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Plane Partitions V: The TSSCPP Conjecture

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This paper is devoted to proving the conjecture by Mills, Robbins, and Rumsey that the number of totally symmetric, self-complementary plane partitions in $[1, 2n]^3$ is given by $\prod_{i=0}^{n-1} (3i+1)!/(n+i)!$. © 1994 Academic Press, Inc.

1. INTRODUCTION

In [6], Mills, Robbins, and Rumsey define "*a totally symmetric plane partition* of size n (to be) a plane partition whose three-dimensional Ferrers graph is contained in the box

$$X_n = [1, n] \times [1, n] \times [1, n]$$

and which is mapped to itself under all permutations of the coordinate axes. The complement of the Ferrers graph of such a plane partition (that is, the set of lattice points in the box X_n that does not belong to the Ferrers graph) is again totally symmetric when viewed from the vantage point of vertex $(n+1, n+1, n+1)$. A totally symmetric plane partition is self complementary if it is congruent (in the geometrical sense) to its complement. This cannot occur unless $n = 2m$ is even."

If we define

$$\lambda_n = \frac{(3n+1)!\, n!}{(2n)!\,(2n+1)!} = \binom{3n+1}{n} \Big/ \binom{2n}{n}, \tag{1.1}$$

then Mills *et al.* conjecture that t_n, the total number of TSSCPP's (totally symmetric, self-complementary plane partitions) in X_{2n}, is given by

$$t_n = \prod_{i=0}^{n-1} \lambda_i = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \tag{1.2}$$

(see also [8, Conjecture 2; 9, Case 10]).

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In [10], Stembridge made a major advance on the problem by proving that

$$t_n = \begin{cases} pf[a_{ij}]_{0 \leqq i < j \leqq n-1} & \text{if } n \text{ is even} \\ pf[a_{ij}]_{1 \leqq i < j \leqq n-1} & \text{if } n \text{ is odd,} \end{cases} \tag{1.3}$$

where for $0 \leqq i < j$,

$$a_{ij} = \sum_{r=2i-j+1}^{2j-i} \binom{i+j}{r}. \tag{1.4}$$

Our object is to prove (1.2). We proceed as follows. In Section 2 we reformulate Stembridge's Pfaffian representation of t_n. In Section 3 we prove a number of lemmas related to hypergeometric series. The proof then follows in a straightforward manner in Section 4.

2. A Reformulation of Stembridge's Theorem

The methods developed in the previous papers in this series [1–4] were concerned with the evaluation of determinants, and in this section we restate Stembridge's result (1.3) as a slightly modified skew-symmetric determinant.

Recalling that the Pfaffian is the square root of the related skew-symmetric determinant [7, p. 394], and noting that $t_n \geqq 0$, we see immediately that (1.3) is equivalent to the assertion that

$$t_n^2 = \begin{cases} \text{determinant } (a_{ij})_{0 \leqq i, j \leqq n-1} & \text{if } n \text{ is even} \\ \text{determinant } (a_{ij})_{1 \leqq i, j \leqq n-1} & \text{if } n \text{ is odd,} \end{cases} \tag{2.1}$$

where we extend (1.4) by requiring $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ for $i > j$.

Lemma 1.

$$(-1)^n \, t_n^2 = \text{determinant } (a_{ij}^*)_{0 \leqq i, j \leqq n-1},$$

where $a_{00}^* = -1$ *and* $a_{ij}^* = a_{ij}$ *otherwise.*

Proof. If n is even, this assertion reduces to the top line of (2.1). To see this we expand the determinant by minors along the top row. The result is clearly.

$$\begin{aligned} &\text{determinant } (a_{ij})_{0 \leqq i, j \leqq n-1} + a_{00}^* \cdot \text{determinant } (a_{ij})_{1 \leqq i, j \leqq n-1} \\ &\quad = \text{determinant } (a_{ij})_{0 \leqq i, j \leqq n-1} \\ &\quad = t_n^2, \end{aligned}$$

since a skew-symmetric matrix of odd order has determinant 0 [7, p. 392].

If n is odd, precisely the same expansion yields

$$\begin{aligned}&\text{determinant } (a_{ij})_{0\leqq i,\, j\leqq n-1}+a^*_{00}\cdot\text{determinant } (a_{ij})_{1\leqq i,\, j\leqq n-1}\\ &\quad=-\text{determinant } (a_{ij})_{1\leqq i,\, j\leqq n-1}\\ &\quad=-t_n^2\end{aligned}$$

since, as before, a skew-symmetric matrix of odd order has determinant 0. ▮

3. Hypergeometric Background and Lemmas

The material in this section is inordinately complicated. The purpose of the various lemmas becomes clearer if you read Section 4 first.

We begin with six definitions.

$$e(i,j)=\begin{cases}0, & \text{if } i<j\\ 1, & \text{if } i=j>0\\ \dfrac{(3j+1)!\,j!}{(2j)!\,(2j+1)!}, & \text{if } i=0\\ \dfrac{(-1)^{j-i}\,(2j-i-1)!\,(3i+2)(3j+1)!\,(i+j)!\,j!\,(2i+1)!}{(j-i)!\,(i-1)!\,i!\,(i+2j+2)!\,(2j+1)!\,(2j)!}, & \\ \quad\text{if } 0<i<j, & \end{cases} \tag{3.1}$$

$$f(i,j)=\frac{\binom{4i-j+1}{j}\binom{3i+1}{j}\binom{i}{j}}{\binom{2i}{j}^2}, \tag{3.2}$$

$$\begin{aligned}\phi(i,j)&=2\binom{3i-j}{j-1}+\binom{3i-j}{j}\\ &=\binom{3i-j}{j}\frac{(3i+1)}{(3i-2j+1)},\end{aligned} \tag{3.2}$$

$$C_j(i)=\frac{\binom{i}{j}\binom{i+1}{2j}\binom{3i-2j}{j}(3i+1)}{(3i-3j+1)\binom{2i-j+1}{j}\binom{2i}{2j}}, \tag{3.4}$$

$$b(i, j) = \sum_{s=0}^{j} \binom{3i-j+1}{s}, \tag{3.5}$$

$$T(i, j, s) = \sum_{k=0}^{2i-1-s} \binom{3i-k-s+1}{k} e(2i-k-s, j). \tag{3.6}$$

LEMMA 2. *For each* $j \geqq 0$,

$$\sum_{k=0}^{j} 2^{k+1} e(k+1, j+1) = e(0, j+1), \tag{3.7}$$

Proof. By (3.1) we may rewrite (3.7) as follows after dividing both sides by $e(0, j+1)$:

$$\sum_{k=1}^{j} \frac{2^{k+1}(-1)^{j-k}(2j-k)!\,(3k+5)(k+j+2)!\,(2k+3)!}{(j-k)!\,(k+1)!\,k!\,(k+2j+5)!} = 1. \tag{3.8}$$

The underlying hypergeometric series [5, p. 8] is (after finite products have been removed)

$$_4F_3\left(\begin{matrix} -j, 8/3, j+3, 5/2; -8 \\ 5/3, -2j, 2j+6 \end{matrix}\right), \tag{3.9}$$

which is not one of the well-known summable series. Consequently, we must rely on the WZ method [11] to prove (3.8). We define

$$F(j, k) = \frac{2^{k+1}(-1)^{j-k}(2j-k)!\,(3k+5)(k+j+2)!\,(2k+3)!}{(j-k)!\,(k+1)!\,k!\,(k+2j+5)!}, \tag{3.10}$$

and

$$G(j, k) = \frac{2^{k+3}(-1)^{j-k-1}(j+2)(2j-k)!\,(k+j+3)!\,(2k+5)!}{(j-k)!\,(k+2)!\,k!\,(k+2j+7)!}, \tag{3.11}$$

and we easily determine that

$$\begin{aligned} &F(j+1, k) - F(j, k) \\ &\quad = \frac{F(j, k)(j+2)(6k(k+3) - 8(j+1)(j+3))}{(j-k+1)(k+2j+7)(k+2j+6)} \\ &\quad = G(j, k) - G(j, k-1). \end{aligned} \tag{3.12}$$

We now let

$$S(j) = \sum_{k=0}^{j} F(j, k). \tag{3.13}$$

Then since $F(j, j+1)=0$

$$\begin{aligned} S(j+1)-S(j) &= \sum_{k=0}^{j+1} (F(j+1, k)-F(j, k)) \\ &= \sum_{k=0}^{j+s} (G(j, k)-G(j, k-1)) \\ &= G(j, j+1)-G(j, -1) \\ &= 0. \end{aligned}$$

Therefore $S(j)=S(0)$ for all j, and $S(0)=1$. This proves (3.8) and consequently (3.7). ▮

LEMMA 3. *For* $0<r\leqslant j$

$$\sum_{\lambda=0}^{r} f(r, \lambda)\, e(2r-\lambda, j)=\begin{cases} 0, & \textit{if} \quad 0<r<j \\ \binom{3j+1}{j}^2 \Big/ \binom{2j}{j}^2 & \textit{if} \quad r=j. \end{cases} \tag{3.15}$$

Proof. We first note that the terms with $\lambda<2r-j$ are identically zero by (3.1). We may now prove by mathematical induction applied to v that for $0\leqq v\leqq j-r$,

$$\sum_{\lambda=0}^{2r-j+v} f(r, \lambda)\, e(2r-\lambda, j) = \frac{(2r+j-v+1)!\,(3r+1)!\,r!\,(j+v)!\,j!\,(3j-v+2)_v\,(r+j-v+1)_v}{v!\,(2r-j+v)!\,(2j+1)!\,(r+j+1)!\,(j-r)!\,(2j-v+1)_v\,(2r)!^2}, \tag{3.16}$$

where $(\alpha)_v=\alpha(\alpha+1)\cdots(\alpha+v-1)$. This is merely a restatement of the fact that

$$\begin{aligned} &f(r, 2r-j+v)\, e(2r-(2r-j+v), j) \\ &\qquad -f(r, 2r-j+v-1)\, e(2r-(2r-j+v-1), j) \end{aligned}$$

equals the right-hand side of (3.16), and that (3.16) is immediate when $v=0$.

Identity (3.15) follows from (3.16) by setting $v=j-r$ and noting that $(r-j+1)_{j-r}=0$ if $r<j$. If $r=j$ the only nonvanishing term of the sum occurs for $r=j$, which is $f(j, j)$ as desired. ▮

LEMMA 4. *For* $s \leqq 2i-1$,

$$T(i, j, s) = T(i-1, j, s-1) + T(i-1, j, s-2). \tag{3.17}$$

Proof.

$$\begin{aligned}
&T(i-1, j, s-1) + T(i-1, j, s-2) \\
&\quad = \sum_{k=0}^{2i-2-s} \binom{3i-k-s-1}{k} e(2i-k-s-1, j) \\
&\qquad + \sum_{k=0}^{2i-1-s} \binom{3i-k-s}{k} e(2i-k-s, j) \\
&\quad = \sum_{k=0}^{2i-1-s} \binom{3i-k-s}{k-1} e(2i-k-s, j) \\
&\qquad + \sum_{k=0}^{2i-1-s} \binom{3i-k-s}{k} e(2i-k-s, j) \\
&\quad = \sum_{k=0}^{2i-1-s} \binom{3i-k-s+1}{k} e(2i-k-s, j) \\
&\quad = T(i, j, s). \quad \blacksquare
\end{aligned}$$

LEMMA 5. *For* $m \leqq 2i$

$$\sum_{r=0}^{i} c_r(i) f(i-r, m-2r) = \phi(i, m). \tag{3.18}$$

Proof. We note that the terms of the sum are identically zero if $r < m-i$. In any event if we begin by assuming m is nonintegral and $z! = \Gamma(z+1)$, then

$$\begin{aligned}
&\sum_{r=0}^{i} c_r(i) f(i-4, m-2r) \\
&\quad = \sum_{r=0}^{i} \frac{\binom{i}{r}\binom{i+1}{2r}\binom{3i-2r}{r}(3i+1)}{(3i-3r+1)\binom{2i-r+1}{r}\binom{2i}{2r}} \\
&\qquad \times \frac{\binom{4i-m-2r+1}{m-2r}\binom{3i-3r+1}{m-2r}\binom{i-r}{m-2r}}{\binom{2i-2r}{m-2r}^2}
\end{aligned}$$

$$=\frac{(2i-m)!^2\, i!\,(i+1)!\,(3i+1)}{(4i-2m+1)!\,(2i)!}$$
$$\times \sum_{r=0}^{i} \frac{(3i-2r)!\,(2i-2r+1)(4i-2r-m+1)!}{\left(\begin{array}{l} r!\,(i+1-2r)!\,(2i-r+1)! \\ \qquad (m-2r)!\,(3i-r-m+1)!\,(i+r-m)! \end{array}\right)}$$
$$=\frac{(2i-m)!^2\, i!\,(3i+1)!\,(4i-m+1)!}{(4i-2m+1)!\,(2i)!^2\,(2i)!^2\, m!\,(3i-m+1)!\,(i-m)!}$$
$$\times {}_7F_6\left(\begin{array}{c} -2i-1,\ -i+1/2,\ -i/2,\ -\frac{(i+1)}{2},\ -3i+m-1,\ -\frac{m+1}{2},\ -\frac{m}{2};\ 1 \\ -i-1/2,\ -3i/2,\ -\frac{3i+1}{2},\ i-m+1,\ \frac{m-1}{2}-2i,\ \frac{m}{2}-2i \end{array}\right)$$
$$=\frac{(3i-m)!\,(3i+1)}{m!\,(3i-2m)!\,(3i-2m+1)}$$
$$=\phi(i,m), \tag{3.19}$$

where we have summed the ${}_7F_6$ by Dougall's summation [5, p. 26, Eq. (5)]. We now extend the result to integral $m \leqq 2i$ by analytic continuation. ∎

LEMMA 6. *For* $0 \leqq i \leqq j$,

$$2T(i-1,j,-1)+T(i-1,j,-2)=\begin{cases} 0, & \text{if } i<j \\ \binom{3i+1}{i}^2 \Big/ \binom{2i}{i}^2 & \text{if } i=j. \end{cases}$$

Proof.

$$2T(i-1,j,-1)+T(i-1,j,-2)$$
$$=\sum_{k=0}^{2i-2} 2\binom{3i-k-1}{k} e(2i-1-k,j)$$
$$+\sum_{k=0}^{2i-1}\binom{3i-k}{k} e(2i-1-k,j)$$
$$=\sum_{k=0}^{2i-1}\left(2\binom{3i-k}{k-1}+\binom{3i-k}{k}\right) e(2i-k,j)$$
$$=\sum_{k=0}^{2i-1} \phi(i,k)\, e(2i-k,j)$$
$$=\sum_{k=0}^{2i-1}\sum_{r=0}^{i} c_r(i)\, f(i-r,k-2r)\, e(2i-k,j)$$
$$=\sum_{r=0}^{i} c_r(i) \sum_{k=0}^{2i-1} f(i-r,k-2r)\, e(2(i-r)-(k-2r),j)$$

$$=\begin{cases} 0, & \text{if } i<j \text{ by (3.15), top line} \\ c_0(i)\binom{3j+1}{j}^2\Big/\binom{2j}{j}^2, & \text{if } i=j \text{ by (3.15), bottom line,} \end{cases}$$

which is (3.20) since $c_0(i)=1$ by (3.4). ▌

LEMMA 7. *For* $0\leqq i\leqq j$

$$\sum_{\mu=0}^{2i-1} b(i,\mu)\,e(2i-\mu,j)=\begin{cases} 0, & \textit{for} \quad i<j \\ \binom{3i+1}{i}^2\Big/\binom{2i}{i}^2 & \textit{for} \quad i=j. \end{cases} \tag{3.21}$$

Proof. Let

$$\begin{aligned} U(i,j) &= \sum_{\mu=0}^{2i-1} b(i,\mu)\,e(2i-\mu,j) \\ &= \sum_{\mu=0}^{2i-1}\sum_{k=0}^{\mu}\binom{3i-\mu+1}{k} e(2i-\mu,j) \\ &= \sum_{\substack{k+s\leqq 2i-1 \\ k\geqq 0,\, s\geqq 0}} \binom{3i-(k+s)+s}{k} e(2i-k-s,j) \\ &= \sum_{s=0}^{2i-1}\sum_{k=0}^{2i-s-1}\binom{3i-k-s+1}{k} e(2i-k-s,j) \\ &= \sum_{s=0}^{2i-1} T(i,j,s). \end{aligned} \tag{3.22}$$

Hence by Lemma 4,

$$\begin{aligned} U(i,j) &= \sum_{s=0}^{2i-1} \left(T(i-1,j,s-1)+T(i-1,j,s-2)\right) \\ &= \sum_{s=-1}^{2i-2} T(i-1,j,s) + \sum_{s=-2}^{2i-3} T(i-1,j,s) \\ &= 2U(i-1,j)+2T(i-1,j,-1)+T(i-1,j,-2) \\ &= 2U(i-1,j)+\begin{cases} 0, & \text{if} \quad 0\leqq i<j \\ \binom{3i+1}{i}^2\Big/\binom{2i}{i}^2 & \text{if} \quad i=j \end{cases} \end{aligned} \tag{3.23}$$

by Lemma 6.

We now proceed by induction on i. If $i=1$,

$$\begin{aligned} U(1, j) &= b(1,0)\, e(2,j) + b(1,1)\, e(1,j) \\ &= e(2,j) + 4e(1,j) \\ &= \begin{cases} 0 & \text{if } j>1 \\ 4 & \text{if } j=1 \end{cases} \end{aligned} \tag{3.24}$$

by Lemma 3 in the case $r=1$ since $f(1,0)=1$ and $f(1,1)=4$.

If $i=2$,

$$\begin{aligned} U(2, j) &= b(2,0)\, e(4,j) + b(2,1)\, e(3,j) \\ &\quad + b(2,2)\, e(2,j) + b(2,3)\, e(1,j) \\ &= e(4,j) + 7e(3,j) + 16e(2,j) + 15e(1,j) \\ &= (e(4,j) + 7e(3,j) + \tfrac{49}{4}\, e(2,j)) \\ &\quad + \tfrac{15}{4}\,(e(2,j) + 4e(1,j)) \\ &= \begin{cases} 0 & \text{if } j>2 \\ 49/4 & \text{if } j=2 \end{cases} \end{aligned} \tag{3.25}$$

by Lemma 3 in the case $r=1$ and $r=2$, since $f(r,0)=1$, $f(r,1)=3r+1$, and $f(2,2)=49/4$.

Now we assume that (3.21) is valid up to but not including a particular $i>2$. Then by (3.23) with $i \leqq j$,

$$\begin{aligned} U(i, j) &= U(i-1, j) + U(i-2, j) \\ &\quad + \begin{cases} 0 & \text{if } i<j \\ \binom{3i+1}{i}^2 \Big/ \binom{2i}{i}^2 & \text{if } i=j \end{cases} \\ &= \begin{cases} 0 & \text{if } i<j \\ \binom{3i+1}{i}^2 \Big/ \binom{2i}{i}^2 & \text{if } i=j \end{cases} \end{aligned}$$

since $i-1<j$. Hence (3.21) holds for all i. ∎

4. The Main Result

Theorem.

$$t_n = \prod_{i=0}^{n-1} \lambda_i = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Proof. We begin by defining three matrices:

$$M(n) = (a_{ij}^{*})_{0 \leqq i,\, j \leqq n-1} \tag{4.1}$$

$$R(n) = (e(i, j))_{0 \leqq i,\, j \leqq n-1} \tag{4.2}$$

$$\begin{aligned} V(n) &= M(n) \cdot R(n) \\ &= (v(i, j))_{0 \leqq i,\, j \leq n-1} \end{aligned} \tag{4.3}$$

We show that $V(n)$ is a lower triangular matrix and that $v(i, i) = -\lambda_i^2 = -\binom{3i+1}{i}^2 / \binom{2i}{i}^2$. If we can establish this last assertion, then we are done because then

$$(-1)^2\, t_n^2 = \text{determinant } (M(n)) \tag{4.4}$$

$$1 = \text{determinant } (R(n)). \tag{4.5}$$

Hence

$$\begin{aligned} (-1)^n\, t_n^2 &= \text{determinant } (M(n)) \cdot \text{determinant } (R(n)) \\ &= \text{determinant } (V(n)) \\ &= \prod_{i=0}^{n-1} v(i, i) \\ &= (-1)^n \prod_{i=0}^{n-1} \lambda_i^2. \end{aligned} \tag{4.6}$$

But since both t_n and λ_i are nonnegative, this proves the theorem.

We now proceed to evaluate $v(i, j)$ for $i \leqq j$. We define

$$A(i, j) = \sum_{s=0}^{2j-i} \binom{i+j}{s}, \tag{4.7}$$

so that

$$a_{ij} = A(i, j) - A(j, i) \tag{4.8}$$

for all nonnegative i and j.

Furthermore,

$$\begin{aligned} A(i, j) &= \sum_{s=0}^{2j-i} \binom{i+j}{s} \\ &= 2^{i+j} - \sum_{s=2j-i+1}^{i+j} \binom{i+j}{s} \\ &= 2^{i+j} - \sum_{s=0}^{2i-j-1} \binom{i+j}{i+j-s} \\ &= 2^{i+j} - \left(A(j, i) - \binom{i+j}{2i-j} \right), \end{aligned}$$

so for all nonnegative i and j

$$A(i, j) + A(j, i) = 2^{i+j} + \binom{i+j}{2i-j}. \tag{4.9}$$

Now $v(0, 0) = a^*_{0,0} e(0, 0) = -1 = -\lambda_0^2$, and for $j > 0$ by (1.4)

$$\begin{aligned} v(0, j) &= \sum_{k=0}^{j} a^*_{0,k} e(k, j) \\ &= -e(0, j) + \sum_{k=1}^{j} 2^k e(k, j) \\ &= 0 \end{aligned} \tag{4.10}$$

by Lemma 2.

Now for $0 < i \leqq j$

$$\begin{aligned} v(i, j) &= \sum_{k=0}^{j} a_{i,k} e(k, j) \\ &= \sum_{k=0}^{j} (A(i, k) - A(k, i))\, e(k, j) \\ &= \sum_{k=0}^{j} \left(2^{i+k} + \binom{i+k}{2i-k} - 2A(k, i)\right) e(k, j) \\ &= 2^i \sum_{k=0}^{j} 2^k e(k, j) - \sum_{k=0}^{j} \left(2A(k, i) - \binom{i+k}{2i-k}\right) e(k, j) \\ &= -\sum_{k=1}^{j} \left(\sum_{s=0}^{2i-k} \binom{i+k+1}{s}\right) e(k, j), \end{aligned}$$

where the first sum has vanished by Lemma 2, and the coefficient of $e(k, j)$ in the second sum has been rewritten using the recurrence $\binom{A}{B} = \binom{A-1}{B} + \binom{A-1}{B-1}$.

Hence for $0 < i \leqq j$,

$$\begin{aligned} v(i, j) &= -\sum_{k=1}^{2i} \sum_{s=0}^{2i-k} \binom{i+k+1}{s} e(k, j) \\ &= -\sum_{k=0}^{2i-1} \sum_{s=0}^{k} \binom{3i-k+1}{s} e(2i-k, j) \\ &= -\sum_{k=0}^{2i-1} b(i, k)\, e(2i-k, j) \\ &= \begin{cases} 0 & \text{for} \quad i < j \\ -\lambda_i^2 & \text{for} \quad i = j \end{cases} \end{aligned} \tag{4.12}$$

by Lemma 7. Hence $V(n)$ is lower triangular and $v(i, i) = -\lambda_i^2$ as desired. Therefore our theorem is proved. ∎

5. Conclusion

The most important observation in cracking this problem was that -1 should be inserted in the upper left-hand corner of the matrix $M(n)$. Without a nonzero entry there, the methods developed in [1]–[4] break down completely. Whether this attack will be succesful in other such problems remains to be seen.

The great value of the WZ method [11] is brought home forcefully in Lemma 2.

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