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Powerful p-Groups. II. p-Adic Analytic Groups

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We apply our results from the first part [LM] to p-adic analytic pro-p groups, i.e., pro-p groups which are Lie groups over the field of p-adic numbers. For a systematic study of these groups see [La, B, S1]. Lazard [La] characterized the pro-p groups which are p-adic analytic as (in the terminology of [LM]) the finitely generated virtually powerful pro-p groups. Our detailed study of finite powerful groups enables us to get a new characterization:

THEOREM A. *A pro-p group is p-adic analytic if and only if the ranks of its open subgroups are bounded.*

To state the second theorem we need to present more notions, which seem to have an independent interest (see also [L2]).

Let G be a discrete (resp: pro-finite) group and $\mathbf{S}(G)$ the family of finite index (f.i.) subgroups of G (resp. open subgroups of G). $\mathbf{S}(G)$ is a directed system with respect to the partial order of anti-inclusion (i.e., $A \geqslant B$ iff $A \subseteq B$). The family $\mathbf{NS}(G)$ of the normal subgroups in $\mathbf{S}(G)$ is a cofinal subsystem of $\mathbf{S}(G)$. If f is a real-valued function on $\mathbf{S}(G)$ then $\{f(A)\}_{A \in \mathbf{S}(G)}$ is a net and $\{f(A)\}_{A \in \mathbf{NS}(G)}$ is a subnet so

$$\underline{L}_f(G) = \lim\inf\{f(A) | A \in \mathbf{S}(G)\},\ \overline{L}_f(G) = \lim\sup\{f(A) | A \in \mathbf{S}(G)\},$$

$$\underline{NL}_f(G) = \lim\inf\{f(A) | A \in \mathbf{NS}(G)\}$$

and

$$\overline{NL}_f(G) = \lim\sup\{f(A) | A \in \mathbf{NS}(G)\} \qquad \text{are well defined.}$$

In general these four numbers (possibly $+\infty$ and $-\infty$) can be all different from each other. Quite surprisingly, if G is a finitely generated pro-p group and d the function "number of generators" we obtain

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THEOREM B. *Let G be a finitely generated* (f.g.) *pro-p group. Then*:

(1) $r=\underline{NL}_d(G)=\overline{NL}_d(G)=\bar{L}_d(G)$ (*but "usually"* $\underline{L}_d(G)<r$).

(2) *G is analytic if and only if $r<\infty$, in which case r is equal to the dimension of G* (*as a p*-adic *manifold*).

It is interesting to note that this result is valid neither for discrete groups nor for general pro-finite groups.

The paper is organized as follows: In Section 1, we located all the results of [LM] needed in this paper to make this paper almost self-contained to those readers who are willing to believe in results on finite groups. In Section 2 we deduce Theorem A and discuss related results and problems. In Section 3 we prove Theorem B. Section 4 is devoted to a more careful analysis of $\underline{L}_d(G)$. In particular we prove that $\underline{L}_d(G)$ is equal to the number of generators of the Lie algebra of G. We conclude in Section 5 with some applications of the above results to discrete groups.

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NOTATIONS AND CONVENTIONS

If G is a (topological) group then $d(G)$ denotes its minimal number of (topological) generators. If G is a pro-p group then $v_1(G)=\langle X^p | X\in G\rangle$, i.e., the closed subgroup generated by the p-powers in G, $v_{(i)}(G)=v_1(v_{(i-1)}(G))$ for $i\geqslant 2$ and $v_i(G)=\langle X^{p^i} | X\in G\rangle$. $\Phi(G)=\Phi_1(G)=[G,G]\,v_1(G)$ is the Frattini subgroup of G and $\Phi_i(G)=\Phi_1(\Phi_{i-1}(G))$ for $i\geqslant 2$.

$\Gamma_{\hat{p}}$ denotes the pro-p completion of a discrete group Γ, $\mathbf{Z}_{\hat{p}}=\hat{\mathbf{Z}}_p$ the ring of p-adic integers, and $\mathbf{Q}_p$ the field of p-adic numbers.

1. FINITE POWERFUL p-GROUPS

Let p be a prime and G a finite p-group.

DEFINITION. G is called powerful if $v_1(G)\geqslant [G,G]$ when $p>2$ and $v_2(G)\geqslant [G,G]$ when $p=2$.

Let G be a finite powerful p-group. The following results are proved in [LM]:

(1.1). [LM, 1.12 and 4.1.12] $d(H)\leqslant d(G)$ for every $H\leqslant G$.

(1.2) [LM, 1.13 and 4.1.13] If N is a normal subgroup of a p-group M contained in $\Phi_{r+1}(M)$ and $d(N)\leqslant r$ then N is powerful.

(1.3) [LM, 3.2 and 4.1.12] If H is a maximal subgroup of G then $|H: v_1(H)| \leqslant p^{d(G)}$.

(1.4). [LM, 3.1 and 4.3.1] If every subgroup of a p-group M is powerful then M is a modular group and in particular it is meta-abelian.

(1.5) [LM, 1.2, 1.3, 4.1.2 and 4.1.3] For every i, $v_{(i)}(G) = v_i(G)$ and it is a powerful group.

2. A Criterion for Analycity

In this section we apply the results in Section 1 to p-adic analytic pro-p groups. Recall that a pro-p group is a group which is an inverse limit of finite p-groups or equivalently it is a compact, totally disconnected Hausdorf group in which the index of every open subgroup is a power of p. Note that every open subgroup is closed and a closed subgroup is open iff it has finite index (f.i.). For more details about pro-finite and pro-p groups the reader is referred to [S2] and [R]. A pro-p group is called analytic (or p-adic Lie group) if it has the structure of a manifold over $\mathbf{Q}_p$ and the group operations are analytic with respect to this structure. For more information see [S1, La, B].

While considering a pro-p group G (or pro-finite groups in general) we shall be interested only in closed subgroups and continuous homomorphisms. So by the commutator subgroup $G' = [G, G]$ we mean the closed commutator subgroup, $v_1(G)$ means the closed subgroup generated by the p-powers, $d(G)$ denotes the minimal number of topological generators, etc. Recall that for a pro-p group $d(G) = d(G/[G, G]\, v_1(G))$ because $[G, G]\, v_1(G)$ is equal to the Frattini subgroup $\Phi(G)$ (cf. [G]).

All the definitions and results of Section 1 are actually valid for pro-p groups. Thus, a powerful pro-p group is a pro-p group for which $v_1(G) \geqslant G'$ (resp. $v_2(G) \geqslant G'$ if $p = 2$). Standard limit arguments show that G is powerful if and only if it is an inverse limit of powerful finite p-groups.

In analogy with the Hilbert fifth problem for real Lie groups, Lazard characterized the pro-p groups which have analytic structure (if such a structure exists it is unique!). In our notions his theorem has the following form.

THEOREM 2.1. (*Lazard* [La, III, 3.4.3]). *A finitely generated pro-p group is analytic if and only if it has a powerful subgroup of finite index.*

We want to give another characterization of analytic pro-p groups. We begin with

PROPOSITION 2.2. *Let G be a finitely generated pro-finite group. Then*

(a) $d(G) = \sup\{d(S) | S$ *is a finite quotient of* $G\}$.

(b) *The following conditions are equivalent*:

(i) $d(H) \leqslant n$ *for every open* (f.i.) *subgroup* H *of* G.

(ii) $d(H) \leqslant n$ *for every closed subgroup* H *of* G.

Proof. (a) Clearly $d(G)$ is greater or equal than this supremum. So assume that every finite quotient S of G is generated by k elements. Let $A_S \subseteq S^k = S \times S \times \cdots \times S$ be the set of all the k-tuples of elements of S which generate S. $A_S \neq \phi$ by assumption and the family $\{A_S\}$ forms an inverse system of finite sets. The inverse limit is, therefore, not empty. If $(y_1, ..., y_k) \in \varprojlim_S A_S$ then $(y_1, ..., y_k) \in G^k$. The images of $y_1, ..., y_k$ in every finite quotient S, generate S, so the abstract subgroup generated by $y_1, ..., y_k$ is dense in G and $d(G) \leqslant k$.

(b) We only have to prove (i) $\Rightarrow$ (ii); by (a) it is sufficient to show that if S is a finite quotient of some closed subgroup K of G then it is a quotient of some open subgroup. So let $N \lhd K$ s.t. $K/N \simeq S$. Then there exists (cf. [R, Proposition 3.1]) an open normal subgroup L of G s.t. $L \cap K \subseteq N$. Thus $LK/L \simeq K/K \cap L \twoheadrightarrow K/N \simeq S$, i.e., S is a quotient of the open subgroup LK.

THEOREM 2.3. *A pro-p group G is analytic if and only if there exists* $M \in \mathbf{N}$ s.t. $d(H) \leqslant M$ *for every finite index subgroup* H *of* G.

Proof. Assume G is analytic, then by Theorem 2.1, G contains a powerful subgroup L of index p^l. By (1.1) $d(H) \leqslant d(L)$ for every subgroup H of L, and so for every subgroup H of G, $d(H) \leqslant d(L) + l$.

Now assume that there exists such an M. Take $N = \Phi_{M+1}(G)$, then by (1.2) N is powerful and so G is analytic again by Theorem 2.1.

As an immediate corollary we obtain the following result of Serre[La, III, 3.4.5].

COROLLARY 2.4. *If* $1 \to H \to G \to K \to 1$ *is an exact sequence of pro-p groups, then G is analytic* iff *H and K are such.*

Theorem 2.3 and Proposition 2.2 show that every (closed) subgroup of an analytic group is finitely generated.

PROPOSITION 2.5. *For a pro-p group G the following two conditions are equivalent*:

(i) *Every* (*closed*) *subgroup of G is finitely generated.*

(ii) *G satisfies the maximum condition on* (*closed*) *subgroups.*

Proof. (ii) $\Rightarrow$ (i) is standard, as for discrete groups.

(i) $\Rightarrow$ (ii) Let $\{K_\alpha\}$ be an increasing sequence of subgroups, $K = \bigcup K_\alpha$ and $H = \bar{K}$. H is finitely generated so $\tilde{H} = H/\Phi(H)$ is finite. If $\tilde{K}_\alpha = \mathrm{Im}(K_\alpha \to H/\Phi(H))$ then $\bigcup \tilde{K}_\alpha = \tilde{H}$ hence there exists α_0 such that $\tilde{K}_{\alpha_0} = \tilde{H}$, i.e., $H = K_{\alpha_0}\Phi(H)$. This implies $K_{\alpha_0} = H$ (cf. [G]).

So analytic pro-p groups satisfy the maximum condition. This, actually, has been proved by Lazard [La, III, 3.2.3] using the fact that $\mathrm{gr}\, G$ is noetherian.

PROBLEM. Is a pro-p group with the maximum condition on closed subgroups analytic?

Remarks. (A) It is interesting to observe that contrary to Proposition 2.5, a pro-finite group all of whose subgroups are finitely generated does not necessarily satify the maximum condition: The group $\hat{\mathbf{Z}} = \prod_p \hat{\mathbf{Z}}_p$, the pro-finite completion of $\mathbf{Z}$, is pro-cyclic (i.e., generated by one element) and every subgroup is pro-cyclic. Still being an infinite direct product it does not satify the maximum condition.

(B) Theorem 2.3 shows that the function d defined on the f.i. subgroups of a pro-p group determines the analycity of G. This is a similar phenomenon to [L1, Proposition 4.1] where d determines the freeness of G; G is free iff $d(H) = (G: H)(d(G) - 1) + 1$ for every f.i. subgroup H of G. In what follows we will show that in case G is analytic, its dimension as a manifold can also be recovered from d.

3. THE LIMITS OF d

Let G be a group $\mathbf{S}(G)$ the set of all f.i. subgroups of G. $\mathbf{S}(G)$ is a directed system if we define $H_1 \geqslant H_2$ when $H_1 \subseteq H_2$. If f is a real-valued function with domain $\mathbf{S}(G)$ then $\{f(H)\}_{H \in \mathbf{S}(G)}$ is a *net* (in the sense of [K, p. 65]) so we can talk about the limits of it: lim inf, lim sup, and lim if it exists. Denote $\underline{L}_f(G) = \liminf \{f(H)\}_{H \in \mathbf{S}(G)}$, $\bar{L}_f(G) = \limsup \{f(H)\}_{H \in \mathbf{S}(G)}$, and if $\underline{L}_f(G) = \bar{L}_f(G)$ call this number $L_f(G)$.

The subset $\mathbf{NS}(G)$ of $\mathbf{S}(G)$ of all the normal f.i. subgroups is also a directed system and $\{f(N)\}_{N \in \mathbf{NS}(G)}$ is a subnet of $\{f(H)\}_{H \in \mathbf{S}(G)}$. This time the limits will be denoted $\underline{NL}_f(G)$, $\overline{NL}_f(G)$ and $NL_f(G)$ if they are equal.

If K is a f.i. subgroup of G, $\mathbf{S}(K)$ is confinal in $\mathbf{S}(G)$. $\mathbf{NS}(G)$ is also a cofinal subset of $\mathbf{S}(G)$ but $\mathbf{NS}(K)$ is, in general, not even a subset of $\mathbf{NS}(G)$. Still $\mathbf{NS}(K) \cap \mathbf{NS}(G)$ is a confinal subset of $\mathbf{NS}(K)$. We can easily deduce

LEMMA 3.1. *Let G be a group, K a* f.i. *subgroup and f a function defined on* $\mathbf{S}(G)$. *Then*:

(a) $\underline{NL}_f(G) \geqslant \underline{L}_f(G)$, $\overline{NL}_f(G) \leqslant \bar{L}_f(G)$.

(b) $\underline{L}_f(K) = \underline{L}_f(G)$, $\bar{L}_f(K) = \bar{L}_f(G)$,

(c) $\underline{NL}_f(K) \leqslant \underline{NL}_f(G)$, $\overline{NL}_f(K) \geqslant \overline{NL}_f(G)$.

Let us restrict ourselves now to the case that G is a finitely generated pro-p group (so $\mathbf{S}(G)$ is the set of open subgroups) and $f(H) = d(H) =$ the number of (topological) generators of H. A f.i. subgroup of G is also f.g. (cf. [L1]) so $d: \mathbf{S}(G) \to \mathbf{N}$.

THEOREM 3.2. *Let G be a finitely generated pro-p group. Then*:

(a) $\underline{NL}_d(G) = \overline{NL}_d(G) = \bar{L}_d(G)$ *in particular* $r = NL_d(G)$ *exists.*

(b) *G is analytic if and only if $r < \infty$, in which case* $r = \dim G$.

Proof. Let $r = \underline{NL}_d(G)$. Clearly $\underline{NL}_d(G) \leqslant \overline{NL}_d(G) \leqslant \bar{L}_d(G)$. If $r = \infty$ then (a) is trivial and G is not analytic because of Theorem 2.3. So assume $r < \infty$, since $\Phi_{r+1}(G)$ is of finite index in G, there exists a f.i. normal subgroup N of G s.t. $N \subseteq \Phi_{r+1}(G)$ and $d(N) = r$. By (1.2) this N is powerful. So G is analytic by Theorem 2.1.

Among all the f.i. normal powerful subgroups K of G choose one with $d(K)$ minimal, say $k = d(K)$. Clearly, $k \leqslant r = d(N)$. By (1.1) $d(H) \leqslant k$ for every f.i. subgroup H of K. Hence by Lemma 3.1.:

$$r = \underline{NL}_d(G) \leqslant \overline{NL}_d(G) \leqslant \bar{L}_d(G) = \bar{L}_d(K) \leqslant k \leqslant r.$$

Thus $r = k = \underline{NL}_d(G) = \overline{NL}_d(G) = \bar{L}_d(G)$.

It remains to show that $r = \dim G$. Note that $\dim G = \dim K$ so we will show that $r = \dim K$. Let $K_i = v_{(i)}(K)$ for $i \geqslant 1$. By (1.5) and (1.1) K_i is powerful, equals to $v_i(K)$ and $d(K_i) \leqslant d(K) \leqslant r$. By the way we have chosen K, we may conclude that $d(K_i) = r$ for every i. Thus $[K_i: v_1(K_i)] = p^r$. On the other hand $v_1(K_i) = v_1(v_{(i)}(K)) = v_{(i+1)}(K) = K_{i+1}$ so $[K_i: K_{i+1}] = p^r$ and by induction $[K: K_i] = p^{ri}$. By [La, III, 3.1.8] $\dim K = \lim_{i \to \infty} (1/i) \log_p [K: v_i(K)]$ so $\dim K = \lim_{i \to \infty} (1/i) \log_p [K: K_i] = r$. Q.E.D.

COROLLARY 3.3. *If H is a* f.i. *subgroup of a* f.g. *pro-p group G then* $NL_d(H) = NL_d(G)$.

Proof. H is analytic iff G is, and if so they have the same dimension.

4. SOME RESULTS ON $\underline{L}_d(G)$

Theorem 3.2 shows that three of the four limits we have defined are equal in case G is a f.g. pro-p group and $f = d$. In this section we will consider the fourth one $\underline{L}_d(G)$. We show that in case the other limits are finite (i.e, G is analytic $\underline{L}_d(G)$ is "rarely" equal to them and in fact it is equal to the number of generators of the Lie algebra of G. We leave open the following problem:

PROBLEM. Does there exist a f.g. pro-p group G for which $\bar{L}_d(G) = \infty$ while $\underline{L}_d(G) < \infty$?

PROPOSITION 4.1. *If G is an analytic pro-p group then $\underline{L}_d(G) = d(L(G))$ where $L(G)$ is the Lie algebra of G and $d(L(G))$ denotes its number of generators.*

Proof. Let $V \subseteq L(G)$ be a neighbourhood of 0 in $L(G)$ for which $\exp\colon V \to \exp(V) = U \subseteq G$ is defined and is an isomorphism onto its image and for which the Hausdorf series $h(X, Y)$ converges on $V \times V$, $h(X, Y) \in V$ for every $X, Y \in V$ and $\exp X \cdot \exp Y = \exp h(X, Y)$ for every $X, Y \in V$ (see [B, chap. II, Sect. 8; Chap. III, Sect. 7]).

(I) $l = \underline{L}_d \geqslant d(L(G)) = r$: U is an open neighbourhood of 1 in G so it contains an open subgroup K generated by l elements, say $g_1, \ldots, g_l$. Let $X_1, \ldots, X_l$ be elements of V such $g_i = \exp X_i$ $i = 1, \ldots, l$.

Claim. $\{X_1, \ldots, X_l\}$ generate $L(G)$.

Let L_1 be the Lie subalgebra of $L(G)$ generated by $X_1, \ldots, X_l$. Being a subspace of a finite dimensional vector space, L_1 is closed. We will show that L_1 contains a basis of $L(G)$.

Let $F = \exp^{-1}(K)$. So F is an open neighbourhood of 0 in $L(G)$ hence it contains a basis of $L(G)$. Let X be an element of F. Then $g = \exp X$ is in K and thus $g = \lim w_j(g_1, \ldots, g_l)$ where $w_j(-)$ is a word in $g_1, \ldots, g_l$. If $W_j = \exp^{-1}(w_j(g_1, \ldots, g_l))$ then $X = \lim W_j$. It is, therefore, sufficient to prove that $W = W_j$ is in L_1.

$\exp W$ is a word in $g_1, \ldots, g_l$. All the g_i's, as well as their products are in $K \subseteq U$ and so these products are translated via $\exp^{-1}$ to applying the Hausdorf series h on V. This shows that W is in the closed subalgebra generated by $X_1, \ldots, X_l$, i.e., $W \in L_1$.

(II) $r = d(L(G)) \geqslant \underline{L}_d(G) = l$: Let $X_1, \ldots, X_r$ be a set of generators of $L(G)$, and let K be an open subgroup of G, $R = K \cap U$ contains an open subgroup K_1 of G. Let $F_1 = \exp^{-1}(K_1)$, F_1 is an open neighbourhood of 0 in $L(G)$. Let α be a scalar such that $Y_i = \alpha X_i$ is in F_1 for every $i = 1, \ldots, r$.

Claim. $\{g_i = \exp Y_i \mid i = 1, \ldots, r\}$ generates an open subgroup of K_1.

Let M be the closed subgroup generated by $g_1, \ldots, g_r$, and L_1 its Lie subalgebra of L. So $L_1 = \{X \in L \mid \exp(tX) \in M$ for every t is some neighbourhood of 0 in $\hat{Z}_p\}$. But $\exp(tX_i) = (\exp(X_i))^t = g_i^t$, hence $X_i \in L_1$. This shows that $L_1 = L$. This proves that M is an open subgroup of G and the Proposition is now proved.

PROPOSITION 4.2. *Let G be a* f.g. *pro-p group. If $\underline{L}_d(G) = \bar{L}_d(G) < \infty$ then G is meta-abelian by finite.*

We will give two proofs for this proposition; the first is valid for $p \neq 2$ and the other for general p:

Proof A. As $\bar{L}_d(G) < \infty$ it follows from Theorem 3.2 that G is analytic of dimension $r = \bar{L}_d(G)$. Since $\underline{L}_d(G) = \bar{L}_d(G) = r$ there exists a powerful subgroup K of finite index on G such that $d(H) = r$ for every f.i. subgroup H of K. In particular if H is a maximal subgroup of K then by (1.3) $p^r = |H/\Phi(H)| \leqslant |H/v_1(H)| \leqslant |K/v_1(K)| = p^r$. This shows that H is also powerful. Replace K by H and apply the same argument to deduce that every open subgroup of K is powerful, so K is a modular pro-p group (1.4), i.e., an inverse limit of finite modular p-groups. So G is meta-abelian by finite.

Proof B. Theorem 3.2 and Proposition 4.1 show that $n = \dim L(G) = d(L(G))$. The following lemma shows that this implies that $L(G)$ is meta-abelian. Thus, by the correspondence between Lie groups and their Lie algebras, G is meta-abelian by finite.

LEMMA 4.3. *Let L be a finite dimensional Lie algebra over a field F such that* $\dim L = d(L) = n$. *Then L is meta-abelian* (*or abelian*).

Proof. If $X, Y \in L$ then $Z = [X, Y]$ lies in the subspace spanned by X and Y. Otherwise, we can complete X, Y, Z to a basis of n elements X, Y, Z, $T_4, ..., T_n$ while X, Y, $T_4, ..., T_n$ is a set of $n-1$ generators for L as a Lie algebra.

Now, assume L is not abelian and let $0 \neq Z = [X, Y] = aX + bY$. Then $[X, Z] = bZ$ and $[Y, Z] = -aZ$. Without loss of generality we can assume $a \neq 0$.

If T is independent of Y and Z, then $[T + Y, Z] = \alpha(T + Y) + \beta Z = \alpha T + \alpha Y + \beta Z$ but also: $[T + Y, Z] = [T, Z] + [Y, Z] = \gamma T + \delta Z - aZ$. Thus $\alpha = 0$ and hence $[T, Z] = (\beta + a)Z$. So $[A, Z] = tZ$ for every $A \in L$. If W is another commutator in L, independent of Z, then the same is true for W. Hence $[Z, W] \in \operatorname{span}(Z) \cap \operatorname{span}(W) = \{0\}$. This show that $[L, L]$ is abelian.

EXAMPLE. $\underline{L}_d(SL_2(\hat{\mathbf{Z}}_p)) = 2$.

This can be deduced either from Proposition 4.1 since the Lie algebra $sl_2(\mathbf{Q}_p)$ is generated by two elements, or from Proposition 4.2 since $\underline{L}_d(SL_2(\hat{\mathbf{Z}}_p)) < \bar{L}_d(SL_2(\hat{\mathbf{Z}}_p)) = \dim(SL_2(\hat{\mathbf{Z}}_p)) = 3$, but clearly $\underline{L}_d(SL_2(\hat{\mathbf{Z}}_p)) > 1$ since $SL_2(\hat{\mathbf{Z}}_p)$ is not virtually cyclic.

5. Applications to Discrete Groups

In this section we will show how our results can be used to give some insight into problems in discrete (finite or infinite) group theory.

Following Humphreys and McCutcheon [HM], let χ_n be the class of solvable groups in which every subgroup is generated by n elements. Groups in χ_n are polycyclic. Let χ_n^p (resp. χ_n^t) be the family of finite p-groups (resp. torsion-free groups) in χ_n. The following dichotomy is proved in [HM]: while the derived length of the groups in χ_n^p is not bounded (for $n \geqslant 3$ and $p \geqslant 3$) the derived length of groups in χ_n^t is bounded. From our point of view this dichotomy is clear as it expresses the difference between semi-simple and solvable analytic group, as the following two proofs will show.

Proposition 5.1. *For every prime p, there is no bound on the derived length of groups in* χ_3^p.

Proof. The group $SL_2(\hat{\mathbf{Z}}_p)$ is an analytic pro-p group of dimension three, it contains, therefore, an open powerful pro-p subgroup P of rank 3. So P is a pro-p group whose quotients are powerful finite p-groups of rank 3 and so are all in χ_3^p. But their derived length is not bounded since P, an open subgroup of a simple p-adic Lie group, is not a solvable group.

Proposition 5.2. *There is a function* $f\colon \mathbf{N} \to \mathbf{N}$ *such that the derived length of every group in* χ_n^t *is bounded by* $f(n)$.

Proof. Let G be a group in χ_n^t. Then G is residually finite p-groups for some prime p (in fact for all of them; see [S, p. 19]). Let $G_{\hat{p}}$ be its pro-p completion. As G is in χ_n, $G_{\hat{p}}$ satisfies the condition of Theorem 2.3 and so it is analytic. In fact by Theorem 3.2 it is analytic of dimension less or equal to n. Let Ad be the adjoint representation of $G_{\hat{p}}$ on its Lie algebra. So $\mathrm{Ad}(G_{\hat{p}})$, and in particular $\mathrm{Ad}(G)$ is a solvable linear group of degree n. Thus its derived length is bounded by a function of n, say $l(n)$. Ker Ad is the set of all elements whose centralizers are of finite index. Thus in Ker Ad every element has only finitely many conjugates. It is well known that a torsion-free group with this property is abelian. We can, therefore, take $f(n) = l(n) + 1$.

Remark. Our method slightly improves the results in [HM]. In Proposition 5.1 we prove the result for every prime and in Proposition 5.2, we can deduce a somewhat better bound.

In another paper McCutcheon [M] proves that the Hirsch rank of the polycyclic groups in χ_n is also bounded. This can also be deduced by our methods but we will not go into details. Instead we shall explain how the following result of Hall [H] is obtained by our method and in fact for every p (Hall proved it for $p \neq 2$).

PROPOSITION 5.3. *Every free group is residually tricyclic finite p-groups. (A tricyclic group is a group which can be written as a product of three cyclic subgroups).*

Proof. Let P be the group used in the proof of Proposition 5.1. It is a powerful rank-3 pro-p group and it contains discrete finitely generated free groups. Rank-3 powerful p-groups are tricycle [LM] so f.g. free groups are residually tricycle p-groups. An arbitrary free group is residually f.g. free group, so the proposition is proved.

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