta < p < 4 + \delta$ and $n = 2$, and for $(3/2) - \delta < p < 3 + \delta$ and $n \geqslant 3$. The ranges of p's in [20] are known to be sharp (even for the Laplacian [14]). It follows that the ranges of p's in Theorems 1.1 and 1.2 are sharp for $n = 2$ or 3. The question of sharp ranges of p's for which the $W^{1,p}$ estimate holds in Lipschitz domains remains open in the case $n \geqslant 4$ (even for elliptic systems with constant coefficients). We remark that in the non-periodic setting the $W^{1,p}$ estimates for second-order elliptic equations and systems have been studied extensively in recent years. We refer the reader to [1,6,5,19,17,7,11] and their references for various results on elliptic operators with nonsmooth coefficients in nonsmooth domains.

For a ball $B = B(x, r)$, we will use tB to denote $B(x, tr)$. Recall that Ω is a Lipschitz domain if there exists $r_0 > 0$ such that for any $Q \in \partial\Omega$, there exists a Lipschitz function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Omega \cap B(Q, 8r_0)$ is given by $\{(x', x_n) \in \mathbb{R}^n\colon x_n > \psi(x')\} \cap B(Q, 8r_0)$, after some possible translation and rotation of the coordinate system. The proofs of Theorems 1.1 and 1.2 rely on the following.

Theorem 1.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $q > 2$. Let $\mathcal{L} = -\operatorname{div}(A\nabla)$ be a second-order elliptic system with $A = (a_{ij}^{\alpha\beta}(x))$ and $1 \leqslant i, j \leqslant n$, $1 \leqslant \alpha, \beta \leqslant m$. Suppose that* (1) $\|A\|_{L^\infty(\mathbb{R}^n)} \leqslant \mu^{-1}$; (2) *for any $\phi \in W_0^{1,2}(\mathbb{R}^n)$ and some $\mu > 0$,*

$$\mu \int_{\mathbb{R}^n} |\nabla\phi|^2\,dx \leqslant \int_{\mathbb{R}^n} a_{ij}^{\alpha\beta}(x) \frac{\partial\phi^\alpha}{\partial x_i} \cdot \frac{\partial\phi^\beta}{\partial x_j}\,dx; \tag{1.8}$$

(3) *for any $w \in W^{1,2}(3B \cap \Omega)$ with the property that $\mathcal{L}(w) = 0$ in $3B \cap \Omega$ and $w = 0$ on $3B \cap \partial\Omega$ (if $3B \cap \partial\Omega \neq \emptyset$), where either $3B \subset \Omega$ or $B = B(y, r)$ with $y \in \partial\Omega$ and $0 < r < r_0$, one has $|\nabla w| \in L^q(B \cap \Omega)$ and*

$$\left\{ \frac{1}{|B \cap \Omega|} \int_{B\cap\Omega} |\nabla w|^q\,dx \right\}^{1/q} \leqslant N \left\{ \frac{1}{|2B \cap \Omega|} \int_{2B\cap\Omega} |\nabla w|^2\,dx \right\}^{1/2}. \tag{1.9}$$

Then there exists $\delta > 0$, depending only on n, m, μ, q, N and the Lipschitz character of Ω, such that for any $f \in L^p(\Omega)$ with $2 < p < q + \delta$, the unique solution to $\mathcal{L}(u) = \operatorname{div}(f)$ in $W_0^{1,2}(\Omega)$ satisfies $\|\nabla u\|_{L^p(\Omega)} \leqslant C_p \|f\|_{L^p(\Omega)}$, where C_p depends only on n, m, μ, p, q, N and the Lipschitz character of Ω.

Theorem 1.3, which is proved in Section 2, follows by a real variable argument originated in [8] and further developed in [19]. As an application of Theorem 1.3, in Section 3, we establish the $W^{1,p}$ estimate in the non-periodic setting for elliptic systems with VMO coefficients in Lipschitz domains. Observe that by Lax–Milgram Theorem, the conditions (1) and (2) in Theorem 1.3 give the existence and uniqueness of $W^{1,2}$ solutions for any $f \in L^2(\Omega)$. Clearly, the ellipticity condition in Theorem 1.2 implies the coercive estimate (1.8). By the first Korn inequality this is also the case for Theorem 1.1. Consequently, to prove Theorems 1.1 and 1.2, as in [20], it suffices to establish the weak reverse Hölder inequality (1.9) with $q = p_n = \frac{2n}{n-1}$ for local $W^{1,2}$ solutions. We further note that under the assumption $A \in \Lambda(\mu, \lambda, \tau)$ or $A \in \mathcal{M}(\mu, \lambda, \tau)$, it follows from [2] that

$$\|\nabla u_\varepsilon\|_{L^\infty(B)} \leqslant C\left\{\frac{1}{|2B|}\int_{2B} |\nabla u_\varepsilon|^2\,dx\right\}^{1/2}, \tag{1.10}$$

if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $3B$. As a result we only need to establish (1.9) for $w \in W^{1,2}(3B \cap \Omega)$ satisfying $\mathcal{L}_\varepsilon(w) = 0$ in $3B \cap \Omega$ and $w = 0$ on $3B \cap \partial\Omega$, where $B = B(Q, r)$ for some $Q \in \partial\Omega$ and $0 < r < cr_0$, with constants c and N *independent* of the parameter $\varepsilon > 0$. We will present two different approaches to this boundary reverse Hölder estimate.

The proof of Theorem 1.2, given in Section 4, uses the recently established non-tangential maximal function estimates for the L^p Dirichlet and regularity problems in [16] for some $p = q_0 > 2$, under the conditions $A \in \Lambda(\mu, \lambda, \tau)$ and $A = A^*$. Let $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. To see (1.9), the basic idea is to write

$$\int_{B\cap\Omega} |\nabla w|^q\,dx = \int_{B\cap\Omega} |\nabla w|^{q_0} \cdot |\nabla w|^{q-q_0}\,dx$$

and estimate $|\nabla w|^{q_0}$ by its (local) non-tangential maximal function and $|\nabla w|^{q-q_0}$ by

$$\bigl|\nabla w(x)\bigr| \leqslant C\bigl[\rho(x)\bigr]^{-n/2}\left\{\int_{2B\cap\Omega} |\nabla w|^2\,dy\right\}^{1/2} \tag{1.11}$$

for any $x \in B \cap \Omega$, which follows from the interior estimate (1.10). This gives (1.9) for any $q < q_0 + \frac{2}{n}$, which can be used to improve the exponent of $\rho(x)$ in (1.11). The desired estimate (1.9) with $q = \frac{2n}{n-1}$ follows by an iteration argument.

In the case of elliptic systems of linear elasticity, the non-tangential maximal function estimates used in the proof of Theorem 1.2 are not known. To prove Theorem 1.1, we will instead adapt the approach used in [20] for single equations ($m = 1$). The idea is to reduce the estimate (1.9) to a decay estimate of an integral of $|w|^q$ (not $|\nabla w|^q$) on a boundary layer and apply a compactness argument. See Section 5 for details.

The summation convention is used throughout this paper. Unless indicated otherwise Ω will always be a bounded Lipschitz domain in $\mathbb{R}^n$. Finally, we will make no effort to distinguish vector-valued functions or function spaces from their real-valued counterparts. This should be clear from the context.

2. Proof of Theorem 1.3

By Lax–Milgram Theorem, under the conditions (1) and (2) in Theorem 1.3, given any $f \in L^2(\Omega)$, the system $\mathcal{L}(u) = \operatorname{div}(f)$ has a unique solution in $W_0^{1,2}(\Omega)$. Moreover, the solution satisfies the estimate $\|\nabla u\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}$, where C depends only on μ. Consider now the linear operator $T(f) = \nabla u$. Clearly, T is bounded on $L^2(\Omega)$. To show that T is bounded on $L^p(\Omega)$ for $2 < p < q + \delta$, we use the following theorem in [19, Theorem 3.3].

Theorem 2.1. *Let T be a bounded sublinear operator on $L^2(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $q > 2$. Suppose that there exists a constant $N > 1$ such that for any bounded measurable function f with* $\operatorname{supp}(f) \subset \Omega \setminus 3B$,

$$\left\{\frac{1}{r^n}\int_{\Omega\cap B} |Tf|^q\,dx\right\}^{1/p} \leqslant N\left\{\left(\frac{1}{r^n}\int_{\Omega\cap 2B} |Tf|^2\,dx\right)^{1/2} + \sup_{B'\supset B}\left(\frac{1}{|B'|}\int_{B'} |f|^q\,dx\right)^{1/q}\right\}, \tag{2.1}$$

where $B = B(x_0, r)$ is a ball with the property that $0 < r < c_0 r_0$ and either $x_0 \in \partial\Omega$ or $B(x_0, 3r) \subset \Omega$. Then T is bounded on $L^p(\Omega)$ for any $2 < p < q$.

It also follows from the proof of Theorem 2.1 in [19] that if $\|T\|_{L^2(\Omega)\to L^2(\Omega)} \leqslant C_0$, then $\|T\|_{L^p(\Omega)\to L^p(\Omega)}$ is bounded by a constant depending only on p, q, N, C_0, c_0 and the Lipschitz character of Ω. Therefore, to prove Theorem 1.3 for $2 < p < q$, it suffices to verify the condition (2.1) with $T(f) = \nabla u$. However, if $\operatorname{supp}(f) \subset \Omega \setminus 3B$, one has $\mathcal{L}(u) = 0$ in $3B \cap \Omega$. Thus the weak reverse Hölder inequality (1.9) with exponent q implies (2.1) with the same exponent q (without the supremum term in the right hand). Finally, we observe that the weak reverse Hölder condition (1.9) is self-improving (see e.g. [13]). That is, if $\mathcal{L}$ has the property (1.9) for some $q = q_1 > 2$, then it has the property for some $q = q_1 + \delta$, where $\delta > 0$ depends only on n, q_1, N and the Lipschitz character of Ω. Consequently, by Theorem 2.1, we obtain $\|\nabla u\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}$ for any $2 < p < q + \delta$. This completes the proof of Theorem 1.3. $\square$

Remark 2.2. Let $\mathcal{L}^*$ denote the adjoint of $\mathcal{L}$. Suppose that $u, v \in W_0^{1,2}(\Omega)$ and $\mathcal{L}(u) = \operatorname{div}(f)$ and $\mathcal{L}^*(v) = \operatorname{div}(g)$ in Ω for some $f = (f_i^\alpha)$, $g = (g_i^\alpha) \in L^2(\Omega)$. Then

$$\int_\Omega f_i^\alpha \cdot \frac{\partial v^\alpha}{\partial x_i}\,dx = -\int_\Omega a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\cdot\frac{\partial v^\alpha}{\partial x_i}\,dx = \int_\Omega g_i^\alpha \cdot \frac{\partial u^\alpha}{\partial x_i}\,dx. \tag{2.2}$$

It follows from (2.2) by duality that if the estimate $\|\nabla u\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}$ holds for any $f \in L^p(\Omega)$ and some $p > 2$, then $\|\nabla v\|_{L^q(\Omega)} \leqslant C\|g\|_{L^q(\Omega)}$ for any $g \in L^2(\Omega)$, where $q = p'$. By a density argument one may deduce that for any $g \in L^q(\Omega)$, there exists $v \in W_0^{1,q}(\Omega)$ such that $\mathcal{L}^*(v) = \operatorname{div}(g)$ in Ω and $\|\nabla v\|_{L^q(\Omega)} \leqslant C\|g\|_{L^q(\Omega)}$. The duality argument above also gives the uniqueness of such solutions.

Remark 2.3. Under the conditions (1) and (2) in Theorem 1.3, the well-known Caccioppoli's inequality

$$\int_{B\cap\Omega} |\nabla u|^2\,dx \leqslant \frac{C}{r^2}\int_{2B\cap\Omega} |u|^2\,dx \tag{2.3}$$

holds for any $u \in W^{1,2}(3B \cap \Omega)$ satisfying $\mathcal{L}(u) = 0$ in $3B \cap \Omega$ and $u = 0$ in $3B \cap \partial\Omega$, where $B = B(y, r)$ with $y \in \overline{\Omega}$ and $0 < r < cr_0$. By the Sobolev inequality this implies that

$$\left\{\frac{1}{r^n}\int_{B\cap\Omega} |\nabla u|^2\,dx\right\}^{1/2} \leqslant C\left\{\frac{1}{r^n}\int_{2B\cap\Omega} |\nabla u|^q\,dx\right\}^{1/q}, \tag{2.4}$$

for any $2n/(n+2) \leqslant q < 2$ if $n \geqslant 3$, and for any $1 < q < 2$ if $n = 2$. It follows that the weak reverse Hölder inequality (1.9) holds for some $q > 2$ and $N > 0$, which depend only on n, m, μ and the Lipschitz character of Ω [13].

3. $W^{1,p}$ estimates in the non-periodic setting

Following [19], as an application of Theorem 1.3, we obtain the $W^{1,p}$ estimate in the non-periodic setting for elliptic systems with VMO coefficients. Recall that $A \in \mathrm{VMO}(\mathbb{R}^n)$ if $\lim_{t\to 0} \omega(t) = 0$, where

$$\omega(t) = \sup_{\substack{x\in\mathbb{R}^n\\ 0<r<t}} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left| A(y) - \frac{1}{|B(x,r)|} \int_{B(x,r)} A(z)\,dz \right| dy. \tag{3.1}$$

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\mathcal{L} = -\operatorname{div}(A(x)\nabla)$ with $A(x) = (a_{ij}^{\alpha\beta}(x))$ and $1 \leqslant i,j \leqslant n$, $1 \leqslant \alpha,\beta \leqslant m$. Suppose that* (1) $\|A\|_{L^\infty(\mathbb{R}^n)} \leqslant \mu^{-1}$; (2) *estimate* (1.8) *holds for any $\phi \in W_0^{1,2}(\mathbb{R}^n)$*; (3) $A = A^*$; *and* (4) $A \in \mathrm{VMO}(\mathbb{R}^n)$. *Then there exists $\delta > 0$ such that for any $f \in L^p(\Omega)$ with $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n} + \delta$, there exists a unique $u \in W_0^{1,p}(\Omega)$ satisfying $\mathcal{L}(u) = \operatorname{div}(f)$ in Ω. Moreover, the solution u satisfies $\|\nabla u\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}$.*

In view of Remark 2.2 and the assumption $A^* = A$, it suffices to prove Theorem 3.1 for $2 < p < p_n + \delta$, where $p_n = \frac{2n}{n-1}$. Furthermore, by Theorem 1.3, we only need to establish the weak reverse Hölder estimate in condition (3) in Theorem 1.3 for $q = p_n$. Note that under the condition $A \in \mathrm{VMO}(\mathbb{R}^n)$, the estimate (1.9) in the case $3B \subset \Omega$ is well known and in fact holds for any $2 < q < \infty$. As a result Theorem 3.1 follows from the following.

Theorem 3.2. *Let $\mathcal{L} = -\operatorname{div}(A(x)\nabla)$ with $A(x)$ satisfying the same conditions as in Theorem* 3.1. *Suppose that $w \in W^{1,2}(3B \cap \Omega)$, $\mathcal{L}(w) = 0$ in $3B \cap \Omega$ and $w = 0$ on $3B \cap \partial\Omega$, where $B = B(Q,r)$ with $Q \in \partial\Omega$ and $0 < r < cr_0$. Then $|\nabla w| \in L^{p_n}(B \cap \Omega)$ and estimate* (1.9) *holds.*

Theorem 3.2 is proved by a perturbation argument. We first show that the desired estimate holds for elliptic systems $L = -\operatorname{div}(\overline{A}\nabla)$ with constant coefficients $\overline{A} = (\bar{a}_{ij}^{\alpha\beta})$ satisfying the Legendre–Hadamard ellipticity condition:

$$\mu|\xi|^2|\eta|^2 \leqslant \bar{a}_{ij}^{\alpha\beta}\xi_i\xi_j\eta^\alpha\eta^\beta \leqslant \mu^{-1}|\xi|^2|\eta|^2, \tag{3.2}$$

for any $\xi = (\xi_i) \in \mathbb{R}^n$, $\eta = (\eta^\alpha) \in \mathbb{R}^m$. It is known that the coercive estimate (1.8) and $\|A\|_\infty < \infty$ imply the Legendre–Hadamard condition. In particular, if $\bar{a}_{ij}^{\alpha\beta} = \frac{1}{|E|}\int_E a_{ij}^{\alpha\beta}(x)\,dx$ and $(a_{ij}^{\alpha\beta})$ satisfies the conditions in Theorem 3.1, then $\overline{A} = (\bar{a}_{ij}^{\alpha\beta})$ satisfies (3.2) and $(\overline{A})^* = \overline{A}$.

Lemma 3.3. *Let $L = \operatorname{div}(\overline{A}\nabla)$ with constant coefficient matrix $\overline{A} = (\bar{a}_{ij}^{\alpha\beta})$ satisfying* (3.2) *and $\overline{A}^* = \overline{A}$. Suppose that $w \in W^{1,2}(3B \cap \Omega)$, $L(w) = 0$ in $3B \cap \Omega$ and $w = 0$ on $3B \cap \partial\Omega$, where $B = B(y,r)$ with $y \in \overline{\Omega}$ and $0 < r < cr_0$. Then $|\nabla w| \in L^{p_n+\delta}(B \cap \Omega)$ and estimate* (1.9) *holds for $q = p_n + \delta$, where δ and N in* (1.9) *are positive constants depending only on n, m, μ and the Lipschitz character of Ω.*

Proof. Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function such that $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \leqslant M$. For $r > 0$, let

$$\Delta_r = \{(x', \psi(x')) \in \mathbb{R}^n \colon |x'| < r\},$$
$$D_r = \{(x', t) \in \mathbb{R}^n \colon |x'| < r \text{ and } \psi(x') < t < \psi(x') + (M + 10n)r\}. \tag{3.3}$$

Suppose that $w \in W^{1,2}(D_{3r})$, $L(w) = 0$ in D_{3r} and $w = 0$ on Δ_{3r}. We will show that

$$\left\{\frac{1}{r^n}\int_{D_r} |\nabla w|^{p_n}\, dx\right\}^{1/p_n} \leqslant C\left\{\frac{1}{r^n}\int_{D_{2r}} |\nabla w|^2\, dx\right\}^{1/2}, \tag{3.4}$$

where C depends only on n, m, μ and M. This, together with the interior estimates, yields (1.9) for $q = p_n$ by a change of coordinates. The case $q = p_n + \delta$ follows by the self-improvement property of the weak reverse Hölder inequality.

To see (3.4), we apply the L^2 estimates in [12] as well as square function estimates in [10] in the Lipschitz domain D_{tr}, where $t \in (1, 2)$. It follows that $\nabla w \in W^{1/2,2}(D_{tr})$ and by Sobolev imbedding, $|\nabla w| \in L^{p_n}(D_{tr})$. Moreover, we obtain

$$\begin{aligned}\left\{\int_{D_{tr}} |\nabla w|^{p_n}\, dx\right\}^{1/p_n} &\leqslant C\left\{\int_{\partial D_{tr}} |\nabla w|^2\, d\sigma\right\}^{1/2}\\ &\leqslant C\left\{\int_{\partial D_{tr}} |\nabla_{tan} w|^2\, d\sigma\right\}^{1/2},\end{aligned} \tag{3.5}$$

where $\nabla_{tan} w$ denotes the tangential gradient of w on ∂D_{tr} and C depends only on n, m, μ and M. Since $w = 0$ on Δ_{3r}, this gives

$$\left\{\int_{D_r} |\nabla w|^{p_n}\, dx\right\}^{2/p_n} \leqslant C \int_{\partial D_{tr}\setminus\Delta_{3r}} |\nabla w|^2\, d\sigma. \tag{3.6}$$

Finally, we integrate both sides of (3.6) with respect to t over $(1, 2)$ to obtain

$$\left\{\int_{D_r} |\nabla w|^{p_n}\, dx\right\}^{2/p_n} \leqslant \frac{C}{r}\int_{D_{2r}} |\nabla w|^2\, dx, \tag{3.7}$$

from which estimate (3.4) follows. □

Lemma 3.4. *Let $\mathcal{L} = -\operatorname{div}(A(x)\nabla)$ with $A(x)$ satisfying the same conditions as in Theorem* 3.1. *Then there exist a function $\eta(r)$ and some constants $N > 0$ and $p > p_n$ with the following properties*:

(1) $\lim_{r\to 0} \eta(r) = 0$;
(2) *if $u \in W^{1,2}(3B \cap \Omega)$, $\mathcal{L}u = 0$ in $3B \cap \Omega$ and $u = 0$ on $3B \cap \partial\Omega$, where $B = B(x_0, r)$ with $x_0 \in \overline{\Omega}$ and $0 < r < cr_0$, then there exists a function $v \in W^{1,p}(B \cap \Omega)$ such that*

$$\left\{\frac{1}{r^n}\int_{B\cap\Omega}\left|\nabla(u-v)\right|^2 dx\right\}^{1/2} \leqslant \eta(r)\left\{\frac{1}{r^n}\int_{3B\cap\Omega}|\nabla u|^2 dx\right\}^{1/2}, \tag{3.8}$$

$$\left\{\frac{1}{r^n}\int_{B\cap\Omega}|\nabla v|^p dx\right\}^{1/p} \leqslant N\left\{\frac{1}{r^n}\int_{3B\cap\Omega}|\nabla u|^2 dx\right\}^{1/2}. \tag{3.9}$$

Proof. The proof is similar to that of Lemma 4.7 in [19]. Suppose that u satisfies the conditions of the lemma. We define the operator $L(w) = -D_i b_{ij}^{\alpha\beta} D_j w^\beta$, where $D_i = \partial/\partial x_i$ and $b_{ij}^{\alpha\beta}$ is a constant given by

$$b_{ij}^{\alpha\beta} = \frac{1}{|B(x_0,3r)|}\int_{B(x_0,3r)} a_{ij}^{\alpha\beta}(x)\,dx. \tag{3.10}$$

Then $(b_{ij}^{\alpha\beta})$ satisfies the ellipticity condition (3.2) and $b_{ij}^{\alpha\beta} = b_{ji}^{\beta\alpha}$. Let v be a weak solution of $L(v) = 0$ in $2B\cap\Omega$ such that $u - v \in W_0^{1,2}(2B\cap\Omega)$. We will prove that v satisfies estimates (3.8) and (3.9).

We first prove (3.8). Note that

$$L(u-v) = (L-\mathcal{L})u = -D_i\big(b_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}\big)D_j u^\beta \quad \text{in } 2B\cap\Omega. \tag{3.11}$$

It follows that

$$\int_{2B\cap\Omega} b_{ij}^{\alpha\beta} D_j(u-v)^\beta D_i(u-v)^\alpha\,dx \leqslant C\sum_{i,j,\alpha,\beta}\int_{2B\cap\Omega}\left|b_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}\right||\nabla u|\left|\nabla(u-v)\right|dx. \tag{3.12}$$

Since $(b_{ij}^{\alpha\beta})$ is a constant matrix satisfying the Legendre–Hadamard condition (3.2) and $u - v \in W_0^{1,2}(2B\cap\Omega)$, we have

$$\int_{2B\cap\Omega} b_{ij}^{\alpha\beta} D_j(u-v)^\beta D_i(u-v)^\alpha\,dx \geqslant \mu\int_{2B\cap\Omega}\left|\nabla(u-v)\right|^2 dx, \tag{3.13}$$

which, together with (3.12), gives

$$\left\{\frac{1}{r^n}\int_{2B\cap\Omega}\left|\nabla(u-v)\right|^2 dx\right\}^{1/2} \leqslant C\sum_{i,j,\alpha,\beta}\left\{\frac{1}{r^n}\int_{2B\cap\Omega}\left|b_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}\right|^2|\nabla u|^2 dx\right\}^{1/2}. \tag{3.14}$$

By Hölder's inequality we have

$$\left\{\frac{1}{r^n}\int_{2B\cap\Omega}\left|\nabla(u-v)\right|^2 dx\right\}^{1/2}$$

$$\leqslant C\sum_{i,j,\alpha,\beta}\left\{\frac{1}{r^n}\int_{2B\cap\Omega}\left|b_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}\right|^{2q_0'} dx\right\}^{1/(2q_0')}\left\{\frac{1}{r^n}\int_{2B\cap\Omega}|\nabla u|^{2q_0} dx\right\}^{1/(2q_0)}$$

$$\leqslant \eta(r)\left\{\frac{1}{r^n}\int\limits_{3B\cap\Omega}|\nabla u|^2\,dx\right\}^{1/2}, \tag{3.15}$$

where $q_0 > 1$ and we have used the weak reverse Hölder inequality

$$\left\{\frac{1}{r^n}\int\limits_{2B\cap\Omega}|\nabla u|^{2q_0}\,dx\right\}^{1/(2q_0)} \leqslant C\left\{\frac{1}{r^n}\int\limits_{3B\cap\Omega}|\nabla u|^2\,dx\right\}^{1/2}. \tag{3.16}$$

Also, the function $\eta(r)$ above is defined by

$$\eta(r) = C\sup_{x_0\in\overline{\Omega}}\sum_{i,j,\alpha,\beta}\left\{\frac{1}{r^n}\int\limits_{B(x_0,2r)}\left|b_{ij}^{\alpha\beta}-a_{ij}^{\alpha\beta}\right|^{2q_0'}\,dx\right\}^{1/(2q_0')}. \tag{3.17}$$

We recall that the well-known Cacciopoli's inequality holds under the conditions (1) and (2) on $A(x)$ in Theorem 3.1. As a consequence the weak reverse Hölder inequality (3.16) holds for some $q_0 > 1$ (see Remark 2.3). Since $a_{ij}^{\alpha\beta} \in \mathrm{VMO}$, by the John–Nirenberg inequality, we have $\eta(r) \to 0$ as $r \to 0$. This completes the proof of (3.8).

Finally, we note that since $L(v) = 0$ in $3B\cap\Omega$ and $v = u = 0$ on $3B\cap\partial\Omega$, we may deduce from Lemma 3.3 that

$$\begin{aligned}\left\{\frac{1}{r^n}\int\limits_{B\cap\Omega}|\nabla v|^p\,dx\right\}^{1/p} &\leqslant C\left\{\frac{1}{r^n}\int\limits_{2B\cap\Omega}|\nabla v|^2\,dx\right\}^{1/2}\\ &\leqslant C\left\{\frac{1}{r^n}\int\limits_{2B\cap\Omega}|\nabla u|^2\,dx\right\}^{1/2},\end{aligned}$$

for some $p = p_n + \delta$, where the last inequality follows from (3.14). This gives (3.9). We point out that $\delta > 0$ depends only on n, m, μ and the Lipschitz character of Ω. $\square$

With Lemma 3.4 at our disposal, Theorem 3.2 follows from the following theorem, as in the proof of Theorem C in [19, p. 192]. We omit the details.

Theorem 3.5. *Let $f : E \to \mathbb{R}^m$ be a locally square integrable function, where E is an open set of $\mathbb{R}^n$. Let $p > 2$. Suppose that there exist three constants $\varepsilon > 0$ and $\alpha, N > 1$ such that for every ball $B = B(x_0, r)$ with $\alpha B \subset E$, there exists a function $h = h_B \in L^p(B)$ with the properties*:

$$\left\{\frac{1}{|B|}\int\limits_B|f-h|^2\,dx\right\}^{1/2} \leqslant \varepsilon\left\{\frac{1}{|\alpha B|}\int\limits_{\alpha B}|f|^2\,dx\right\}^{1/2}, \tag{3.18}$$

$$\left\{\frac{1}{|B|}\int\limits_B|h|^p\,dx\right\}^{1/p} \leqslant N\left\{\frac{1}{|\alpha B|}\int\limits_{\alpha B}|f|^2\,dx\right\}^{1/2}. \tag{3.19}$$

Then, if $2 < q < p$ *and* $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(n, m, p, q, \alpha, N)$*, we have*

$$\left\{ \frac{1}{|B|} \int_B |f|^q \, dx \right\}^{1/q} \leqslant C \left\{ \frac{1}{|\alpha B|} \int_{\alpha B} |f|^2 \, dx \right\}^{1/2}, \tag{3.20}$$

for any ball B *with* $\alpha B \subset E$*, where* C *depends only on* n, m, p, q, α *and* N*.*

We remark that Theorem 3.5, which was stated in [19, p. 191], was proved essentially in [8].

4. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. In view of Remark 2.2 and Theorem 1.3, it suffices to show that if $u_\varepsilon \in W^{1,2}(3B \cap \Omega)$ is a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $3B \cap \Omega$ and $u_\varepsilon = 0$ on $3B \cap \partial\Omega$, where $B = B(x_0, r)$ with $x_0 \in \overline{\Omega}$ and $0 < r < cr_0$, then $|\nabla u_\varepsilon| \in L^{p_n}(B \cap \Omega)$ and estimate (1.9) holds for $q = p_n = \frac{2n}{n-1}$ with a constant N independent of ε. By the interior estimate (1.10) we may assume that $B = B(Q, r)$ for some $Q \in \partial\Omega$. Furthermore, by a change of coordinates, it is enough to show that

$$\left\{ \frac{1}{r^n} \int_{D_r} |\nabla u_\varepsilon|^{p_n} \, dx \right\}^{1/p_n} \leqslant C \left\{ \frac{1}{r^n} \int_{D_{3r}} |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2}, \tag{4.1}$$

whenever $u_\varepsilon \in W^{1,2}(D_{3r})$ is a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{3r} and $u_\varepsilon = 0$ on Δ_{3r}.

Throughout this section we assume that $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$.

Lemma 4.1. *Let* $u_\varepsilon \in W^{1,2}(D_{3r})$ *be a weak solution to* $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ *in* D_{3r} *and* $u_\varepsilon = 0$ *on* Δ_{3r}*. Suppose that for some* $q = q_1 > 2$*,*

$$\left\{ \frac{1}{\rho^n} \int_{D_\rho} |\nabla u_\varepsilon|^q \, dx \right\}^{1/q} \leqslant C_q \left\{ \frac{1}{\rho^n} \int_{D_{3\rho}} |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2}, \tag{4.2}$$

for all $0 < \rho \leqslant r$*. Then there exists* $\delta > 0$*, depending only on* n*,* m*,* μ*,* λ*,* τ*,* C_{q_1} *and* M*, such that the estimate* (4.2) *holds for* $2 < q < 2 + \delta + \frac{q_1}{n}$ *and* $C_q = C(n, m, \mu, \lambda, \tau, q, C_{q_1}, M)$*.*

Proof. Let $d(x) = |x_n - \psi(x')|$ for $x = (x', x_n)$. It follows from (1.10) and (4.2) with $q = q_1$ that

$$\begin{aligned} \left|\nabla u_\varepsilon(x)\right| &\leqslant C \left\{ \frac{1}{[d(x)]^n} \int_{B(x, cd(x))} \left|\nabla u_\varepsilon(y)\right|^{q_1} dy \right\}^{1/q_1} \\ &\leqslant C \left\{ \frac{\rho}{d(x)} \right\}^{\frac{n}{q_1}} \left\{ \frac{1}{\rho^n} \int_{D_{3\rho}} \left|\nabla u_\varepsilon(y)\right|^2 dy \right\}^{1/2}, \end{aligned} \tag{4.3}$$

for any $x \in D_{2\rho}$. Since $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$, it follows from [16] that there exists $\delta > 0$ such that the unique weak solution to the Dirichlet problem $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ with boundary data in

$W^{1,2+\delta}(\partial\Omega)$ in a Lipschitz domain Ω with connected boundary satisfies $\|(\nabla u_\varepsilon)^*\|_{L^{2+\delta}(\partial\Omega)} \leqslant C\|\nabla_{tan} u_\varepsilon\|_{L^{2+\delta}(\partial\Omega)}$. Here $(\nabla u_\varepsilon)^*$ denotes the non-tangential maximal function of ∇u_ε. By applying this estimate to u_ε on the Lipschitz domain $D_{t\rho}$ for $t \in (3/2, 2)$ and using an integration argument, one may obtain

$$\int_{\Delta_\rho} \left|(\nabla u_\varepsilon)^*_\rho\right|^{2+\delta} d\sigma \leqslant \frac{C}{\rho} \int_{D_{2\rho}} |\nabla u_\varepsilon|^{2+\delta}\, dx, \tag{4.4}$$

where

$$(\nabla u_\varepsilon)^*_\rho\big(x', \psi(x')\big) = \sup\big\{\big|\nabla u_\varepsilon\big(x', x_n\big)\big|:\ \big(x', x_n\big) \in D_\rho\big\}. \tag{4.5}$$

Let $q_0 = 2 + \delta$. Note that, if δ is sufficiently small,

$$\left\{\frac{1}{\rho^n}\int_{D_{2\rho}} |\nabla u_\varepsilon|^{q_0}\, dx\right\}^{1/q_0} \leqslant C\left\{\frac{1}{\rho^n}\int_{D_{3\rho}} |\nabla u_\varepsilon|^{2}\, dx\right\}^{1/2} \tag{4.6}$$

(see Remark 2.3). Hence,

$$\left\{\frac{1}{\rho^{n-1}}\int_{\Delta_\rho} \left|(\nabla u_\varepsilon)^*_\rho\right|^{q_0} d\sigma\right\}^{1/q_0} \leqslant C\left\{\frac{1}{\rho^n}\int_{D_{3\rho}} |\nabla u_\varepsilon|^{2}\, dx\right\}^{1/2}. \tag{4.7}$$

Now, using estimates (4.3) and (4.7), we see that

$$\begin{aligned}
&\left\{\frac{1}{\rho^n}\int_{D_\rho} |\nabla u_\varepsilon|^{q}\, dx\right\}^{1/q} \\
&= \left\{\frac{1}{\rho^n}\int_{D_\rho} |\nabla u_\varepsilon|^{q_0} |\nabla u_\varepsilon|^{q-q_0}\, dx\right\}^{1/q} \\
&\leqslant C\left\{\frac{1}{\rho^{n-1}}\int_{\Delta_\rho} \left|(\nabla u_\varepsilon)^*_\rho\right|^{q_0} d\sigma \cdot \frac{1}{\rho}\int_0^{c\rho} \left(\frac{\rho}{t}\right)^{\frac{n(q-q_0)}{q_1}} dt\right\}^{1/q} \cdot \left\{\frac{1}{\rho^n}\int_{D_{3\rho}} |\nabla u_\varepsilon|^{2}\, dx\right\}^{\frac{q-q_0}{2q}} \\
&\leqslant C\left\{\frac{1}{\rho^n}\int_{D_{3\rho}} |\nabla u_\varepsilon|^{2}\, dx\right\}^{1/2},
\end{aligned}$$

if $0 < n(q - q_0) < q_1$. Note that $n(q - q_0) < q_1$ is equivalent to $q < 2 + \delta + \frac{q_1}{n}$. This finishes the proof. $\square$

Proof of Theorem 1.2. Let $u_\varepsilon \in W^{1,2}(D_{3r})$ be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{3r} and $u_\varepsilon = 0$ on Δ_{3r}. It follows from Cacciopoli's inequality that the weak reverse Hölder inequality (4.2) always holds for some $q_1 > 2$ under the ellipticity condition (1.8) (see Remark 2.3; smoothness and periodicity conditions are not needed). Suppose that $q_1 < \frac{2n}{n-1}$. By Lemma 4.1 estimate (4.2) holds for some $q = q_2 > 2 + \frac{\delta}{2} + \frac{q_1}{n} > q_1$. If $q_2 < \frac{2n}{n-1}$, then the same argument would give (4.2) for $q = q_3 > 2 + \frac{\delta}{2} + \frac{q_2}{n} > q_2$. Continuing this process, we claim that there exists some j such that estimate (4.2) holds for some $q = q_j > \frac{2n}{n-1}$. For otherwise we would have a bounded increasing sequence $\{q_j\}$ such that $q_{j+1} > 2 + \frac{\delta}{2} + \frac{q_j}{n} > q_j$. Let q be the limit of $\{q_j\}$. Then $q \geqslant 2 + \frac{\delta}{2} + \frac{q}{n}$, which implies that $q > p_n = \frac{2n}{n-1}$. It follows that $q_j > p_n$ if j is sufficiently large. Thus (4.2) must hold for some $q = q_j > p_n$. This completes the proof. □

5. Proof of Theorem 1.1

Let D_r and Δ_r be defined as in (3.3) with $\|\nabla\psi\|_\infty \leqslant M$. In view of Remark 2.2 and Theorem 1.3, as in the case of Theorem 1.2, Theorem 1.1 is a consequence of the following.

Theorem 5.1. *Let $\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla)$ with $A \in \mathcal{M}(\mu, \lambda, \tau)$. Suppose that $u_\varepsilon \in W^{1,2}(D_{3r})$, $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{3r} and $u_\varepsilon = 0$ on Δ_{3r}. Then the estimate* (4.2) *holds for $q = p_n = \frac{2n}{n-1}$ with a constant C depending only on n, m, μ, λ, τ and M.*

Since the non-tangential maximal function estimates used in Lemma 4.1 are not available under the assumption $A \in \mathcal{M}(\mu, \lambda, \tau)$, the proof of Theorem 5.1 relies on a compactness method motivated by [2]. In [20] the same approach was used to establish Theorem 1.2 in the case $m = 1$.

Throughout the rest of this section we will assume that $A \in \mathcal{M}(\mu, \lambda, \tau)$.

Lemma 5.2. *Let $u_\varepsilon \in W^{1,2}(D_{3r})$, $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_{3r} and $u_\varepsilon = 0$ on Δ_{3r}. Then for any $p > 1$,*

$$\int_0^{cr} \int_{|x'|<r} \left|\nabla u_\varepsilon\big(x', \psi(x') + s\big)\right|^p dx'\,ds \leqslant C_p \int_0^{2cr} \int_{|x'|<2r} \left|\frac{u_\varepsilon(x', \psi(x') + s)}{s}\right|^p dx'\,ds, \quad (5.1)$$

where $c = (M + 10n)$ and $C_p > 0$ depends only on n, p, μ, τ, λ and M.

Proof. This follows from the interior estimate (1.10). The proof is similar to that of Lemma 3.2 in [20] and thus omitted. □

Lemma 5.3. *Let $L = -\operatorname{div}(\overline{A}\nabla)$, where $\overline{A} = (a_{ij}^{\alpha\beta})$ with $1 \leqslant i, j \leqslant n$ and $1 \leqslant \alpha, \beta \leqslant m$ is a constant matrix satisfying $\overline{A}^* = \overline{A}$ and the Legendre–Hadamard condition* (3.2). *Suppose that $u_0 \in W^{1,2}(D_{3/2})$, $L(u_0) = 0$ in $D_{3/2}$ and $u_0 = 0$ on $\Delta_{3/2}$. Then*

$$\int_0^t \int_{|x'|<1} \left|u_0\big(x', \psi(x') + s\big)\right|^{p_n} dx'\,ds \leqslant C_0 t^{p_n + 2\sigma} \int_0^{3/2} \int_{|x'|<\frac{3}{2}} \left|u_0\big(x', \psi(x') + s\big)\right|^{p_n} dx'\,ds \quad (5.2)$$

for any $0 < t < 1$, where C_0 and σ are positive constants depending only on n, m, μ and M.

Proof. Since $u_0 = 0$ on Δ_{3r}, it follows by the fundamental theorem of calculus that

$$\int_0^t \int_{|x'|<1} \left|u_0\big(x', \psi(x') + s\big)\right|^{p_n} dx'\,ds \leqslant C t^{p_n} \int_0^t \int_{|x'|<1} \left|\nabla u_0\big(x', \psi(x') + s\big)\right|^{p_n} dx'\,ds. \quad (5.3)$$

By Hölder's inequality the right hand side of (5.3) is bounded by

$$Ct^{p_n+\frac{\delta}{p_n+\delta}}\left\{\int_0^1\int_{|x'|<1}\left|\nabla u_0\big(x',\psi\big(x'\big)+s\big)\right|^{p_n+\delta}dx'\,ds\right\}^{\frac{p_n}{p_n+\delta}}.$$

This, together with Lemma 3.3, implies that

$$\int_0^t\int_{|x'|<1}\left|u_0\big(x',\psi\big(x'\big)+s\big)\right|^{p_n}dx'\,ds\leqslant Ct^{p_n+\frac{\delta}{p_n+\delta}}\left\{\int_0^{\frac{5}{4}}\int_{|x'|<\frac{5}{4}}\left|\nabla u_0\big(x',\psi\big(x'\big)+s\big)\right|^2dx'\,ds\right\}^{p_n/2}. \tag{5.4}$$

Let $2\sigma=\frac{\delta}{p_n+\delta}$. Estimate (5.2) now follows from (5.4) by Cacciopoli and Hölder's inequalities. $\square$

Let C_0 and σ be given by Lemma 5.3. Choose $t_0\in(0,1/2)$ so small that $C_0t_0^\sigma<(1/2)$. Then $C_0t_0^{p_n+2\sigma}<(1/2)t_0^{p_n+\sigma}$.

Lemma 5.4. *There exists $\varepsilon_0>0$, depending only on n, μ, λ, τ and M, such that for any $0<\varepsilon\leqslant\varepsilon_0$,*

$$\int_0^{t_0}\int_{|x'|<1}\left|u_\varepsilon\big(x',\psi\big(x'\big)+t\big)\right|^{p_n}dx'\,dt\leqslant t_0^{p_n+\sigma}\int_0^{3c}\int_{|x'|<3}\left|u_\varepsilon\big(x',\psi\big(x'\big)+t\big)\right|^{p_n}dx'\,dt, \tag{5.5}$$

where $c=(M+10n)$, if $u_\varepsilon\in W^{1,2}(D_3)$, $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in D_3 and $u_\varepsilon=0$ on Δ_3.

Proof. We will prove the lemma by contradiction. For any $k\in\mathbb{N}$, denote

$$D_r^k=\big\{\big(x',x_n\big):\ \big|x'\big|<r\text{ and }\psi_k\big(x'\big)<x_n<\psi_k\big(x'\big)+(M+10n)r\big\},$$
$$\Delta_r^k=\big\{\big(x',x_n\big):\ \big|x'\big|<r\text{ and }x_n=\psi_k\big(x'\big)\big\},$$

where $\|\nabla\psi_k\|_\infty\leqslant M$ and $\psi_k(0)=0$. Assume that there exist $\{\mathcal{L}^{(k)}\}$, $\{\varepsilon_k\}$, $\{\psi_k\}$ and $\{u_{\varepsilon_k}\}$ such that $\varepsilon_k\to0$ as $k\to\infty$,

$$\mathcal{L}_{\varepsilon_k}^{(k)}(u_{\varepsilon_k})=-\operatorname{div}\left(A^k\left(\frac{x}{\varepsilon_k}\right)\nabla u_k\right)=0\quad\text{in }D_3^k,\qquad u_{\varepsilon_k}=0\quad\text{on }\Delta_3^k, \tag{5.6}$$

$$\int_0^{3c}\int_{|x'|<3}\left|u_{\varepsilon_k}\big(x',\psi_k\big(x'\big)+t\big)\right|^{p_n}dx'dt=1, \tag{5.7}$$

and

$$\int_0^{t_0} \int_{|x'|<1} \left|u_{\varepsilon_k}\big(x', \psi_k(x') + t\big)\right|^{p_n} dx'\,dt > t_0^{p_n+\sigma}, \tag{5.8}$$

where the coefficient matrices $A^k = (a_{ij}^{\alpha\beta,k}(y)) \in \mathcal{M}(\mu, \lambda, \tau)$.

Let

$$b_{ij}^{\alpha\beta,k} = \int_{[0,1]^n} \left[a_{ij}^{\alpha\beta,k} + a_{i\ell}^{\alpha\gamma,k} \frac{\partial}{\partial y_\ell}\big(\chi_j^{\gamma\beta,k}\big)\right] dy, \tag{5.9}$$

where $\chi^k(y) = (\chi_j^{\alpha\beta,k}(y))_{1\leqslant\alpha,\beta,j\leqslant n}$ are correctors for $\mathcal{L}_\varepsilon^{(k)}$. Note that $b_{ij}^{\alpha\beta,k}$ are bounded. Hence, by passing to a subsequence, we may suppose that

$$b_{ij}^{\alpha\beta} = \lim_{k\to\infty} b_{ij}^{\alpha\beta,k} \tag{5.10}$$

exists for $1 \leqslant i, j, \alpha, \beta \leqslant n$. Since each $(b_{ij}^{\alpha\beta,k}) \in \mathcal{M}(\tilde{\mu}, \lambda, \tau)$ for some $\tilde{\mu}$ depending only on μ (see e.g. [9, p. 202]), so does the matrix $(b_{ij}^{\alpha\beta})$. We remark that t_0 and σ should be chosen for this $\tilde{\mu}$.

Since the sequence $\{\psi_k\}$ is equi-continuous on $\{x' \in \mathbb{R}^{n-1}\colon |x'| \leqslant 5\}$ and $\psi_k(0) = 0$, by the Ascoli–Arzela theorem, we may assume that ψ_k converges uniformly to ψ_0 on $\{x'\colon |x'| \leqslant 5\}$. We also have that $\|\nabla\psi_0\|_\infty \leqslant M$ and $\psi_0(0) = 0$. Let $v_k(x', t) = u_k(x', \psi_k(x') + t)$ and $Q_r = \{(x', t)\colon |x'| < r \text{ and } 0 < t < cr\}$. Note that by Cacciopoli's inequality and (5.7), $\{v_k\}$ is uniformly bounded in $W^{1,2}(Q_2)$. Thus, by passing to a subsequence, we may assume that $v_k \to v_0$ weakly in $W^{1,2}(Q_2)$. Since $W^{1,2}(Q_2)$ is compactly embedded in $L^{p_n}(Q_2)$, we may assume that $v_k \to v_0$ strongly in $L^{p_n}(Q_2)$. In view of (5.7) and (5.8) we obtain

$$\begin{gathered}\int_0^2 \int_{|x'|<2} \left|v_0\big(x', t\big)\right|^{p_n} dx'\,dt \leqslant 1,\\ \int_0^{t_0} \int_{|x'|<1} \left|v_0\big(x', t\big)\right|^{p_n} dx'\,dt \geqslant t_0^{p_n+\sigma}.\end{gathered} \tag{5.11}$$

Now, let $w(x', x_n) = v_0(x', x_n - \psi_0(x'))$. Then $w \in W^{1,2}(\widetilde{D}_2)$ and $w = 0$ on $\widetilde{\Delta}_2$, where $\widetilde{D}_r$ and $\widetilde{\Delta}_r$ are defined as in (3.3), but with ψ replaced by ψ_0. Let $L = -\operatorname{div}(\overline{A}\nabla)$, where $\overline{A} = (b_{ij}^{\alpha\beta})$. It follows from the theory of homogenization that $L(w) = 0$ in D_2 (see e.g. [15, Lemma 2.1]). In view of Lemma 5.3 and (5.11) we obtain

$$\begin{aligned}\int_0^{t_0} \int_{|x'|<1} \left|w\big(x', \psi_0(x') + t\big)\right|^{p_n} dx'\,dt &\leqslant C_0 t_0^{p_n+2\sigma} \int_0^2 \int_{|x'|<2} \left|w\big(x', \psi_0(x') + t\big)\right|^{p_n} dx'\,dt\\ &\leqslant (1/2) t_0^{p_n+\sigma},\end{aligned} \tag{5.12}$$

which contradicts the second inequality in (5.11). This completes the proof. □

Lemma 5.5. *Let $\varepsilon_0 > 0$ be given by Lemma* 5.4*. There exist positive constants δ and C, depending only on n, μ, τ, λ and M, such that for $(\varepsilon/\varepsilon_0) < t < 1$,*

$$\int_0^t \int_{|x'|<1} \left|u_\varepsilon\left(x', \psi\left(x'\right) + s\right)\right|^{p_n} dx'\, ds \leqslant C t^{p_n+\delta} \int_0^{3c} \int_{|x'|<3} \left|u_\varepsilon\left(x', \psi\left(x'\right) + s\right)\right|^{p_n} dx'\, ds, \tag{5.13}$$

whenever $u_\varepsilon \in W^{1,2}(D_3)$, $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D_3 and $u_\varepsilon = 0$ on Δ_3.

Proof. Lemma 5.5 follows from Lemma 5.4 by a rescaling-iteration argument. We refer the reader to [20, pp. 2294–2295] for details. □

Finally we are in a position to give the proof of Theorem 5.1.

Proof of Theorem 5.1. By rescaling we may assume that $r = 1$. Let ε_0 be given by Lemma 5.4. If $\varepsilon \geqslant \epsilon_0/4$, estimate (4.2) follows directly from Theorem 3.2. Now we suppose that $\varepsilon < \varepsilon_0/4$. Observe that $v(x) = u_\varepsilon(\varepsilon x)$ is a weak solution of $\mathcal{L}_1(v) = 0$. Thus by Hardy's inequality and Theorem 3.2,

$$\begin{aligned} \int_0^{\varepsilon/\epsilon_0} \int_{|x'|<\varepsilon/\varepsilon_0} \left|\frac{u_\varepsilon(x', \psi(x') + s)}{s}\right|^{p_n} dx'\, ds &\leqslant C \int_0^{\varepsilon/\varepsilon_0} \int_{|x'|<\varepsilon/\varepsilon_0} \left|\nabla u_\varepsilon\left(x', \psi\left(x'\right) + s\right)\right|^{p_n} dx'\, ds \\ &\leqslant \frac{C}{(\varepsilon)^{p_n}} \int_0^{c\varepsilon/\varepsilon_0} \int_{|x'|<2\varepsilon/\varepsilon_0} \left|u_\varepsilon\left(x', \psi\left(x'\right) + s\right)\right|^{p_n} dx'\, ds. \end{aligned} \tag{5.14}$$

By covering Δ_1 with surface balls of radius $\varepsilon/\varepsilon_0$, we can deduce from (5.14) that

$$\begin{aligned} \int_0^{\varepsilon/\varepsilon_0} \int_{|x'|<1} \left|\frac{u_\varepsilon(x', \psi(x') + s)}{s}\right|^{p_n} dx'\, ds &\leqslant \frac{C}{\varepsilon^{p_n}} \int_0^{c\varepsilon/\varepsilon_0} \int_{|x'|<2} \left|u_\varepsilon\left(x', \psi\left(x'\right) + s\right)\right|^{p_n} dx'\, ds \\ &\leqslant C \int_0^{3c} \int_{|x'|<3} \left|u_\varepsilon\left(x', \psi\left(x'\right) + s\right)\right|^{p_n} dx'\, ds, \end{aligned} \tag{5.15}$$

where we have used Lemma 5.5 for the last inequality.

Next, we denote $f(x', s) = s^{-1} u_\varepsilon(x', \psi(x') + s)$ and write

$$\int_0^c \int_{|x'|<1} \left|f\left(x', s\right)\right|^{p_n} dx'\, ds = \left\{ \int_0^{\varepsilon/\varepsilon_0} \int_{|x'|<1} + \sum_{j=1}^{j_0} \int_{2^{j-1}\varepsilon/\epsilon_0}^{2^j \varepsilon/\varepsilon_0} \int_{|x'|<1} + \int_{2^{j_0}\varepsilon/\varepsilon_0}^{c} \int_{|x'|<1} \right\} \left|f\left(x', s\right)\right|^{p_n} dx'\, ds, \tag{5.16}$$

where $2^{-j_0-1} \leqslant \varepsilon/\varepsilon_0 \leqslant 2^{-j_0}$. The first term in the right hand side of (5.16) is handled by (5.15). Now we apply (5.13) to estimate the second term. This gives

$$\begin{aligned}
&\sum_{j=1}^{j_0} \int_{2^{j-1}\varepsilon/\varepsilon_0}^{2^j\varepsilon/\varepsilon_0} \int_{|x'|<1} \left|f\left(x', s\right)\right|^{p_n} dx'\, ds \\
&\quad \leqslant C \sum_{j=1}^{j_0} \left(2^{j-1}\frac{\varepsilon}{\varepsilon_0}\right)^{-p_n} \left(2^j \frac{\varepsilon}{\varepsilon_0}\right)^{p_n+\delta} \int_0^{3c} \int_{|x'|<3} \left|u_\varepsilon\left(x', \psi\left(x'\right)+s\right)\right|^{p_n} dx'\, ds \\
&\quad \leqslant C \int_0^{3c} \int_{|x'|<3} \left|u_\varepsilon\left(x', \psi\left(x'\right)+s\right)\right|^{p_n} dx'\, ds,
\end{aligned} \tag{5.17}$$

where in the last inequality we have used $2^{-j_0-1} \leqslant \varepsilon/\varepsilon_0 \leqslant 2^{-j_0}$.

Finally, the last term in (5.16) is controlled by

$$C \int_0^{c} \int_{|x'|<1} \left|u_\varepsilon\left(x', \psi\left(x'\right)+s\right)\right|^{p_n} dx'\, ds. \tag{5.18}$$

Therefore, we have shown that

$$\int_0^1 \int_{|x'|<1} \left|\frac{u_\varepsilon(x', \psi(x')+s)}{s}\right|^{p_n} dx'\, ds \leqslant C \int_{D_3} \left|u_\varepsilon(x)\right|^{p_n} dx. \tag{5.19}$$

In view of Lemma 5.2 this implies that

$$\int_{D_1} |\nabla u_\varepsilon|^{p_n}\, dx \leqslant C \int_{D_3} |u_\varepsilon|^{p_n}\, dx \leqslant C \left\{ \int_{D_3} |\nabla u_\varepsilon|^2\, dx \right\}^{p_n/2}, \tag{5.20}$$

where the last step follows from the Sobolev inequality. This completes the proof of Theorem 5.1. □

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