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Left-symmetric algebras from linear functions

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Abstract

In this paper, some left-symmetric algebras are constructed from linear functions. They include a kind of simple left-symmetric algebras and some examples appearing in mathematical physics. Their complete classification is also given, which shows that they can be regarded as generalization of certain two-dimensional left-symmetric algebras.

Keywords: Left-symmetric algebra; Linear function; Lie algebra

1. Introduction

A left-symmetric algebra is an algebra whose associator is left-symmetric: let A be a vector space over a field $\mathbf{F}$ with a bilinear product $(x, y) \to xy$. A is called a left-symmetric algebra if for any $x, y, z \in A$, the associator

$$(x, y, z) = (xy)z - x(yz) \tag{1.1}$$

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is symmetric in x, y, that is,

$$(x, y, z) = (y, x, z) \quad \text{or equivalently} \quad (xy)z - x(yz) = (yx)z - y(xz). \tag{1.2}$$

Left-symmetric algebras are a class of non-associative algebras arising from the study of convex homogenous cones, affine manifolds and affine structures on Lie groups [13,19,22]. Moreover, they have very close relations with many problems in mathematical physics. For example, they appear as an underlying structure of those Lie algebras that possess a phase space ([15–18], thus they form a natural category from the point of view of classical and quantum mechanics) and there is a close relation between them and classical Yang–Baxter equation [9,10,12].

However, due to the non-associativity, there is not a suitable representation theory of left-symmetric algebras. It is also known that the definition identity (1.2) of left-symmetric algebras involves the quadric forms of structure constants, which is not linear in general [13]. Hence it is quite difficult to study them. Therefore, one of the most important problems is how to construct interesting left-symmetric algebras. One way is to construct them through some well-known algebras and algebraic structures. This can be regarded as a kind of "realization theory." For example, there is a study of realization of Novikov algebras (they are left-symmetric algebras with commuting right multiplications) from commutative associative algebras and Lie algebras in [3–5]. Another way is to try to reduce the "non-linearity" in certain sense. Combining these two ways, a natural and simple way is to construct left-symmetric algebras from linear functions, which is the main content of this paper.

On the other hand, there are many examples of left-symmetric algebras appearing in mathematical physics ([6,11,12,21], etc.). For example, let V be a vector space over the complex field $\mathbf{C}$ with the ordinary scalar product $(,)$ and a be a fixed vector in V, then

$$u * v = (u, v)a + (u, a)v, \quad \forall u, v \in V, \tag{1.3}$$

defines a left-symmetric algebra on V which gives the integrable (generalized) Burgers equation [20,21]

$$U_t = U_{xx} + 2U * U_x + \big(U * (U * U)\big) - \big((U * U) * U\big). \tag{1.4}$$

However, such examples are often scattered and independent in different references of mathematical physics. And in most of the cases, there is neither a good mathematical motivation nor a further study. In this paper, our construction not only has a natural motivation from the point of view of mathematics, but also can be regarded as a kind of generalization of the examples given by Eq. (1.3). Moreover, a systematic study is given.

The algebras that we consider in this paper are of finite dimension and over $\mathbf{C}$. The paper is organized as follows. In Section 2, we construct left-symmetric algebras from linear functions. In Section 3, we give their classification. In Section 4, we discuss some properties of these left-symmetric algebras and certain application in mathematical physics.

2. Constructing left-symmetric algebras from linear functions

Let A be a vector space in dimension n. In general, we assume $n \geqslant 2$. Just as said in the introduction, motivated by the study of algebraic structure itself and some equations in integrable systems, it is natural to consider the left-symmetric algebras satisfying the following conditions: for any two vectors x, y in A, the product $x * y$ is still in the subspace spanned by x, y, that is, any two vectors make up a subalgebra in A. Thus, it is natural to assume

$$x * y = f_1(x, y)x + f_2(x, y)y, \quad \forall x, y \in A, \tag{2.1}$$

where $f_1, f_2 : A \times A \to \mathbf{C}$ are two functions. In general, f_1 and f_2 are not necessarily linear. However, if they are not linear functions, they cannot be decided by their values at a basis of A. Hence the problem turns to be more complicated, even more complicated than the study of the algebra itself.

Therefore, we can assume that f_1 and f_2 are linear functions. Since the algebra product $*$ is bilinear, for $f_1 \neq 0$, f_1 depends on only y, that is, f_1 is not a linear function depending on x. Otherwise, for any $\lambda \in \mathbf{C}$, we have

$$\begin{aligned}(\lambda x) * y &= f_1(\lambda x, y)\lambda x + f_2(\lambda x, y)y = \lambda^2 f_1(x, y)x + \lambda f_2(x, y)y \\ &= \lambda\big(f_1(x, y)x + f_2(x, y)y\big).\end{aligned} \tag{2.2}$$

Hence $f_1(x, y) = 0$, $\forall x, y \in A$, which is a contradiction. Similarly, f_2 depends on only x. Thus, we can set $f_1(x, y) = f(y)$, $f_2(x, y) = g(x)$, where $f, g : A \to \mathbf{C}$ are two linear functions.

Proposition 2.1. *Let A be a vector space in dimension $n \geqslant 2$. Let $f, g : A \to \mathbf{C}$ be two linear functions. Then the product*

$$x * y = f(y)x + g(x)y, \quad \forall x, y \in A \tag{2.3}$$

defines a left-symmetric algebra if and only if $f = 0$ or $g = 0$. Moreover, when $f = 0$ or $g = 0$, the above equation defines an associative algebra.

Proof. For any $x, y, z \in A$, the associator

$$\begin{aligned}(x, y, z) &= (x * y) * z - x * (y * z) \\ &= f(y)\big(f(z)x + g(x)z\big) + g(x)\big(f(z)y + g(y)z\big) - f(z)\big(f(y)x + g(x)y\big) \\ &\quad - g(y)\big(f(z)x + g(x)z\big) \\ &= f(y)g(x)z - g(y)f(z)x.\end{aligned}$$

Hence $(x, y, z) = (y, x, z)$ if and only if for any $y, z \in A$, $g(y)f(z) = 0$, that is, $f = 0$ or $g = 0$. Moreover, when $f = 0$ or $g = 0$, $(x, y, z) = 0$. Thus the proposition holds. □

Let L_x, R_x denote the left and right multiplication, respectively, i.e., $L_x(y) = xy$, $R_x(y) = yx$, $\forall x, y \in A$.

Corollary 2.2. *With the conditions in above proposition, we have*

(1) *If* $f = 0$, $g \neq 0$, *then there exists a basis* $\{e_1, \ldots, e_n\}$ *in* A *such that* $L_{e_1} = \mathrm{Id}$, $L_{e_i} = 0$, $i = 2, \ldots, n$, *where* Id *is the identity transformation.*
(2) *If* $g = 0$, $f \neq 0$, *then there exists a basis* $\{e_1, \ldots, e_n\}$ *in* A *such that* $R_{e_1} = \mathrm{Id}$, $R_{e_i} = 0$, $i = 2, \ldots, n$.
(3) *If* $f = g = 0$, *then* A *is a trivial algebra, that is, all products are zero.*

Proof. For any linear function $g : A \to \mathbf{C}$, if $g \neq 0$, due to the linearity of g and the direct sum of vector spaces

$$A = \operatorname{Ker} g \oplus g(A) = \operatorname{Ker} g \oplus \mathbf{C},$$

there exists a basis $\{e_1, \ldots, e_n\}$ in A such that $g(e_1) \neq 0$, $g(e_i) = 0$, $i = 2, \ldots, n$. Furthermore, we can normalize g by $g(e_1) = 1$. Hence (1) and (2) follows. (3) is obvious. □

Remark 1. There is a natural matrix representation of above associative algebras [8]. Let $\{E_{ij}\}$ be the canonical basis of $gl(n)$, that is, E_{ij} is a $n \times n$ matrix with 1 at ith row and jth column and zero at other places. Then the algebra in above case (1) (respectively (2)) is an associative subalgebra of $gl(n)$ (under the ordinary matrix product) with $e_i = E_{1i}$ (respectively $e_i = E_{i1}$).

It is well known that the commutator of a left-symmetric algebra $[x, y] = xy - yx$ defines a (sub-adjacent) Lie algebra ([13,19], etc.).

Corollary 2.3. *The sub-adjacent Lie algebras of the associative algebras defined by equation* (2.3) *with* $g = 0$, $f \neq 0$, *or* $f = 0$, $g \neq 0$ *are isomorphic to the following* 2*-step solvable Lie algebra*:

$$A = \left\langle e_i,\ i = 1, \ldots, n \;\middle|\; [e_1, e_i] = e_i,\ i = 2, \ldots, n,\ \text{other products are zero} \right\rangle. \tag{2.4}$$

Proof. For case (1) in Corollary 2.2, the conclusion is obvious. For case (2) in Corollary 2.2, we only need a linear transformation by letting e_1 be $-e_1$ and e_i still be e_i ($i = 2, \ldots, n$), which the conclusion follows. □

Remark 2. The above conclusion also can be obtained from Eq. (2.3) directly. That is, the Lie algebra given by $[x, y] = (f - g)(x)y - (f - g)(y)x$ is isomorphic to the Lie algebra given by Eq. (2.4) for $g \neq f$. In fact, this algebra can be regarded as a (unique!) non-abelian Lie algebra constructed from linear functions: it is easy to show that the product $[x, y] = f(x)y + g(y)x$ defines a Lie algebra if and only if $f(x) = -g(x)$, $\forall x \in A$.

Due to the above discussion, in order to get non-associative left-symmetric algebras, we need to extend the above construction. A simple extension of Eq. (2.3) is to add a fixed vector $c \neq 0$ as follows:

$$x * y = f(x)y + g(y)x + h(x, y)c, \quad \forall x, y \in A, \tag{2.5}$$

where $h : A \times A \to \mathbf{C}$ is a non-zero bilinear function. The above Eq. (2.5) can be understood that for any two vectors x, y, the three vectors x, y, c make up a subalgebra in A. Moreover, if h is symmetric, then its sub-adjacent Lie algebra is isomorphic to the Lie algebra given by Eq. (2.4) ($f \neq g$) or the abelian Lie algebra ($f = g$).

For a further study, we give a lemma on linear functions at first.

Lemma 2.4. *Let A be a vector space in dimension $n \geqslant 2$. Let $f, g : A \to \mathbf{C}$ be two linear functions and $h : A \times A \to \mathbf{C}$ be a symmetric bilinear function.*

(1) *If for any $x, y \in A$, $f(x)g(y) = f(y)g(x)$, then $f = 0$, or $g = 0$, or $f \neq 0$, $g \neq 0$ and there exists $\alpha \in \mathbf{C}$, $\alpha \neq 0$ such that $f(x) = \alpha g(x)$, $\forall x \in A$.*
(2) *If for any $x, y, z \in A$, $f(x)h(y, z) = f(y)h(x, z)$, then $f = 0$, or $h = 0$, or there exists a basis $\{e_1, \ldots, e_n\}$ in A and $\alpha \in \mathbf{C}$, $\alpha \neq 0$ such that $f(x) = h(x, \alpha e_1)$, $\forall x \in A$; $h(e_1, e_1) = 1$, $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$.*

Proof. For a linear function f, if $f \neq 0$, we can choose a basis $\{e_1, \ldots, e_n\}$ in A such that $f(e_1) \neq 0$, $f(e_2) = \cdots = f(e_n) = 0$. If $g \neq 0$, then from $f(x)g(y) = f(y)g(x)$, we can have $g(e_1) \neq 0$, $g(e_i) = 0$, $i = 2, \ldots, n$. Let $\alpha = f(e_1)/g(e_1)$, then by linearity, for any $x \in A$, we have $f(x) = \alpha g(x)$.

Similarly, for $f \neq 0$ and the basis $\{e_1, \ldots, e_n\}$ in A such that $f(e_1) \neq 0$, $f(e_2) = \cdots = f(e_n) = 0$, we have $h = 0$ or $h(e_1, e_1) \neq 0$ and $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$. For the latter case, we can normalize h by $h(e_1, e_1) = 1$. Thus, we still have $f(x) = h(x, \alpha e_1)$, $\forall x \in A$, where $\alpha = f(e_1)/h(e_1, e_1) = f(e_1)$. □

Theorem 2.5. *With the conditions in above lemma and $h \neq 0$, Eq.* (2.5) *defines a left-symmetric algebra if and only if the functions f, g, h belong to one of the following cases*:

(1) $f = g = 0$, $h(x, c) = 0$, $\forall x \in A$;
(2) $f = g = 0$, *and there exists a basis $\{e_1, \ldots, e_n\}$ such that $h(e_1, e_1) = 1$, $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$, and $c = \sum_{i=1}^n a_i e_i$ with $a_1 \neq 0$;*
(3) $g = 0$, $f \neq 0$, $f(x) = h(x, c)$, $\forall x \in A$;
(4) $g = 0$, $f \neq 0$, *and there exists a basis $\{e_1, \ldots, e_n\}$ and $\alpha \in \mathbf{C}$, $\alpha \neq 0$ such that $f(x) = h(x, c - \alpha e_1)$, $h(e_1, e_1) = 1$, $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$, and $c = \sum_{i=1}^n a_i e_i$ with $a_1 \neq \alpha$;*
(5) $f = 0$, $g \neq 0$, $g(x) = -h(x, c)$, $\forall x \in A$ *and* $h(c, c) = 0$;
(6) $f = 0$, $g \neq 0$, $h(x, c) = 0$, $\forall x \in A$, *and there exists a basis $\{e_1, \ldots, e_n\}$ and $\alpha \in \mathbf{C}$, $\alpha \neq 0$ such that $g(x) = h(x, \alpha e_1)$, $h(e_1, e_1) = 1$, $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$;*

(7) *$f \neq 0$, $g \neq 0$, $f(c) \neq 0$ and there exists $\alpha \in \mathbf{C}$, $\alpha \neq 0$ such that $g(x) = \alpha f(x)$, $h(x, y) = -f(x)f(y)/f(c)$, $\forall x \in A$.*

Proof. For any $x, y, z \in A$, the associator

$$\begin{aligned}
(x, y, z) &= (x * y) * z - x * (y * z) \\
&= f(x)\big[f(y)z + g(z)y + h(y, z)c\big] + g(y)\big[f(x)z + g(z)x + h(x, z)c\big] \\
&\quad + h(x, y)\big[f(c)z + g(z)c + h(c, z)c\big] - f(y)\big[f(x)z + g(z)x + h(x, z)c\big] \\
&\quad - g(z)\big[f(x)y + g(y)x + h(x, y)c\big] - h(y, z)\big[f(x)c + g(c)x + h(x, c)c\big] \\
&= \big[-f(y)g(z) - g(c)h(y, z)\big]x + \big[g(y)f(x) + f(c)h(x, y)\big]z \\
&\quad + \big[g(y)h(x, z) - f(y)h(x, z) + h(x, y)h(c, z) - h(y, z)h(x, c)\big]c.
\end{aligned}$$

Then by left-symmetry, we can get the following equations: for any $x, y, z \in A$,

$$f(y)g(x) = g(y)f(x); \tag{2.6}$$

$$f(y)g(z) + g(c)h(y, z) = 0; \tag{2.7}$$

$$\big[(g - f)(y) + h(y, c)\big]h(x, z) = \big[(g - f)(x) + h(x, c)\big]h(y, z). \tag{2.8}$$

From Eq. (2.6) and using Lemma 2.4, we can consider the following cases.

Case (I). $f = g = 0$. There is only one non-trivial equation $h(y, c)h(x, z) = h(x, c)h(y, z)$. Let $h'(x) = h(x, c)$, then by Lemma 2.4, we know that $h'(x) = 0$ or there exists a basis $\{e_1, \ldots, e_n\}$ in A and $\alpha \in \mathbf{C}$, $\alpha \neq 0$ such that $h'(x) = h(x, \alpha e_1)$, $\forall x \in A$; $h(e_1, e_1) = 1$, $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$. The former is the case (1) and the latter is the case (2) since $h(x, c) = h(x, \alpha e_1)$ implies that $a_1 = \alpha \neq 0$ for $c = \sum_{i=1}^n a_i e_i$.

Case (II). $g = 0$, $f \neq 0$. Then Eq. (2.7) is satisfied. From Eq. (2.8) and using Lemma 2.4 again, we have $f(x) = h(x, c)$ or there exists a basis $\{e_1, \ldots, e_n\}$ such that $h(e_1, c) - f(e_1) \neq 0$, $f(e_i) = h(e_i, c) = 0$; $h(e_1, e_1) = 1$, $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$. The former is the case (3) and the latter is the case (4) where $\alpha = -f(e_1) + h(e_1, c)$. Notice for the latter, $f \neq 0$ if and only if $a_1 \neq \alpha$ for $c = \sum_{i=1}^n a_i e_i$.

Case (III). $f = 0$, $g \neq 0$. From Eq. (2.7), we have $g(c) = 0$. As the same as the discussion in Case (II), Eq. (2.8) implies that $g(x) = -h(x, c)$ or there exists a basis $\{e_1, \ldots, e_n\}$ such that $g(e_1) + h(e_1, c) \neq 0$, $g(e_i) = h(e_i, c) = 0$, $h(e_1, e_1) = 1$, $h(e_i, e_j) = 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$. The former is the case (5). For the latter, we have $g(x) = -h(x, c) + \alpha h(x, e_1)$ where $\alpha = g(e_1) + h(e_1, c)$. Set $c = \sum_{i=1}^n a_i e_i$, then $g(c) = -a_1^2 + \alpha a_1 = 0$. Thus $a_1 = \alpha$ or $a_1 = 0$. Therefore if $g \neq 0$, we have $h(x, c) = 0$ and $g(x) = h(x, \alpha e_1)$ which is just the case (6).

Case (IV). $f \neq 0$, $g \neq 0$. Thus there exists $\alpha \neq 0$ such that $g(x) = \alpha f(x)$. Hence from Eq. (2.7) and the assumption $h \neq 0$, we know that $f(c) \neq 0$ and $h(x, y) = -\frac{f(x)f(y)}{f(c)}$, $\forall x \in A$. It is easy to see that Eq. (2.8) holds under these conditions. This is the case (7). □

Corollary 2.6. *The left-symmetric algebras given in Theorem* 2.5 *are commutative* (*hence associative*), *if and only if their sub-adjacent Lie algebras are abelian, if and only if they belong to the case* (1), (2), *and* (7) *with* $\alpha = 1$.

By direct checking, we have

Corollary 2.7. *Let A be a left-symmetric algebra in Theorem* 2.5.

(1) *If A is in the case* (1), (2), (4), (6), (7), *then the corresponding bilinear function h satisfies*

$$h(x * y, z) = h(y * x, z) = h(x * z, y), \quad \forall x, y, z \in A. \tag{2.9}$$

(2) *If A is in the case* (3), *then the corresponding bilinear function h is invariant under the following sense*:

$$h(x * y, z) = h(x, z * y), \quad \forall x, y, z \in A. \tag{2.10}$$

That is, for every $x, y, z \in A$, $h(R_x(y), z) = h(y, R_x(z))$ (R_x is self-adjoint).

(3) *If A is in the case* (5), *then the corresponding bilinear function h satisfies*

$$h(x * y, z) + h(y, x * z) = 0, \quad \forall x, y, z \in A. \tag{2.11}$$

That is, for every $x, y, z \in A$, $h(L_x(y), z) + h(y, L_x(z)) = 0$.

3. Classification of left-symmetric algebras from linear functions

In this section, we discuss the classification of left-symmetric algebras given in Theorem 2.5. Since the bilinear function h appearing in the case (2), (4), (6), and (7) has been (almost) decided completely, we give the classification of these cases at first.

Proposition 3.1. *Let A be a left-symmetric algebra in the case* (2) *with dimension $n \geqslant 2$. Then A is isomorphic to the following algebra* (*we only give the non-zero products*):

$$A_{(2)} = \langle e_i,\ i = 1, \ldots, n \mid e_1 e_1 = e_1 \rangle. \tag{3.1}$$

Proof. For $c = \sum_{i=1}^{n} a_i e_i$ with $a_1 \neq 0$, let

$$e_1' = \frac{1}{a_1} e_1 + \frac{1}{a_1^2} \sum_{i=2}^{n} a_i e_i, \quad e_j' = e_j,\ j = 2, \ldots, n,$$

then under the new basis, Eq. (3.1) follows. □

Proposition 3.2. *Let A be a left-symmetric algebra in the case* (4) *with dimension $n \geqslant 2$. Then A is isomorphic to one of the following algebras*:

$$A^1_{(4)} = \langle e_i,\ i = 1,\dots,n \mid e_1e_1 = e_1 + e_2,\ e_1e_j = e_j,\ j = 2,\dots,n\rangle; \tag{3.2}$$

$$A^\lambda_{(4)} = \langle e_i,\ i = 1,\dots,n \mid e_1e_1 = \lambda e_1,\ e_1e_j = e_j,\ j = 2,\dots,n\rangle, \quad \lambda \neq 1, 2. \tag{3.3}$$

Proof. For the case (4), we have

$$e_1 * e_1 = h\big(e_1, (a_1-\alpha)e_1\big)e_1 + h(e_1,e_1)c = (a_1-\alpha)e_1 + c = (2a_1-\alpha)e_1 + \sum_{i=2}^n a_ie_i,$$

$$e_1 * e_i = (a_1-\alpha)e_i, \qquad e_i * e_j = 0, \quad i = 2,\dots,n,\ j = 1,\dots,n.$$

If $a_1 = 0$, then $c = \sum_{i=2}^n a_ie_i \neq 0$. Without losing generality, we suppose $a_2 \neq 0$. Let

$$e_1' = -\frac{1}{\alpha}e_1, \qquad e_2' = \frac{1}{\alpha^2}c, \qquad e_j' = e_j,\ j = 3,\dots,n,$$

then under the new basis, we can get Eq. (3.2).

If $a_1 \neq 0$ and $a_1 \neq \alpha$, then let

$$e_1' = \frac{1}{a_1-\alpha}e_1 + \frac{1}{(a_1-\alpha)a_1}\sum_{i=2}^n a_ie_i, \quad e_j' = e_j,\ j = 2,\dots,n.$$

Hence under the new basis, we have

$$e_1' * e_1' = \frac{2a_1-\alpha}{a_1-\alpha}e_1', \qquad e_1' * e_i' = e_i', \qquad e_i' * e_j' = 0, \quad i = 2,\dots,n,\ j = 1,\dots,n.$$

Set $\lambda = \frac{2a_1-\alpha}{a_1-\alpha}$ which gives Eq. (3.3). Notice that $\lambda \neq 1, 2$ since $a_1 \neq 0, a_1 \neq \alpha$ and $\alpha \neq 0$. □

As the same as the proof of Eq. (3.2), we have the following proposition.

Proposition 3.3. *Let A be a left-symmetric algebra in the case* (6) *with dimension $n \geqslant 2$. Then A is isomorphic to*

$$A_{(6)} = \langle e_i,\ i = 1,\dots,n \mid e_1e_1 = e_1 + e_2,\ e_je_1 = e_j,\ j = 2,\dots,n\rangle. \tag{3.4}$$

Proposition 3.4. *Let A be a left-symmetric algebra in the case* (7) *with dimension $n \geqslant 2$. Then A is isomorphic to one of the following algebras*:

$$A^\alpha_{(7)} = \langle e_i,\ i = 1,\dots,n \mid e_1e_1 = \alpha e_1,\ e_1e_j = e_j,\ e_je_1 = \alpha e_j,\ j = 2,\dots,n\rangle, \quad \alpha \neq 0. \tag{3.5}$$

Proof. Without losing generality, we can choose a basis $\{e_1, \dots, e_n\}$ such that $e_1 = c$ and $f(e_2) = \cdots = f(e_n) = 0$. Hence

$$e_1 * e_1 = \alpha f(e_1)e_1, \qquad e_1 * e_j = f(e_1)e_j, \qquad e_j * e_1 = \alpha f(e_1)e_j, \quad j = 2, \dots, n.$$

The conclusion follows by the basis transformation

$$e_1' = \frac{1}{f(e_1)}e_1, \qquad e_j' = e_j, \quad j = 2, \dots, n. \qquad \square$$

In order to classify the left-symmetric algebras in other cases, we need the following lemma.

Lemma 3.5. *Let A be a finite-dimensional algebra over $\mathbf{C}$. Let $A = A_1 \oplus A_2$ as the direct sum of two subspaces and A_1 be a subalgebra. Assume that, for every $x \in A_1$, L_x and R_x acts on A_2 is zero or* Id. *If there exists a non-zero vector $v \in A_1$ such that for any two vectors $x, y \in A_2$, $xy = yx \in \mathbf{C}v$, then the classification of the algebraic operation in A_2 (without changing other products) is given by the classification of symmetric bilinear forms on a n-dimensional vector space over $\mathbf{C}$, where $n = \dim A_2$. That is, there exists a basis $\{e_1, \dots, e_n\}$ in A_2 such that the classification is given as follows: A_2 is trivial or for every $k = 1, \dots, n$:*

$$e_i e_j = \begin{cases} \delta_{ij} v, & i, j = 1, \dots, k; \\ 0, & \textit{otherwise}. \end{cases} \tag{3.6}$$

Proof. From the assumption, there exists a symmetric bilinear form $f : A_2 \times A_2 \to \mathbf{C}$ such that

$$xy = f(x, y)v, \quad \forall x, y \in A_2.$$

Moreover, any linear transformation of A_2 does not change the operation relations between A_1 and A_2, hence the whole algebra $A = A_1 \oplus A_2$. Thus the classification of A_2 is decided completely by the classification of symmetric bilinear forms on a vector space in dimension $\dim A_2$. Therefore there exists a basis $\{e_1, \dots, e_n\}$ in A_2 such that the matrix $(f(e_i, e_j))$ is zero or a diagonal matrix with the first k ($k = 1, \dots, n$) elements are 1 and the others are zero on the diagonal, which gives Eq. (3.6). It is easy to show that for different k, the algebras are not mutually isomorphic. $\square$

Proposition 3.6. *The classification of left-symmetric algebras in the case* (1) *with dimension $n \geqslant 2$ is equivalent to the classification of symmetric bilinear forms on a $(n-1)$-dimensional vector space. The classification is given as follows: for every $k = 0, \dots, n-1$,*

$$A_{(1)}^{(k)} = \langle e_i,\ i = 1, \dots, n \mid e_j e_j = e_1,\ j = 2, \dots, k+1 \rangle. \tag{3.7}$$

Proof. Let A be a left-symmetric algebra in the case (1) with dimension $n \geqslant 2$. We can choose a basis $\{e_1, \dots, e_n\}$ such that $e_1 = c$. Thus we have

$$e_1 * e_1 = e_1 * e_j = e_j * e_1 = 0, \qquad e_j * e_k = h(e_j, e_k)e_1, \quad j,k = 2, \dots, n.$$

Let A_1 be a subspace spanned by e_1 and A_2 be a subspace spanned by $e_2, \dots, e_n$. Then by Lemma 3.5, the proposition holds. □

Proposition 3.7. *The classification of left-symmetric algebras in the case* (3) *with dimension $n \geqslant 2$ is given by the following matrices* ($F = (h(e_i, e_j))$, *where* $\{e_1, \dots, e_n\}$ *is a basis*):

$$F_1 = I, \qquad F_2^{(k)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A^{(k)} \end{pmatrix}, \qquad F_3^{(k)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & A^{(k)} \end{pmatrix}, \tag{3.8}$$

where $A^{(k)} = \operatorname{diag}(1, \dots, 1, 0, \dots, 0)$ *is a* $(n-2) \times (n-2)$ *diagonal matrix with the first* k *elements are* 1 *and the others are zero on the diagonal,* $k = 0, 1 \dots, n-2$. *The corresponding left-symmetric algebras are*

$$A^1_{(3)} = \langle e_i,\ i = 1, \dots, n \mid e_1e_1 = 2e_1,\ e_1e_j = e_j,\ e_je_j = e_1,\ j = 2, \dots, n \rangle; \tag{3.9}$$

$$A^{(k),2}_{(3)} = \langle e_i,\ i = 1, \dots, n \mid e_1e_1 = 2e_1,\ e_1e_j = e_j,\ e_le_l = e_1,\ j = 2, \dots, n,\ l = 3, \dots, k+2 \rangle; \tag{3.10}$$

$$A^{(k),3}_{(3)} = \langle e_i,\ i = 1, \dots, n \mid e_1e_2 = e_1,\ e_2e_1 = 2e_1,\ e_2e_2 = e_2,\ e_2e_j = e_j,\ e_le_l = e_1,\ j = 3, \dots, n,\ l = 3, \dots, k+2 \rangle. \tag{3.11}$$

Proof. Let A be a left-symmetric algebra in the case (3) with dimension $n \geqslant 2$. Without losing generality, we can assume $c = e_1$. At first we consider the case $h(e_1, e_1) \neq 0$. Thus we can choose $e_2, \dots, e_n$ such that $\{e_1, \dots, e_n\}$ is a basis and $h(e_1, e_j) = 0$, $j = 2, \dots, n$. Set $h_{ij} = h(e_i, e_j)$. Therefore the product of A is given by

$$e_1 * e_1 = 2h_{11}e_1, \qquad e_1 * e_j = h_{11}e_j, \qquad e_j * e_1 = 0,$$
$$e_j * e_l = h_{jl}e_1, \quad j,l = 2, \dots, n.$$

Moreover, we can assume $h_{11} = 1$ by letting

$$e_1' = \frac{1}{h_{11}}e_1, \qquad e_j' = \frac{1}{\sqrt{h_{11}}}e_j, \quad j = 2, \dots, n.$$

Let $A_1 = \mathbf{C}e_1$ and A_2 be a subspace spanned by $e_2, \dots, e_n$, then from Lemma 3.5, we know the classification of above algebras is just given by the matrix F_1 and $F_2^{(k)}$, respectively, which corresponds to the left-symmetric algebra given by Eqs. (3.9) and (3.10), respectively.

Next assume $h(c,c) = h(e_1,e_1) = 0$. Since there exists an element $u \in A$ such that $h(u,c) \neq 0$, we can let $u = e_2$. Then we can choose $e_3,\dots,e_n$ such that $\{e_1,\dots,e_n\}$ is a basis and $h(e_1,e_j) = 0$, $j = 1,3,\dots,n$. Hence we have

$$e_1 * e_1 = 0, \qquad e_1 * e_2 = h_{12}e_1, \qquad e_2 * e_1 = 2h_{12}e_1,$$
$$e_2 * e_2 = h_{12}e_2 + h_{22}e_1, \qquad e_j * e_1 = e_1 * e_j = 0, \qquad e_2 * e_j = h_{12}e_j + h_{2j}e_1,$$
$$e_j * e_2 = h_{2j}e_1, \qquad e_j * e_l = h_{jl}e_1, \quad j,l = 3,\dots,n.$$

Let

$$e_1' = e_1, \qquad e_2' = \frac{1}{h_{12}}e_2 - \frac{h_{22}}{2h_{12}}e_1, \qquad e_j' = e_j - \frac{h_{2j}}{h_{12}^2}e_1, \quad j = 3,\dots,n.$$

Under the new basis, we have

$$e_1' * e_1' = 0, \qquad e_1' * e_2' = e_1', \qquad e_2' * e_1' = 2e_1', \qquad e_2' * e_2' = e_2',$$
$$e_j' * e_1' = e_1' * e_j' = 0, \qquad e_2' * e_j' = e_j', \qquad e_j' * e_2' = 0,$$
$$e_j' * e_l' = h_{jl}e_1', \quad j,l = 3,\dots,n.$$

Let A_1 be a subspace spanned by e_1, e_2 and A_2 be a subspace spanned by $e_3,\dots,e_n$, then from Lemma 3.5, we know the classification of above algebras is just given by the matrix $F_3^{(k)}$, which corresponds to the left-symmetric algebra given by Eq. (3.11). □

As the same as the proof of the case $A_{(3)}^{(k),3}$ in above proposition, we have

Proposition 3.8. *The classification of left-symmetric algebras in the case* (5) *with dimension $n \geqslant 2$ is given by the matrix $F_3^{(k)}$. The corresponding left-symmetric algebras is* ($k = 0,1,\dots,n-2$)

$$A_{(5)}^{(k)} = \langle e_i,\ i = 1,\dots,n \mid e_2e_1 = -e_1,\ e_2e_2 = e_2,\ e_je_2 = e_j,\ e_le_l = e_1,$$
$$3 \leqslant j \leqslant n,\ 3 \leqslant l \leqslant k+2\rangle. \tag{3.12}$$

Corollary 3.9. *Let A be a left-symmetric algebra in dimension $n \geqslant 2$ given in Theorem* 2.5*. If the bilinear function h is non-degenerate, then A is isomorphic to one of the following algebras*: $A_{(3)}^1$; $A_{(3)}^{(n-2),3}$; $A_{(5)}^{(n-2)}$.

Theorem 3.10. *When the dimension $n = 2$, the left-symmetric algebras given in Theorem* 2.5 *are not* (*mutually*) *isomorphic except for*

$$A_{(3)}^{(0),3} \sim A_{(7)}^{1/2}, \qquad A_{(5)}^{(0)} \sim A_{(4)}^{-1}. \tag{3.13}$$

Moreover, with the associative algebras given in Corollary 2.2 *together, they include all two-dimensional non-commutative left-symmetric algebras.*

Proof. Comparing the classification of two-dimensional left-symmetric algebras given in [1] or [8], the conclusion follows immediately. Notice that $A_{(3)}^{(0),3}$ is isomorphic to $A_{(7)}^{1/2}$ by $e_1 \to e_2, e_2 \to 2e_1$ and $A_{(5)}^{(0)}$ is isomorphic to $A_{(4)}^{-1}$ by $e_1 \to e_2$, $e_2 \to -e_1$, which the order of e_1, e_2 is changed respectively. □

Remark 3. Obviously, some commutative associative algebras such as the direct sum of two fields $\mathbf{C} \oplus \mathbf{C}$ are not included in above algebras. Moreover, we would like to point out that the above conclusion is not obvious since for a general algebra, the "linear" construction like in this paper has certain restriction conditions for the corresponding structure constants, which could not contain all (non-trivial) examples.

Corollary 3.11. *When $n > 2$, the left-symmetric algebras given in Theorem* 2.5 *and Corollary* 2.2 *are not mutually isomorphic.*

Proof. It is easy to see that when $n > 2$, $A_{(3)}^{(k),3}$ is not isomorphic to $A_{(7)}^{1/2}$ and $A_{(5)}^{(k)}$ is not isomorphic to $A_{(4)}^{-1}$. With the special roles of e_1, e_2 in the algebraic operation and similar to the classification of two-dimensional left-symmetric algebras in [1] or [8], the conclusion follows by a straightforward analysis. □

4. Further discussion

In this section, we discuss some properties of the algebras given in the previous sections and certain application in mathematical physics.

Theorem 4.1. *The left-symmetric algebras given by Eq.* (1.3) *are isomorphic to the left-symmetric algebra $A_{(3)}^{1}$. Moreover, it is a simple left-symmetric algebra, that is, it has no ideals except itself and zero.*

Proof. The first half of the above conclusion follows directly from the proof of Proposition 3.7 and the fact that for every $c \neq 0$, $h(c, c) \neq 0$ since h is the ordinary scalar product. The simplicity of the algebra is proved in [8]. □

Remark 4. The simple left-symmetric algebra $A_3^{(1)}$ is firstly constructed in [8]. In certain sense, our re-construction gives it an interesting (geometric) interpretation.

Due to Corollary 2.7, we have

Corollary 4.2. *The scalar product appearing in Eq.* (1.3) *is invariant under the sense of Eq.* (2.10).

Corollary 4.3. *The* (*generalized*) *Burgers Eq.* (1.4) *is just the following equation*:

$$u_t^1 = u_{xx}^1 + 4u^1u_x^1 + 2\sum_{j=2}^{n} u^j u_x^j;$$

$$u_t^k = u_{xx}^k + 2u^1u_x^k - u^1u^1u^k - u^ku^ku^k, \quad k = 2, \ldots, n. \tag{4.1}$$

Proof. Let C_{ij}^k be the structure constants. Hence Eq. (1.4) gives

$$u_t^i = u_{xx}^i + 2\sum_{j,k=1}^{n} C_{jk}^i u^j u_x^k + \sum_{k,j,l,m=1}^{n} \left(C_{ml}^i C_{kj}^l - C_{kj}^l C_{lm}^i\right) u^k u^j u^m.$$

For the left-symmetric algebra $A_{(3)}^1$, the non-zero structure constants are $C_{11}^1 = 2$, $C_{jj}^1 = 1$, $C_{1j}^j = 1$, $j = 2, \ldots, n$. Hence Eq. (4.1) follows. $\square$

Besides the simple left-symmetric algebra $A_{(3)}^1$, there are some other algebras appearing in Theorem 2.5 and Corollary 2.2 satisfying certain additional (interesting) conditions, which play important roles in the study of left-symmetric algebras.

Definition. Let A be left-symmetric algebra.

(1) If for every $x \in A$, R_x is nilpotent, then A is said to be transitive or complete. The transitivity corresponds to the completeness of the affine manifolds in geometry [7,13, 19].
(2) If for every $x \in A$, L_x is an interior derivation of the sub-adjacent Lie algebra of A, then A is said to be an interior derivation algebra. Such a structure corresponds to a flat left-invariant connection adapted to the interior automorphism structure of a Lie group [19].
(3) If for every $x, y \in A$, $R_x R_y = R_y R_x$, then A is said to be a Novikov algebra. It was introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus [6,11].
(4) If for every $x, y, z \in A$, the associator (x, y, z) is right-symmetric, that is, $(x, y, z) = (x, z, y)$, then A is said to be bi-symmetric. It is just the assosymmetric ring in the study of near-associative algebras [2,14].

By direct computation, we have

Proposition 4.4. *Let A be a left-symmetric constructed from Theorem* 2.5 *and Corollary* 2.2.

(1) *A is associative if and only if A is isomorphic to one of the following algebras*: *the associative algebras given in Corollary* 2.2; $A_{(1)}^{(k)}$; $A_{(2)}$; $A_{(7)}^1$.
(2) *A is transitive if and only if A is trivial or A is isomorphic to $A_{(1)}^{(k)}$ or $A_{(4)}^0$.*

(3) *Besides the commutative cases, A is an interior derivation algebra if and only if A is isomorphic to $A^0_{(4)}$. Moreover, $A^0_{(4)}$ is the unique left-symmetric interior derivation algebra on the Lie algebra given by Eq.* (2.4) (*cf.* [19]).
(4) *Besides the commutative cases, A is a Novikov algebra if and only if A is isomorphic to one of the following algebras*: *the associative algebra in the case* (2) *of Corollary* 2.2; $A^0_{(4)}$; $A_{(6)}$; $A^{\alpha}_{(7)}$; $A^{(k),3}_{(3)}$.
(5) *Besides the associative cases, A is bi-symmetric if and only if A isomorphic to $A^1_{(4)}$ or $A_{(6)}$.*

Corollary 4.5. *Let A be a left-symmetric constructed from Theorem* 2.5 *and Corollary* 2.2. *Then A with dimension $n > 2$ is associative* (*or transitive, or bi-symmetric, or a interior derivation algebra, or a Novikov algebra*) *if and only if A has such an additional structure when its dimension $n = 2$. Hence, the construction in this paper can be regarded as generalization* (*not extension*!) *of certain two-dimensional left-symmetric algebras.*

At the end of this paper, we give an application of the results in this paper to integrable systems. Recall that a linear transformation R on a Lie algebra $\mathcal{G}$ is called a classical r-matrix if R satisfies

$$\big[R(x), R(y)\big] = R\big(\big[R(x), y\big] + \big[x, R(y)\big]\big), \quad \forall x, y \in \mathcal{G}. \tag{4.2}$$

It corresponds to a solution of classical Yang–Baxter equation [12,18]. Moreover, if R satisfies the above equation, then

$$x * y = \big[R(x), y\big], \quad \forall x, y \in \mathcal{G}, \tag{4.3}$$

defines a left-symmetric algebra on $\mathcal{G}$. Two classical r-matrices are said to be equivalent if their corresponding left-symmetric algebras are isomorphic.

Corollary 4.6. *For the Lie algebra A given by Eq.* (2.4), *there is only one non-zero classical r-matrix under the sense of equivalence such that A is the sub-adjacent Lie algebra of the left-symmetric algebra given by Eq.* (4.3), *which R is given by*

$$R(e_1) = e_1, \qquad R(e_j) = 0, \quad j = 2, \ldots, n. \tag{4.4}$$

The corresponding left-symmetric algebra given by Eq. (4.3) *is isomorphic to $A^0_{(4)}$.*

Proof. Let R satisfy Eq. (4.2). Hence by Eq. (4.3), we know that for every $x \in A$, $L_x = \operatorname{ad} R(x)$, where ad is the adjoint operator of Lie algebra. Hence L_x is an interior derivation of the Lie algebra A. Thus the left-symmetric algebra defined by Eq. (4.3) is an interior derivation algebra. Therefore the conclusion follows from (3) in Proposition 4.4. □

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