



Plane Partitions V: The TSSCPP Conjecture

GEORGE E. ANDREWS*

*The Pennsylvania State University, University Park, Pennsylvania 16802 and
IBM T. J. Watson Research Center, Yorktown Heights, New York 10598*

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This paper is devoted to proving the conjecture by Mills, Robbins, and Rumsey that the number of totally symmetric, self-complementary plane partitions in $[1, 2n]^3$ is given by $\prod_{i=0}^{n-1} (3i+1)!/(n+i)!$. © 1994 Academic Press, Inc.

1. INTRODUCTION

In [6], Mills, Robbins, and Rumsey define “a *totally symmetric plane partition* of size n (to be) a plane partition whose three-dimensional Ferrers graph is contained in the box

$$X_n = [1, n] \times [1, n] \times [1, n]$$

and which is mapped to itself under all permutations of the coordinate axes. The complement of the Ferrers graph of such a plane partition (that is, the set of lattice points in the box X_n that does not belong to the Ferrers graph) is again totally symmetric when viewed from the vantage point of vertex $(n+1, n+1, n+1)$. A totally symmetric plane partition is self-complementary if it is congruent (in the geometrical sense) to its complement. This cannot occur unless $n = 2m$ is even.”

If we define

$$\lambda_n = \frac{(3n+1)! n!}{(2n)! (2n+1)!} = \binom{3n+1}{n} / \binom{2n}{n}, \tag{1.1}$$

then Mills *et al.* conjecture that t_n , the total number of TSSCPP’s (totally symmetric, self-complementary plane partitions) in X_{2n} , is given by

$$t_n = \prod_{i=0}^{n-1} \lambda_i = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \tag{1.2}$$

(see also [8, Conjecture 2; 9, Case 10]).

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In [10], Stembridge made a major advance on the problem by proving that

$$t_n = \begin{cases} pf[a_{ij}]_{0 \leq i < j \leq n-1} & \text{if } n \text{ is even} \\ pf[a_{ij}]_{1 \leq i < j \leq n-1} & \text{if } n \text{ is odd,} \end{cases} \quad (1.3)$$

where for $0 \leq i < j$,

$$a_{ij} = \sum_{r=2i-j+1}^{2j-i} \binom{i+j}{r}. \quad (1.4)$$

Our object is to prove (1.2). We proceed as follows. In Section 2 we reformulate Stembridge's Pfaffian representation of t_n . In Section 3 we prove a number of lemmas related to hypergeometric series. The proof then follows in a straightforward manner in Section 4.

2. A REFORMULATION OF STEMBRIDGE'S THEOREM

The methods developed in the previous papers in this series [1–4] were concerned with the evaluation of determinants, and in this section we restate Stembridge's result (1.3) as a slightly modified skew-symmetric determinant.

Recalling that the Pfaffian is the square root of the related skew-symmetric determinant [7, p. 394], and noting that $t_n \geq 0$, we see immediately that (1.3) is equivalent to the assertion that

$$t_n^2 = \begin{cases} \text{determinant } (a_{ij})_{0 \leq i, j \leq n-1} & \text{if } n \text{ is even} \\ \text{determinant } (a_{ij})_{1 \leq i, j \leq n-1} & \text{if } n \text{ is odd,} \end{cases} \quad (2.1)$$

where we extend (1.4) by requiring $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ for $i > j$.

LEMMA 1.

$$(-1)^n t_n^2 = \text{determinant } (a_{ij}^*)_{0 \leq i, j \leq n-1},$$

where $a_{00}^* = -1$ and $a_{ij}^* = a_{ij}$ otherwise.

Proof. If n is even, this assertion reduces to the top line of (2.1). To see this we expand the determinant by minors along the top row. The result is clearly

$$\begin{aligned} & \text{determinant } (a_{ij})_{0 \leq i, j \leq n-1} + a_{00}^* \cdot \text{determinant } (a_{ij})_{1 \leq i, j \leq n-1} \\ &= \text{determinant } (a_{ij})_{0 \leq i, j \leq n-1} \\ &= t_n^2, \end{aligned}$$

since a skew-symmetric matrix of odd order has determinant 0 [7, p. 392].

If n is odd, precisely the same expansion yields

$$\begin{aligned} & \text{determinant } (a_{ij})_{0 \leq i, j \leq n-1} + a_{00}^* \cdot \text{determinant } (a_{ij})_{1 \leq i, j \leq n-1} \\ &= -\text{determinant } (a_{ij})_{1 \leq i, j \leq n-1} \\ &= -t_n^2 \end{aligned}$$

since, as before, a skew-symmetric matrix of odd order has determinant 0. ■

3. HYPERGEOMETRIC BACKGROUND AND LEMMAS

The material in this section is inordinately complicated. The purpose of the various lemmas becomes clearer if you read Section 4 first.

We begin with six definitions.

$$e(i, j) = \begin{cases} 0, & \text{if } i < j \\ 1, & \text{if } i = j > 0 \\ \frac{(3j+1)! j!}{(2j)! (2j+1)!}, & \text{if } i = 0 \\ \frac{(-1)^{j-i} (2j-i-1)! (3i+2)(3j+1)! (i+j)! j! (2i+1)!}{(j-i)! (i-1)! i! (i+2j+2)! (2j+1)! (2j)!}, & \text{if } 0 < i < j, \end{cases} \quad (3.1)$$

$$f(i, j) = \frac{\binom{4i-j+1}{j} \binom{3i+1}{j} \binom{i}{j}}{\binom{2i}{j}^2}, \quad (3.2)$$

$$\begin{aligned} \phi(i, j) &= 2 \binom{3i-j}{j-1} + \binom{3i-j}{j} \\ &= \binom{3i-j}{j} \frac{(3i+1)}{(3i-2j+1)}, \end{aligned} \quad (3.2)$$

$$C_j(i) = \frac{\binom{i}{j} \binom{i+1}{2j} \binom{3i-2j}{j} (3i+1)}{(3i-3j+1) \binom{2i-j+1}{j} \binom{2i}{2j}}, \quad (3.4)$$

$$b(i, j) = \sum_{s=0}^j \binom{3i-j+1}{s}, \tag{3.5}$$

$$T(i, j, s) = \sum_{k=0}^{2i-1-s} \binom{3i-k-s+1}{k} e(2i-k-s, j). \tag{3.6}$$

LEMMA 2. For each $j \geq 0$,

$$\sum_{k=0}^j 2^{k+1} e(k+1, j+1) = e(0, j+1), \tag{3.7}$$

Proof. By (3.1) we may rewrite (3.7) as follows after dividing both sides by $e(0, j+1)$:

$$\sum_{k=1}^j \frac{2^{k+1} (-1)^{j-k} (2j-k)! (3k+5)(k+j+2)! (2k+3)!}{(j-k)! (k+1)! k! (k+2j+5)!} = 1. \tag{3.8}$$

The underlying hypergeometric series [5, p. 8] is (after finite products have been removed)

$${}_4F_3 \left(\begin{matrix} -j, 8/3, j+3, 5/2; \\ 5/3, -2j, 2j+6 \end{matrix} ; -8 \right), \tag{3.9}$$

which is not one of the well-known summable series. Consequently, we must rely on the WZ method [11] to prove (3.8). We define

$$F(j, k) = \frac{2^{k+1} (-1)^{j-k} (2j-k)! (3k+5)(k+j+2)! (2k+3)!}{(j-k)! (k+1)! k! (k+2j+5)!}, \tag{3.10}$$

and

$$G(j, k) = \frac{2^{k+3} (-1)^{j-k-1} (j+2)(2j-k)! (k+j+3)! (2k+5)!}{(j-k)! (k+2)! k! (k+2j+7)!}, \tag{3.11}$$

and we easily determine that

$$\begin{aligned} & F(j+1, k) - F(j, k) \\ &= \frac{F(j, k)(j+2)(6k(k+3) - 8(j+1)(j+3))}{(j-k+1)(k+2j+7)(k+2j+6)} \\ &= G(j, k) - G(j, k-1). \end{aligned} \tag{3.12}$$

We now let

$$S(j) = \sum_{k=0}^j F(j, k). \tag{3.13}$$

Then since $F(j, j+1) = 0$

$$\begin{aligned} S(j+1) - S(j) &= \sum_{k=0}^{j+1} (F(j+1, k) - F(j, k)) \\ &= \sum_{k=0}^{j+s} (G(j, k) - G(j, k-1)) \\ &= G(j, j+1) - G(j, -1) \\ &= 0. \end{aligned}$$

Therefore $S(j) = S(0)$ for all j , and $S(0) = 1$. This proves (3.8) and consequently (3.7). ■

LEMMA 3. For $0 < r \leq j$

$$\sum_{\lambda=0}^r f(r, \lambda) e(2r - \lambda, j) = \begin{cases} 0, & \text{if } 0 < r < j \\ \binom{3j+1}{j}^2 / \binom{2j}{j}^2 & \text{if } r = j. \end{cases} \quad (3.15)$$

Proof. We first note that the terms with $\lambda < 2r - j$ are identically zero by (3.1). We may now prove by mathematical induction applied to v that for $0 \leq v \leq j - r$,

$$\begin{aligned} &\sum_{\lambda=0}^{2r-j+v} f(r, \lambda) e(2r - \lambda, j) \\ &= \frac{(2r + j - v + 1)! (3r + 1)! r! (j + v)! j! (3j - v + 2)_v (r + j - v + 1)_v}{v! (2r - j + v)! (2j + 1)! (r + j + 1)! (j - r)! (2j - v + 1)_v (2r)!^2}, \end{aligned} \quad (3.16)$$

where $(\alpha)_v = \alpha(\alpha + 1) \cdots (\alpha + v - 1)$. This is merely a restatement of the fact that

$$\begin{aligned} &f(r, 2r - j + v) e(2r - (2r - j + v), j) \\ &\quad - f(r, 2r - j + v - 1) e(2r - (2r - j + v - 1), j) \end{aligned}$$

equals the right-hand side of (3.16), and that (3.16) is immediate when $v = 0$.

Identity (3.15) follows from (3.16) by setting $v = j - r$ and noting that $(r - j + 1)_{j-r} = 0$ if $r < j$. If $r = j$ the only nonvanishing term of the sum occurs for $r = j$, which is $f(j, j)$ as desired. ■

LEMMA 4. For $s \leq 2i - 1$,

$$T(i, j, s) = T(i - 1, j, s - 1) + T(i - 1, j, s - 2). \quad (3.17)$$

Proof.

$$\begin{aligned} & T(i - 1, j, s - 1) + T(i - 1, j, s - 2) \\ &= \sum_{k=0}^{2i-2-s} \binom{3i-k-s-1}{k} e(2i-k-s-1, j) \\ &\quad + \sum_{k=0}^{2i-1-s} \binom{3i-k-s}{k} e(2i-k-s, j) \\ &= \sum_{k=0}^{2i-1-s} \binom{3i-k-s}{k-1} e(2i-k-s, j) \\ &\quad + \sum_{k=0}^{2i-1-s} \binom{3i-k-s}{k} e(2i-k-s, j) \\ &= \sum_{k=0}^{2i-1-s} \binom{3i-k-s+1}{k} e(2i-k-s, j) \\ &= T(i, j, s). \quad \blacksquare \end{aligned}$$

LEMMA 5. For $m \leq 2i$

$$\sum_{r=0}^i c_r(i) f(i-r, m-2r) = \phi(i, m). \quad (3.18)$$

Proof. We note that the terms of the sum are identically zero if $r < m - i$. In any event if we begin by assuming m is nonintegral and $z! = \Gamma(z + 1)$, then

$$\begin{aligned} & \sum_{r=0}^i c_r(i) f(i-4, m-2r) \\ &= \sum_{r=0}^i \frac{\binom{i}{r} \binom{i+1}{2r} \binom{3i-2r}{r} (3i+1)}{(3i-3r+1) \binom{2i-r+1}{r} \binom{2i}{2r}} \\ &\quad \times \frac{\binom{4i-m-2r+1}{m-2r} \binom{3i-3r+1}{m-2r} \binom{i-r}{m-2r}}{\binom{2i-2r}{m-2r}^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{(2i-m)!^2 i! (i+1)! (3i+1)}{(4i-2m+1)! (2i)!} \\
&\quad \times \sum_{r=0}^i \frac{(3i-2r)! (2i-2r+1)(4i-2r-m+1)!}{\binom{r}{r! (i+1-2r)! (2i-r+1)!} (m-2r)! (3i-r-m+1)! (i+r-m)!} \\
&= \frac{(2i-m)!^2 i! (3i+1)! (4i-m+1)!}{(4i-2m+1)! (2i)!^2 (2i)!^2 m! (3i-m+1)! (i-m)!} \\
&\quad \times {}_7F_6 \left(\begin{matrix} -2i-1, -i+1/2, -i/2, -\frac{(i+1)}{2}, -3i+m-1, -\frac{m+1}{2}, -\frac{m}{2}; 1 \\ -i-1/2, -3i/2, -\frac{3i+1}{2}, i-m+1, \frac{m-1}{2}, -2i, \frac{m}{2}-2i \end{matrix} \right) \\
&= \frac{(3i-m)! (3i+1)}{m! (3i-2m)! (3i-2m+1)} \\
&= \phi(i, m), \tag{3.19}
\end{aligned}$$

where we have summed the ${}_7F_6$ by Dougall's summation [5, p. 26, Eq. (5)]. We now extend the result to integral $m \leq 2i$ by analytic continuation. ■

LEMMA 6. For $0 \leq i \leq j$,

$$2T(i-1, j, -1) + T(i-1, j, -2) = \begin{cases} 0, & \text{if } i < j \\ \binom{3i+1}{i}^2 / \binom{2i}{i}^2 & \text{if } i = j. \end{cases}$$

Proof.

$$\begin{aligned}
&2T(i-1, j, -1) + T(i-1, j, -2) \\
&= \sum_{k=0}^{2i-2} 2 \binom{3i-k-1}{k} e(2i-1-k, j) \\
&\quad + \sum_{k=0}^{2i-1} \binom{3i-k}{k} e(2i-1-k, j) \\
&= \sum_{k=0}^{2i-1} \left(2 \binom{3i-k}{k-1} + \binom{3i-k}{k} \right) e(2i-k, j) \\
&= \sum_{k=0}^{2i-1} \phi(i, k) e(2i-k, j) \\
&= \sum_{k=0}^{2i-1} \sum_{r=0}^i c_r(i) f(i-r, k-2r) e(2i-k, j) \\
&= \sum_{r=0}^i c_r(i) \sum_{k=0}^{2i-1} f(i-r, k-2r) e(2(i-r) - (k-2r), j)
\end{aligned}$$

$$= \begin{cases} 0, & \text{if } i < j \text{ by (3.15), top line} \\ c_0(i) \binom{3j+1}{j}^2 / \binom{2j}{j}^2, & \text{if } i = j \text{ by (3.15), bottom line,} \end{cases}$$

which is (3.20) since $c_0(i) = 1$ by (3.4). ■

LEMMA 7. For $0 \leq i \leq j$

$$\sum_{\mu=0}^{2i-1} b(i, \mu) e(2i - \mu, j) = \begin{cases} 0, & \text{for } i < j \\ \binom{3i+1}{i}^2 / \binom{2i}{i}^2 & \text{for } i = j. \end{cases} \quad (3.21)$$

Proof. Let

$$\begin{aligned} U(i, j) &= \sum_{\mu=0}^{2i-1} b(i, \mu) e(2i - \mu, j) \\ &= \sum_{\mu=0}^{2i-1} \sum_{k=0}^{\mu} \binom{3i - \mu + 1}{k} e(2i - \mu, j) \\ &= \sum_{\substack{k+s \leq 2i-1 \\ k \geq 0, s \geq 0}} \binom{3i - (k+s) + s}{k} e(2i - k - s, j) \\ &= \sum_{s=0}^{2i-1} \sum_{k=0}^{2i-s-1} \binom{3i - k - s + 1}{k} e(2i - k - s, j) \\ &= \sum_{s=0}^{2i-1} T(i, j, s). \end{aligned} \quad (3.22)$$

Hence by Lemma 4,

$$\begin{aligned} U(i, j) &= \sum_{s=0}^{2i-1} (T(i-1, j, s-1) + T(i-1, j, s-2)) \\ &= \sum_{s=-1}^{2i-2} T(i-1, j, s) + \sum_{s=-2}^{2i-3} T(i-1, j, s) \\ &= 2U(i-1, j) + 2T(i-1, j, -1) + T(i-1, j, -2) \\ &= 2U(i-1, j) + \begin{cases} 0, & \text{if } 0 \leq i < j \\ \binom{3i+1}{i}^2 / \binom{2i}{i}^2 & \text{if } i = j \end{cases} \end{aligned} \quad (3.23)$$

by Lemma 6.

We now proceed by induction on i . If $i = 1$,

$$\begin{aligned} U(1, j) &= b(1, 0) e(2, j) + b(1, 1) e(1, j) \\ &= e(2, j) + 4e(1, j) \\ &= \begin{cases} 0 & \text{if } j > 1 \\ 4 & \text{if } j = 1 \end{cases} \end{aligned} \quad (3.24)$$

by Lemma 3 in the case $r = 1$ since $f(1, 0) = 1$ and $f(1, 1) = 4$.

If $i = 2$,

$$\begin{aligned} U(2, j) &= b(2, 0) e(4, j) + b(2, 1) e(3, j) \\ &\quad + b(2, 2) e(2, j) + b(2, 3) e(1, j) \\ &= e(4, j) + 7e(3, j) + 16e(2, j) + 15e(1, j) \\ &= (e(4, j) + 7e(3, j) + \frac{49}{4} e(2, j)) \\ &\quad + \frac{15}{4} (e(2, j) + 4e(1, j)) \\ &= \begin{cases} 0 & \text{if } j > 2 \\ 49/4 & \text{if } j = 2 \end{cases} \end{aligned} \quad (3.25)$$

by Lemma 3 in the case $r = 1$ and $r = 2$, since $f(r, 0) = 1$, $f(r, 1) = 3r + 1$, and $f(2, 2) = 49/4$.

Now we assume that (3.21) is valid up to but not including a particular $i > 2$. Then by (3.23) with $i \leq j$,

$$\begin{aligned} U(i, j) &= U(i-1, j) + U(i-2, j) \\ &\quad + \begin{cases} 0 & \text{if } i < j \\ \binom{3i+1}{i}^2 / \binom{2i}{i}^2 & \text{if } i = j \end{cases} \\ &= \begin{cases} 0 & \text{if } i < j \\ \binom{3i+1}{i}^2 / \binom{2i}{i}^2 & \text{if } i = j \end{cases} \end{aligned}$$

since $i-1 < j$. Hence (3.21) holds for all i . ■

4. THE MAIN RESULT

THEOREM.

$$t_n = \prod_{i=0}^{n-1} \lambda_i = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Proof. We begin by defining three matrices:

$$M(n) = (a_{ij}^*)_{0 \leq i, j \leq n-1} \quad (4.1)$$

$$R(n) = (e(i, j))_{0 \leq i, j \leq n-1} \quad (4.2)$$

$$\begin{aligned} V(n) &= M(n) \cdot R(n) \\ &= (v(i, j))_{0 \leq i, j \leq n-1} \end{aligned} \quad (4.3)$$

We show that $V(n)$ is a lower triangular matrix and that $v(i, i) = -\lambda_i^2 = -\binom{3i+1}{i}^2 / \binom{2i}{i}^2$. If we can establish this last assertion, then we are done because then

$$(-1)^n t_n^2 = \text{determinant } (M(n)) \quad (4.4)$$

$$1 = \text{determinant } (R(n)). \quad (4.5)$$

Hence

$$\begin{aligned} (-1)^n t_n^2 &= \text{determinant } (M(n)) \cdot \text{determinant } (R(n)) \\ &= \text{determinant } (V(n)) \\ &= \prod_{i=0}^{n-1} v(i, i) \\ &= (-1)^n \prod_{i=0}^{n-1} \lambda_i^2. \end{aligned} \quad (4.6)$$

But since both t_n and λ_i are nonnegative, this proves the theorem.

We now proceed to evaluate $v(i, j)$ for $i \leq j$. We define

$$A(i, j) = \sum_{s=0}^{2j-i} \binom{i+j}{s}, \quad (4.7)$$

so that

$$a_{ij} = A(i, j) - A(j, i) \quad (4.8)$$

for all nonnegative i and j .

Furthermore,

$$\begin{aligned} A(i, j) &= \sum_{s=0}^{2j-i} \binom{i+j}{s} \\ &= 2^{i+j} - \sum_{s=2j-i+1}^{i+j} \binom{i+j}{s} \\ &= 2^{i+j} - \sum_{s=0}^{2i-j-1} \binom{i+j}{i+j-s} \\ &= 2^{i+j} - \left(A(j, i) - \binom{i+j}{2i-j} \right), \end{aligned}$$

so for all nonnegative i and j

$$A(i, j) + A(j, i) = 2^{i+j} + \binom{i+j}{2i-j}. \quad (4.9)$$

Now $v(0, 0) = a_{0,0}^* e(0, 0) = -1 = -\lambda_0^2$, and for $j > 0$ by (1.4)

$$\begin{aligned} v(0, j) &= \sum_{k=0}^j a_{0,k}^* e(k, j) \\ &= -e(0, j) + \sum_{k=1}^j 2^k e(k, j) \\ &= 0 \end{aligned} \quad (4.10)$$

by Lemma 2.

Now for $0 < i \leq j$

$$\begin{aligned} v(i, j) &= \sum_{k=0}^j a_{i,k} e(k, j) \\ &= \sum_{k=0}^j (A(i, k) - A(k, i)) e(k, j) \\ &= \sum_{k=0}^j \left(2^{i+k} + \binom{i+k}{2i-k} - 2A(k, i) \right) e(k, j) \\ &= 2^i \sum_{k=0}^j 2^k e(k, j) - \sum_{k=0}^j \left(2A(k, i) - \binom{i+k}{2i-k} \right) e(k, j) \\ &= - \sum_{k=1}^j \left(\sum_{s=0}^{2i-k} \binom{i+k+1}{s} \right) e(k, j), \end{aligned}$$

where the first sum has vanished by Lemma 2, and the coefficient of $e(k, j)$ in the second sum has been rewritten using the recurrence $\binom{A}{B} = \binom{A-1}{B} + \binom{A-1}{B-1}$.

Hence for $0 < i \leq j$,

$$\begin{aligned} v(i, j) &= - \sum_{k=1}^{2i} \sum_{s=0}^{2i-k} \binom{i+k+1}{s} e(k, j) \\ &= - \sum_{k=0}^{2i-1} \sum_{s=0}^k \binom{3i-k+1}{s} e(2i-k, j) \\ &= - \sum_{k=0}^{2i-1} b(i, k) e(2i-k, j) \\ &= \begin{cases} 0 & \text{for } i < j \\ -\lambda_i^2 & \text{for } i = j \end{cases} \end{aligned} \quad (4.12)$$

by Lemma 7. Hence $V(n)$ is lower triangular and $v(i, i) = -\lambda_i^2$ as desired. Therefore our theorem is proved. ■

5. CONCLUSION

The most important observation in cracking this problem was that -1 should be inserted in the upper left-hand corner of the matrix $M(n)$. Without a nonzero entry there, the methods developed in [1]–[4] break down completely. Whether this attack will be successful in other such problems remains to be seen.

The great value of the WZ method [11] is brought home forcefully in Lemma 2.

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