Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 234 (2007) 582–606
Journal of Differential Equations
www.elsevier.com/locate/jde

The Dirichlet problem for supercritical biharmonic equations with power-type nonlinearity ☆

Alberto Ferrero [a], Hans-Christoph Grunau [b,*]

[a] Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
[b] Fakultät für Mathematik, Otto-von-Guericke-Universität, Postfach 4120, 39016 Magdeburg, Germany

Received 7 September 2006
Available online 8 December 2006

Abstract

For a semilinear biharmonic Dirichlet problem in the ball with supercritical power-type nonlinearity, we study existence/nonexistence, regularity and stability of radial positive minimal solutions. Moreover, qualitative properties, and in particular the precise asymptotic behaviour near $x = 0$ for (possibly existing) singular radial solutions, are deduced. Dynamical systems arguments and a suitable Lyapunov (energy) function are employed.

1. Introduction

Many papers have studied second order elliptic boundary value problems with supercritical growth. We only mention the work by Brezis, Cazenave, Martel, Ramiandrisoa and Brezis, Vazquez [5,6], where the role of singular solutions, the change in the bifurcation diagrams in dependence on the space dimension and the nonlinearity as well as many other interesting features were highlighted. In these works, also many references to previous related important work can be found.

According to [14, Section 4.2(c)], it is an important task to gain also a deeper understanding for related higher order problems. Due to the lack of a general maximum principle and many other strong tools typical for second order equations, up to now only relatively limited results have been available for this case.

☆ Financial support by the Vigoni programme of CRUI (Rome) and DAAD (Bonn) is gratefully acknowledged.
* Corresponding author.
E-mail address: hans-christoph.grunau@mathematik.uni-magdeburg.de (H.-Ch. Grunau).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.11.007

In [2] biharmonic Dirichlet problems with exponential nonlinearities could be studied to some extent, while important questions concerning e.g. the existence of singular solutions are in general still open. A first investigation of biharmonic Dirichlet problems with power-type nonlinearities was done in [3]. It turned out, however, that the power case is technically much more involved than the exponential case, where advantage could be taken of the extreme convexity properties of the nonlinearity. It is the goal of the present paper to develop tools which enable us to prove analogues for the results in [2] also for power-type nonlinearities. Moreover, we improve on some of the achievements for the exponential case.

Let us now describe the scope of the present paper in some detail. We study the Dirichlet problem

$$\begin{cases} \Delta^2 u = \lambda(1+u)^p & \text{in } B, \\ u > 0 & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where $B \subset \mathbb{R}^n$ is the unit ball, $\lambda > 0$ is an eigenvalue parameter, $n \geqslant 5$ and $p > \frac{n+4}{n-4}$. The subcritical case $p < \frac{n+4}{n-4}$ is by now "folklore," where existence and multiplicity results are easily established by means of variational methods. For the critical case $p = \frac{n+4}{n-4}$ (under Navier boundary conditions), we refer to [3].

According to related work on second order equations and on the biharmonic Dirichlet problem with exponential nonlinearity, we address questions concerning existence/nonexistence, smoothness and stability of positive minimal solutions, characterization of radial singular solutions in terms of critical points of associated dynamical systems, and some further qualitative properties of (possibly existing) singular solutions. In the present work, a singular solution is always understood to be singular at the origin $x = 0$.

The first of our results, a precise formulation of which is given in the next section, concerns existence of "minimal" solutions. It is shown that there exists a limiting parameter $\lambda^* = \lambda^*(n, p)$ such that one has existence of stable *regular* minimal solutions to (1) for $\lambda \in (0, \lambda^*)$, while for $\lambda > \lambda^*$, not even singular solutions exist. In second order problems this immediate switch from existence of regular to nonexistence even of singular solutions is established by using suitable functions of possibly existing singular solutions as bounded supersolutions, see [5]. Such techniques seem to fail completely for fourth and higher order problems. Here, we employ dynamical systems arguments, and this is one reason why we have to formulate our results in the ball. A second reason for this restriction is the need to apply comparison principles which are known to fail in general domains but to hold in the ball. As for uniqueness in (1) for $\lambda > 0$ close to 0, see [20], where techniques of [17,18] are substantially generalized and developed.

Further results concern qualitative properties of radial singular solutions. The precise blow-up rate $\sim C|x|^{-4/(p-1)}$ at $x = 0$ is determined and an explicit estimate from below is deduced. For this purpose, in Section 4, we transform the differential equation in (1) into an autonomous system of ordinary differential equations and apply subtle energy estimates. This technique has proved to be very powerful for studying the precise asymptotic behaviour of entire solutions in $\mathbb{R}^n$ in [9]. Moreover, a characterization of singular (respectively regular) radial solutions to (1) in terms of the corresponding dynamical system is given. This system is shown to have two critical points, the unstable manifolds of which are related to singular (respectively regular) radial solutions.

As in the case of an exponential nonlinearity, we cannot yet provide an analytical proof for the existence of singular solutions. In that case, see [2], a computer assisted proof was given for dimensions $n = 5, \dots, 16$. We are convinced that for selected values of n and p, the same should also work in the present situation. However, since we have to consider not only countably many values of n but also uncountably many of p, a computer assisted proof may not be suitable here and we formulate the existence of singular solutions to (1) with a suitable parameter λ_s as an important and presumably difficult open problem. Once the existence of singular solutions has been established, further interesting questions concerning the singular parameter λ_s and the extremal parameter λ^* arise: in which dimensions n and for which exponents p does one have $\lambda_s = \lambda^*$ so that one has a singular extremal solution?

"Large" solutions of the differential equation $\Delta^2 u = \lambda(1+u)^p$ are studied in a recent work of Diaz, Lazzo, Schmidt [8]. The focus of their work, however, is different from ours, since they classify those solutions of the differential equation, which converge to ∞ at ∂B.

The paper is organized as follows: in the next section, a precise formulation of our results is given. In Section 3, the partial differential equation in (1) applied to radial functions is transformed into an autonomous system of ordinary differential equations. A phase space analysis is performed in Section 4 in order to characterize singular and regular radial solutions to (1). Section 5 is devoted to proving existence of regular minimal solutions, the stability of which is studied in Section 7. A refined use of Lyapunov or energy functions in Section 6 will yield precise information of qualitative properties of radial singular solutions.

2. Main results

Definition 1. We say that $u \in L^p(B)$ is a *solution* of (1) if $u \geqslant 0$ and if for all $\varphi \in C^4(\overline{B})$ with $\varphi|\partial B = |\nabla\varphi||\partial B = 0$ one has

$$\int_B u\Delta^2\varphi\,dx = \lambda \int_B (1+u)^p \varphi\,dx.$$

We call u *singular* if $u \notin L^\infty(B)$, and *regular* if $u \in L^\infty(B)$.

A radial singular solution $u = u(r)$ of (1) is called *weakly singular* if $\lim_{r\to 0} r^{4/(p-1)}u(r) \in [0,\infty]$ exists.

Weakly singular solutions display a somehow specified asymptotic behaviour at the origin. Below we shall prove that any radial singular solution is weakly singular. Note that by standard regularity theory for the biharmonic operator (see [1]), any regular solution u of (1) satisfies $u \in C^\infty(\overline{B})$. Note also that by the positivity preserving property of Δ^2 in the ball [4] any solution of (1) is positive, see also [2, Lemmas 16 and 18] for a generalized statement. This property is known to fail in general domains. For this reason, we restrict ourselves to balls also in Theorems 1 and 2. Hence, the sub- and supersolution method applies as well as monotone iterative procedures.

We also need the notion of minimal solution:

Definition 2. We call a solution u of (1) *minimal* if $u \leqslant v$ a.e. in B for any further solution v of (1).

In order to state our results, we denote by $\lambda_1 > 0$ the first eigenvalue for the biharmonic operator with Dirichlet boundary conditions

$$\begin{cases} \Delta^2\varphi_1 = \lambda_1\varphi_1 & \text{in } B, \\ \varphi_1 = \dfrac{\partial\varphi_1}{\partial\mathbf{n}} = 0 & \text{on } \partial B. \end{cases} \tag{2}$$

It is known from the positivity preserving property and Jentzsch's (or Krein–Rutman's) theorem that λ_1 is isolated and simple and that the corresponding eigenfunctions φ_1 do not change sign.

Define

$$\Lambda := \{\lambda > 0\colon \text{ (1) admits a solution}\}; \qquad \lambda^* := \sup\Lambda.$$

It is well known that Λ is a bounded interval (see [2,3]). Our first main result states that on the open interior of this interval, i.e. for any $0 < \lambda < \lambda^* = \sup\Lambda < \infty$, problem (1) admits a regular minimal solution u. Similarly as in second order problems, where much stronger tools are available, we are able to prove the immediate switch from existence of a regular minimal solution to nonexistence even of singular solutions. For biharmonic supercritical equations this was proved earlier [2, Theorem 1] only for an exponential nonlinearity, where advantage could be taken of the extreme convexity of the exponential. Here, much more refined arguments are needed.

Theorem 1. *We have*:

(i) *For* $\lambda \in (0, \lambda^*)$ *problem* (1) *admits a minimal regular solution. This solution is radially symmetric and strictly decreasing in* $r = |x|$.
(ii) *For* $\lambda = \lambda^*$ *problem* (1) *admits at least one not necessarily bounded solution.*
(iii) *For* $\lambda > \lambda^*$ *problem* (1) *admits no* (*not even singular*) *solutions.*

Moreover

$$\lambda^* \in \left[K_0, \frac{\lambda_1}{p}\right), \tag{3}$$

where

$$K_0 = \frac{8}{(p-1)^4}\big((n-2)(n-4)(p-1)^3 + 2(n^2-10n+20)(p-1)^2 - 16(n-4)(p-1) + 32\big). \tag{4}$$

For the proof see Section 5.

The regular minimal solution is stable:

Theorem 2. *Assume* $\lambda \in (0, \lambda^*)$. *Let* u_λ *be the corresponding minimal solution of* (1). *Denote by* $\mu_1(\lambda)$ *the first eigenvalue of the linearized operator* $\Delta^2 - \lambda p(1+u_\lambda)^{p-1}$. *Then,* $\mu_1(\lambda) > 0$.

We remark that, obviously, this theorem also holds in the subcritical and critical range, i.e. for any $p > 1$. For a proof, see Section 7.

The notion of *weakly singular* radial solution is motivated from a somehow technical point of view, because within this class, by definition, the asymptotic behaviour at the origin is in some sense specified. Consequently, here and also in the previous work [2], a number of results can be proven much more easily in this restricted class of singular solutions.

However, our next main result, which is proved using a suitable energy functional in Section 6, states that:

Theorem 3. *Any radial singular solution of* (1) *is weakly singular.*

That means that in what follows we need no longer distinguish between weakly singular and general radial singular solutions. The corresponding question for the Dirichlet problem with exponential nonlinearity had to be left open in [2].

In Section 3 below we shall transform the differential equation in (1) for radial functions into an autonomous system (17) having precisely two critical points O and P. With the help of these critical points we can give a precise characterization of regular and singular solutions of (1).

Theorem 4. *Let $u = u(r)$ be a radial solution of* (1) *and let*

$$W(t) = \big(w_1(t), w_2(t), w_3(t), w_4(t)\big)$$

be the corresponding trajectory relative to (17)*. Then*:

(i) *u is regular* (*i.e. $u \in L^\infty(B)$*) *if and only if*

$$\lim_{t\to-\infty} W(t) = O.$$

(ii) *u is singular if and only if*

$$\lim_{t\to-\infty} W(t) = P.$$

This result is proved in Section 4.

Again, by means of energy considerations, in Section 6 we can give an explicit estimate from below for radial singular solutions and the corresponding singular parameter:

Theorem 5. *Assume that u_s is a singular radial solution of* (1) *with parameter λ_s. Then, $\lambda_s > K_0$ and*

$$u_s(x) > \left(\frac{K_0}{\lambda_s}\right)^{1/(p-1)} |x|^{-4/(p-1)} - 1. \tag{5}$$

In particular, any radial solution to (1) *for $\lambda \leqslant K_0$ is regular.*

3. An autonomous system

In radial coordinates $r = |x|$, the differential equation in (1) reads

$$u^{(4)}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)(n-3)}{r^3}u'(r) = \lambda\bigl(1+u(r)\bigr)^p$$
$$r \in [0,1]. \tag{6}$$

In order to make the results and techniques from [9] accessible to our present work, as there we put first

$$U(x) := (u+1)(x/\sqrt[4]{\lambda}) \quad \bigl(x \in B_{\sqrt[4]{\lambda}}(0)\bigr), \qquad u(x) = U(\sqrt[4]{\lambda}x) - 1 \quad (x \in B).$$

For $x \in B_{\sqrt[4]{\lambda}}(0)$ one has

$$\Delta^2 U(x) = U(x)^p. \tag{7}$$

Our purpose here is to transform (7) first into an autonomous equation and, subsequently, into an autonomous system. For some of the estimates which follow, it is convenient to rewrite the original assumption $p > \frac{n+4}{n-4}$ as

$$(n-4)(p-1) > 8. \tag{8}$$

Inspired by the proof of [22, Proposition 3.7] (see also [10,13]) we set as in [9]

$$U(r) = r^{-4/(p-1)}v(\log r), \quad r \in (0, \sqrt[4]{\lambda}),$$
$$v(t) = e^{4t/(p-1)}U\bigl(e^t\bigr), \quad t \in \left(-\infty, \frac{1}{4}\log\lambda\right). \tag{9}$$

We take from [9] that, after the change (9), Eq. (7) may be rewritten as

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = v^p(t) \quad \left(t < \frac{1}{4}\log\lambda\right), \tag{10}$$

where the constants $K_i = K_i(n,p)$ $(i = 0, \ldots, 3)$ are given by

$$K_0 = \frac{8}{(p-1)^4}\bigl[(n-2)(n-4)(p-1)^3 + 2\bigl(n^2 - 10n + 20\bigr)(p-1)^2$$
$$- 16(n-4)(p-1) + 32\bigr],$$

$$K_1 = -\frac{2}{(p-1)^3}\bigl[(n-2)(n-4)(p-1)^3 + 4\bigl(n^2 - 10n + 20\bigr)(p-1)^2$$
$$- 48(n-4)(p-1) + 128\bigr],$$

$$K_2 = \frac{1}{(p-1)^2}\big[\big(n^2-10n+20\big)(p-1)^2 - 24(n-4)(p-1) + 96\big],$$

$$K_3 = \frac{2}{p-1}\big[(n-4)(p-1) - 8\big].$$

By using (8), it is not difficult to show that $K_1 = K_3 = 0$ if $p = \frac{n+4}{n-4}$ and that for $n \geqslant 5$, $p > \frac{n+4}{n-4}$

$$K_0 > 0, \qquad K_1 < 0, \qquad K_3 > 0. \tag{11}$$

On the other hand, the sign of K_2 depends on n and p. We emphasize that the sign of K_1 and K_3 is due to assumption (8).

Finally, we put

$$z(t) := v(-t), \quad t > -\frac{1}{4}\log\lambda. \tag{12}$$

For z, we have the differential equation analogous to (10):

$$z^{(4)}(t) - K_3 z'''(t) + K_2 z''(t) - K_1 z'(t) + K_0 z(t) = z^p(t) \quad \left(t > -\frac{1}{4}\log\lambda\right). \tag{13}$$

In order to study the possibly singular behaviour of u near $r = 0$, we have to investigate the behaviour of z for $t \to \infty$. Equation (13) has two equilibrium points, namely 0 and $K_0^{1/(p-1)}$. First we show that once the solution converges to an equilibrium point, then all derivatives converge to 0 as $t \to \infty$.

Proposition 1. *Assume that $z : [T_0, \infty) \to \mathbb{R}$ exists for some T_0 and solves a constant coefficient fourth order equation*

$$z^{(4)}(t) - K_3 z'''(t) + K_2 z''(t) - K_1 z'(t) = f\big(z(t)\big) \quad (t > -T_0), \tag{14}$$

where $f \in C^1(\mathbb{R})$ and where the coefficients may be considered as arbitrary real numbers $K_j \in \mathbb{R}$. Moreover, let z_0 be such that $f(z_0) = 0$ and assume that z satisfies $\lim_{t\to\infty} z(t) = z_0$. Then, for $k = 1, \ldots, 4$, one also has:

$$\lim_{t\to\infty} z^{(k)}(t) = 0. \tag{15}$$

If $f \in C^{k_0+1}$ in a neighborhood of $\{z_0\}$, then (15) *holds true for all $k \leqslant k_0 + 4$.*

Proof. By assumption, we have for any $q > 1$ that

$$\lim_{t\to\infty} \int_{t-2}^{t+3} \big|f\big(z(\tau)\big)\big|^q \, d\tau = 0, \qquad \lim_{t\to\infty} \big\|z(\,\cdot\,) - z_0\big\|_{C^0([t-2,t+3])} = 0.$$

We consider (14) as a fourth order "elliptic" equation and apply local L^q-estimates, which could of course be directly obtained in a much easier way for the ordinary differential equation (14), and conclude

$$\lim_{t\to\infty}\big\|z(\,\cdot\,)-z_0\big\|_{W^{4,q}(t-1,t+2)}=0.$$

By combining now Sobolev embedding and classical local Schauder estimates we have that

$$\lim_{t\to\infty}\big\|z(\,\cdot\,)-z_0\big\|_{C^{4,\alpha}(t,t+1)}=0.$$

The differential equation (14) finally shows the claim for any k in the given range. □

We now write (10) as a system in $\mathbb{R}^4$. We obtain from (9)

$$\frac{U'(r)}{r^3}=r^{-4p/(p-1)}\left[v'(t)-\frac{4}{p-1}v(t)\right] \tag{16}$$

so that

$$U'(r)=0 \quad\Longleftrightarrow\quad v'(t)=\frac{4}{p-1}v(t).$$

This fact suggests the definition

$$w_1(t)=v(t),\qquad w_2(t)=v'(t)-\frac{4}{p-1}v(t),\qquad w_3(t)=v''(t)-\frac{4}{p-1}v'(t),$$
$$w_4(t)=v'''(t)-\frac{4}{p-1}v''(t)$$

so that (10) becomes

$$\begin{cases} w_1'(t)=\dfrac{4}{p-1}w_1(t)+w_2(t),\\ w_2'(t)=w_3(t),\\ w_3'(t)=w_4(t),\\ w_4'(t)=C_2w_2(t)+C_3w_3(t)+C_4w_4(t)+w_1^p(t), \end{cases} \tag{17}$$

where

$$C_m=-\sum_{k=m-1}^{4}\frac{K_k4^{k+1-m}}{(p-1)^{k+1-m}}\quad \text{for } m=1,2,3,4 \text{ with } K_4=1. \tag{18}$$

This gives first that $C_1=0$ so that the term $C_1w_1(t)$ does not appear in the last equation of (17). Moreover, we have the explicit formulae:

$$C_2 = \frac{2}{(p-1)^3}\left[(n-2)(n-4)(p-1)^3 + 2\left(n^2-10n+20\right)(p-1)^2 - 16(n-4)(p-1) + 32\right]$$
$$= \frac{p-1}{4}K_0,$$
$$C_3 = -\frac{1}{(p-1)^2}\left[\left(n^2-10n+20\right)(p-1)^2 - 16(n-4)(p-1) + 48\right],$$
$$C_4 = -\frac{2}{p-1}\left[(n-4)(p-1) - 6\right].$$

We recall the phase space analysis performed in [9]. System (17) has the two stationary points (corresponding to $v_0 := 0$ and $v_s := K_0^{1/(p-1)}$)

$$O(0,0,0,0) \quad \text{and} \quad P\left(K_0^{1/(p-1)}, -\frac{4}{p-1}K_0^{1/(p-1)}, 0, 0\right). \tag{19}$$

Let us consider first the "regular point" O. The linearized matrix at O is

$$M_O = \begin{pmatrix} \frac{4}{p-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & C_2 & C_3 & C_4 \end{pmatrix}$$

and the characteristic polynomial is

$$\lambda \mapsto \lambda^4 + K_3\lambda^3 + K_2\lambda^2 + K_1\lambda + K_0.$$

Then, according to MAPLE™, the eigenvalues are given by

$$\lambda_1 = 2\frac{p+1}{p-1}, \qquad \lambda_2 = \frac{4}{p-1}, \qquad \lambda_3 = \frac{4p}{p-1} - n, \qquad \lambda_4 = 2\frac{p+1}{p-1} - n.$$

Since we assume that $p > \frac{n+4}{n-4} > \frac{n}{n-4} > \frac{n+2}{n-2}$, we have

$$\lambda_1 > \lambda_2 > 0 > \lambda_3 > \lambda_4.$$

This means that O is a hyperbolic point and that both the stable and the unstable manifolds are two-dimensional.

Around the "singular point" P the linearized matrix of the system (17) is given by

$$M_P = \begin{pmatrix} \frac{4}{p-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ pK_0 & C_2 & C_3 & C_4 \end{pmatrix}. \tag{20}$$

The corresponding characteristic polynomial is

$$\nu \mapsto \nu^4 + K_3\nu^3 + K_2\nu^2 + K_1\nu + (1-p)K_0$$

and the eigenvalues are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \qquad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)},$$
$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}, \qquad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)},$$

where

$$N_1 := -(n-4)(p-1) + 8, \qquad N_2 := \left(n^2 - 4n + 8\right)(p-1)^2,$$
$$N_3 := (9n-34)(n-2)(p-1)^4 + 8(3n-8)(n-6)(p-1)^3 + \left(16n^2 - 288n + 832\right)(p-1)^2 - 128(n-6)(p-1) + 256.$$

The stability of the stationary point P is described by the following

Proposition 2. *Assume that* $p > \frac{n+4}{n-4}$.

(i) *We have* $\nu_1, \nu_2 \in \mathbb{R}$ *and* $\nu_2 < 0 < \nu_1$.
(ii) *For* $5 \leqslant n \leqslant 12$ *we have* $\nu_3, \nu_4 \notin \mathbb{R}$ *and* $\operatorname{Re}\nu_3 = \operatorname{Re}\nu_4 < 0$.
(iii) *For* $n \geqslant 13$ *there exists* $p_c > \frac{n+4}{n-4}$ *such that*:
if $p < p_c$, *then* $\nu_3, \nu_4 \notin \mathbb{R}$ *and* $\operatorname{Re}\nu_3 = \operatorname{Re}\nu_4 < 0$;
if $p = p_c$, *then* $\nu_3, \nu_4 \in \mathbb{R}$ *and* $\nu_4 = \nu_3 < 0$;
if $p > p_c$, *then* $\nu_3, \nu_4 \in \mathbb{R}$ *and* $\nu_4 < \nu_3 < 0$. *The number* p_c *is the unique value of* $p > \frac{n+4}{n-4}$ *such that*

$$-(n-4)\left(n^3 - 4n^2 - 128n + 256\right)(p-1)^4 + 128(3n-8)(n-6)(p-1)^3 + 256\left(n^2 - 18n + 52\right)(p-1)^2 - 2048(n-6)(p-1) + 4096 = 0.$$

The function $n \mapsto p_c$ *is strictly decreasing and approaches* 1 *as* $n \to \infty$.

According to Proposition 2, in all cases we have

$$\nu_1 > 0, \qquad \nu_2 < 0, \qquad \operatorname{Re}\nu_3 = \operatorname{Re}\nu_4 < 0.$$

This means that P has a three-dimensional stable manifold and a one-dimensional unstable manifold (as in the exponential case, see [2, Section 3.1]).

4. Characterization of regular and weakly singular solutions

Now, we are in position to give a precise formulation of Theorem 4. According to Theorem 3, to be proved below, we may restrict ourselves to weakly singular solutions.

Theorem 6. *Let* $u = u(r)$ *be a radial solution of* (1) *and let*

$$W(t) = \left(w_1(t), w_2(t), w_3(t), w_4(t)\right)$$

be the corresponding trajectory relative to (17). *Then*:

(i) *u is regular* (*i.e.* $u \in L^\infty(B)$) *if and only if*

$$\lim_{t\to-\infty} W(t) = O.$$

(ii) *u is weakly singular if and only if*

$$\lim_{t\to-\infty} W(t) = P.$$

As a first step in proving this theorem we show that there are only a few possible values for $\lim_{t\to-\infty} v(t)$, provided the limit exists. The following proposition holds independently of the signs of the coefficients K_i.

Proposition 3. *Let v be a positive solution of* (10) *on* $(-\infty, \frac{1}{4}\log\lambda)$ *and assume that there exists* $L \in [0, +\infty]$ *such that*

$$\lim_{t\to-\infty} v(t) = L.$$

Then, $L \in \{0, K_0^{1/(p-1)}\}$.

Proof. This is almost the same as in [9, Proposition 5]. We remark that there the arguments are not affected by reversing the time or—equivalently—changing the sign of the coefficients K_1 and K_3. For the reader's convenience, we give the proof in Appendix A. □

Proof of Theorem 6. (i) Assuming that W corresponds to a regular solution, it is obvious that $\lim_{t\to-\infty} W(t) = O$. Let us now conversely assume that $\lim_{t\to-\infty} W(t) = O$; we have to prove that the corresponding solution u of (1) is regular. We calculate the eigenvectors of M_O corresponding to the positive eigenvalues i.e. spanning the unstable manifold. These are

$$\vec{W}_1 = \left(1, 2, 4\frac{p+1}{p-1}, 8\left(\frac{p+1}{p-1}\right)^2\right) \quad \text{for } \lambda_1 = 2\frac{p+1}{p-1};$$
$$\vec{W}_2 = (1, 0, 0, 0) \quad \text{for } \lambda_2 = \frac{4}{p-1}.$$

Since $\lambda_1 > \lambda_2$, all trajectories approaching O as $t \to -\infty$ are tangent to $\vec{W}_2$ except one, which is tangent to $\vec{W}_1$ (see Theorem IX.6.2 in [12]).

But for a solution to (1), the latter case cannot occur, since one always has $u(r) > 0$, $u'(r) \leqslant 0$, i.e. $w_1 > 0$, $w_2 \leqslant 0$. So, for any solution of (1), we may conclude that $ru'(r) = o(u(r))$ for $r \searrow 0$. That means that for any $\varepsilon > 0$ and $r > 0$ close enough to 0 we have

$$-\varepsilon < \frac{ru'(r)}{u(r)} \leqslant 0.$$

Integration yields that for $r \searrow 0$

$$0 \leqslant u(r) \leqslant C r^{-\varepsilon}.$$

Using this information, the differential equation, and $\lim_{t\to\infty} W(t) = 0$, i.e.

$$r^{4/(p-1)} u(r) \to 0, \qquad r^{1+4/(p-1)} u'(r) \to 0, \qquad r^{2+4/(p-1)} \Delta u(r) \to 0,$$
$$r^{3+4/(p-1)} (\Delta u)'(r) \to 0,$$

successive integration of (1) shows that

$$(\Delta u)'(r) = O(1), \qquad \Delta u(r) = O(1), \qquad u'(r) = O(1), \qquad u(r) = O(1)$$

for $r \searrow 0$. This shows that u is regular.

(ii) Let W belong to a weakly singular solution. Then, by definition, part (i), and Propositions 1 and 3, we see that $\lim_{t\to-\infty} W(t) = P$. The converse conclusion is obvious. □

5. Regular minimal solutions

The main goal here is to prove the most difficult part of Theorem 1, namely the immediate switch from existence of regular minimal solutions to nonexistence even of singular solutions.

Theorem 7. *Assume that u_s is a solution of* (1) *with parameter λ_s. Then, for any $\lambda \in (0, \lambda_s)$, the Dirichlet problem* (1) *has a regular radially decreasing minimal solution.*

We start by proving the following

Lemma 1. *Let u be a radial solution of the Dirichlet problem* (1)*, and define the corresponding functions $U = U(r)$ and $v = v(t)$ according to* (7) *and* (9) *respectively for $r \in (0, \sqrt[4]{\lambda})$ and $t \in (-\infty, \frac{1}{4}\log\lambda)$. Then, v is bounded.*

Proof. For contradiction, assume that v is not bounded. In view of Proposition 3 we may exclude that the limit as $t \to -\infty$ exists and equals $+\infty$. Hence we assume that

$$0 \leqslant \liminf_{t\to-\infty} v(t) < \limsup_{t\to-\infty} v(t) = +\infty.$$

This shows that there exists a sequence $t_k \to -\infty$ of local maxima for v such that for all k

$$\lim_{k\to+\infty} v(t_k) = +\infty, \qquad v'(t_k) = 0. \tag{21}$$

Define

$$\lambda_k = v^{p-1}(t_k) \tag{22}$$

so that

$$\lim_{k\to+\infty} \lambda_k = +\infty.$$

Since (10) is an autonomous equation the translated function

$$\tilde{v}_k(t) = v\left(t + t_k - \frac{1}{4}\log\lambda_k\right), \quad t \in \left(-\infty, \frac{1}{4}\log\lambda - t_k + \frac{1}{4}\log\lambda_k\right)$$

also solves (10). In particular, the function

$$\widetilde{U}_k(r) = r^{-4/(p-1)}\tilde{v}_k(\log r)$$

is a radial solution of Eq. (7) which satisfies the conditions

$$\widetilde{U}_k(\sqrt[4]{\lambda_k}) = \lambda_k^{-1/(p-1)}\tilde{v}_k\left(\frac{1}{4}\log\lambda_k\right) = \lambda_k^{-1/(p-1)}v(t_k) = 1 \tag{23}$$

and by (16), (21), (22)

$$\widetilde{U}_k'(\sqrt[4]{\lambda_k}) = -\frac{4}{p-1}\lambda_k^{-1/4} < 0. \tag{24}$$

Next, we define the radial function

$$u_k(r) = \widetilde{U}_k(\sqrt[4]{\lambda_k}r) - 1 = \lambda_k^{-1/(p-1)}e^{4t_k/(p-1)}U\left(re^{t_k}\right) - 1 \tag{25}$$

so that by (23) and (24) we have

$$\begin{cases} \Delta^2 u_k = \lambda_k(1+u_k)^p, \quad u_k > 0 & \text{in } B, \\ u_k = 0 & \text{on } \partial B, \\ -\dfrac{\partial u_k}{\partial \mathbf{n}} = \dfrac{4}{p-1} > 0 & \text{on } \partial B. \end{cases}$$

This boundary value problem is solved in a weak sense, since U is a weak solution of (7). One should observe that one also has a comparison principle in B with respect to the boundary datum $-\frac{\partial u}{\partial \mathbf{n}}$, see [11].

This shows that u_k is a weak supersolution for the problem

$$\begin{cases} \Delta^2 u = \lambda_k(1+u)^p, \quad u > 0 & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B. \end{cases} \tag{26}$$

By standard arguments, see for example Lemma 3.3 in [3], we infer that for any λ_k problem (26) admits a weak solution. Since $\lambda_k \to +\infty$ this contradicts the nonexistence of solutions of (1) for large λ (see the proof of Theorem 3 in [2] for more details). This completes the proof of the lemma. □

Proof of Theorem 7. Suppose now that u_s is a solution of (1) corresponding to $\lambda = \lambda_s$. After possibly replacing u_s by the minimal solution of (1) corresponding to $\lambda = \lambda_s$, we may assume that u_s is radial. We look for a regular solution of (1) for a fixed $\lambda \in (0, \lambda_s)$. Put $u_0 = u_s$ and define $u_1 = \frac{\lambda}{\lambda_s} u_0$ so that u_1 solves

$$\int_B u_1 \Delta^2 \varphi \, dx = \lambda \int_B (1 + u_0)^p \varphi \, dx \quad \forall \varphi \in C^4(\overline{B}) \cap H_0^2(B).$$

We define by iteration u_k as the unique solution of

$$\int_B u_k \Delta^2 \varphi \, dx = \lambda \int_B (1 + u_{k-1})^p \varphi \, dx \quad \forall \varphi \in C^4(\overline{B}) \cap H_0^2(B). \tag{27}$$

By the weak comparison principle (see Lemma 16 in [2]) we deduce that

$$0 < u_{\min} \leqslant u_k \leqslant u_{k-1} \quad \forall k \geqslant 1, \tag{28}$$

where $u_{\min}$ denotes the minimal solution of (1) with respect to the parameter λ. By monotone convergence it follows that there exists $u \in L^p(B)$ such that $u_k \to u$ in $L^p(B)$ as $k \to \infty$, $u \geqslant u_{\min}$. Moreover, passing to the limit in (27) we have

$$\int_B u \Delta^2 \varphi \, dx = \lambda \int_B (1 + u)^p \varphi \, dx \quad \forall \varphi \in C^4(\overline{B}) \cap H_0^2(B).$$

Fix $\bar{\vartheta} \in (\frac{\lambda}{\lambda_s}, 1)$ and introduce a strictly increasing sequence $\{\vartheta_k\}$ with $\frac{\lambda}{\lambda_s} < \vartheta_k < \bar{\vartheta}$ for any $k \geqslant 1$. Note that for any $\alpha > 0$ and for any $\beta > \alpha$ there exists $\gamma > 0$ such that for all $s \geqslant 0$

$$(1 + \alpha s)^p \leqslant \beta^p (1 + s)^p + \gamma. \tag{29}$$

By (29) there exists $C_1 > 0$ such that for all $\varphi \in C^4(\overline{B})$, $\varphi \geqslant 0$, $\varphi = |\nabla \varphi| = 0$ on ∂B

$$\begin{aligned} \int_B u_2 \Delta^2 \varphi \, dx &= \lambda \int_B (1 + u_1)^p \varphi \, dx = \lambda \int_B \left(1 + \frac{\lambda}{\lambda_s} u_0\right)^p \varphi \, dx \\ &\leqslant \int_B \lambda \big[\vartheta_1^p (1 + u_0)^p + C_1\big] \varphi \, dx = \int_B \big(\vartheta_1^p u_1 + \lambda C_1 \psi\big) \Delta^2 \varphi \, dx, \end{aligned}$$

where ψ is the unique solution of the Dirichlet problem

$$\begin{cases} \Delta^2 \psi = 1 & \text{in } B, \\ \psi = |\nabla \psi| = 0 & \text{on } \partial B. \end{cases}$$

The weak comparison principle yields

$$u_2 \leqslant \vartheta_1^p u_1 + \lambda C_1 \psi \leqslant \vartheta_1 u_1 + \lambda C_1 \psi.$$

Iterating this procedure we prove that for any $k \geqslant 1$ there exists $C_k > 0$ such that

$$u_{k+1} \leqslant \vartheta_k u_k + \lambda C_k \psi. \tag{30}$$

Since we chose $\frac{\lambda}{\lambda_s} < \vartheta_k < \bar{\vartheta} < 1$ for any $k \geqslant 1$, by (30) it follows that

$$u_k \leqslant (\bar{\vartheta})^k u_0 + D_k \quad \forall k \geqslant 1 \tag{31}$$

for a suitable $D_k > 0$. Therefore, for any $\varepsilon > 0$ there exists $\bar{k}$ such that $(\bar{\vartheta})^{\bar{k}} < \varepsilon$ and hence by (28) and (31) we have

$$0 \leqslant u \leqslant u_{\bar{k}} \leqslant \varepsilon u_0 + D_{\bar{k}}. \tag{32}$$

Making use of (9), (32), and Lemma 1 we deduce that for any $\varepsilon > 0$

$$0 \leqslant \limsup_{r \to 0^+} r^{4/(p-1)} u(r) \leqslant \limsup_{r \to 0^+} \bigl(\varepsilon r^{4/(p-1)} u_0(r) + r^{4/(p-1)} D_{\bar{k}}\bigr) = \varepsilon L,$$

where $L = \limsup_{r \to 0^+} r^{4/(p-1)} u_0(r) < +\infty$. This proves that

$$\lim_{r \to 0^+} r^{4/(p-1)} u(r) = 0.$$

Finally by (9), Proposition 1 and Theorem 4 we conclude that $u \in L^\infty(B)$.

The minimal solution $u_{\min}$ may now be obtained by means of an iterative procedure starting with 0. Radial symmetry is so obvious. For monotonicity we refer to [7, Lemma 2.2] or [21, Proposition 1]. □

Proof of Theorem 1. First, we remark that (i) and (iii) are proved by Theorem 7. As for (ii), i.e. existence of a possibly singular solution for the extremal parameter λ^*, we can proceed as outlined in [2, Lemma 22]. By means of a generalized Pohožaev identity (cf. [18,19]) one can obtain uniform bounds for the minimal *regular* solutions to (1) ($\lambda \in (0, \lambda^*)$) in $H_0^2(B) \cap L^{p+1}(B)$, which allow to perform a monotone limit as $\lambda \nearrow \lambda^*$. Cf. also [15].

Alternatively, one may refer to [3, Proposition 3.6] and Theorem 2 the proof of which requires only Theorem 7. In [3], Navier instead of Dirichlet boundary conditions are considered. However, since here we are working in the ball, no changes in the argument are needed.

It remains to prove the estimate (3) for λ^*. The explicit singular solution of the differential equation

$$u(x) := |x|^{-4/(p-1)} - 1$$

is also a weak supersolution for (1) with parameter $\lambda = K_0$. To see this one observes that u is only weakly singular near the origin, and that also for biharmonic equations, one has a kind of Hopf lemma for the boundary data, see [11]. This shows $\lambda^* \geqslant K_0$.

In order to show $\lambda^* < \lambda_1/p$, we multiply (1) by the positive first eigenfunction φ_1 of (2) and obtain

$$\lambda_1 \int_B u\varphi_1 \, dx = \int_B u \Delta^2 \varphi_1 \, dx = \lambda \int_B (1+u)^p \varphi_1 \, dx > p\lambda \int_B u\varphi_1 \, dx,$$

thereby proving the desired inequality. □

6. Energy considerations

Let u be a radial singular solution of (1), and let $v = v(t)$ be the corresponding function defined in (9). Let $z(t) = v(-t)$ so that $z(t)$ solves Eq. (13) for $t > -\frac{1}{4}\log\lambda$.

The following energy functional

$$E(t) := \frac{1}{p+1}z(t)^{p+1} - \frac{K_0}{2}z(t)^2 - \frac{K_2}{2}\left|z'(t)\right|^2 + \frac{1}{2}\left|z''(t)\right|^2 \tag{33}$$

will help to show that every singular solution is weakly singular, i.e. to prove Theorem 3. Moreover, in the second part of this section, we shall specify the asymptotic behaviour of any (weakly) singular solution near $r = 0$, i.e. of $z(t)$ for $t \to \infty$.

The first result is analogous to Proposition 1.

Lemma 2. *Let $z : (-\frac{1}{4}\log\lambda, \infty) \to \mathbb{R}$ be the solution of* (13) *corresponding to a radial singular solution of* (1). *Then, for $k = 1, \dots, 4$ the functions z and $z^{(k)}$ are bounded in $(-\frac{1}{4}\log\lambda, \infty)$.*

Proof. By Lemma 1 it follows immediately that $z(t) = v(-t)$ is bounded in $(-\frac{1}{4}\log\lambda, \infty)$. Put $I = (-\frac{1}{4}\log\lambda, \infty)$ and $t_0 = -\frac{1}{4}\log\lambda$. Then, $z^p(t) - K_0 z(t)$ is bounded in I and hence, by local L^q-estimates for fourth order elliptic equations, we infer that for any $q > 1$ there exists a constant $C_q > 0$ such that for any $t > t_0 + 1$ we have

$$\left\|z(\cdot)\right\|_{W^{4,q}(t-1,t+2)} \leqslant C_q \|z\|_{L^\infty(I)}.$$

By combining Sobolev embeddings and local Schauder estimates we conclude that there exists a positive constant independent of t, still denoted by C_q, such that

$$\left\|z(\cdot)\right\|_{C^{4,\alpha}(t,t+1)} \leqslant C_q \|z\|_{L^\infty(I)}. \qquad \square$$

Arguing as in [9], in the next four lemmas we prove some summability properties for the function z and its derivatives.

Lemma 3. *Let $t_0 = -\frac{1}{4}\log\lambda$. Then*

$$\int_{t_0}^{\infty}\left|z'(s)\right|^2 ds + \int_{t_0}^{\infty}\left|z''(s)\right|^2 ds < \infty.$$

Proof. Let $E(t)$ be the function defined in (33). For any $t > t_0$ we obtain by integration by parts and exploiting (13)

$$E(t) - E(t_0) = \int_{t_0}^{t} E'(s)\,ds = \int_{t_0}^{t}\left(z^p z' - K_0 z z' - K_2 z' z'' + z'' z'''\right) ds$$

$$= z'(t)z'''(t) - z'(t_0)z'''(t_0) + \int_{t_0}^{t} z'\left(z^p - K_0 z - K_2 z'' - z^{(4)}\right) ds$$

$$= z'(t)z'''(t) - z'(t_0)z'''(t_0) + \int_{t_0}^{t} z'(-K_3 z''' - K_1 z')\,ds$$
$$= z'(t)z'''(t) - z'(t_0)z'''(t_0) - K_3 z'(t)z''(t) + K_3 z'(t_0)z''(t_0)$$
$$+ \int_{t_0}^{t} \big(K_3 z''(s)^2 - K_1 z'(s)^2\big)\,ds. \tag{34}$$

By Lemma 2 it follows that $E(t)$ and the functions $z'(t)$, $z''(t)$, $z'''(t)$ are bounded in $I = (t_0, \infty)$, while around t_0, they are obviously smooth. This together with (34) and the fact that $K_3 > 0$, $K_1 < 0$ proves the claim. □

Lemma 4. *We have*

$$\int_{t_0}^{\infty} \big|z'''(s)\big|^2\,ds < \infty.$$

Proof. We multiply Eq. (13) by z'' and integrate over (t_0, t) to obtain

$$\int_{t_0}^{t} \big(z^{(4)}(s) - K_3 z'''(s) + K_2 z''(s) - K_1 z'(s) + K_0 z(s)\big) z''(s)\,ds = \int_{t_0}^{t} z^p(s) z''(s)\,ds. \tag{35}$$

First, we prove that all the lower order terms in the integral identity (35) are bounded. By Lemmas 2, 3, and integration by parts we have

$$\left| \int_{t_0}^{t} z(s)z''(s)\,ds \right| \leqslant \big|z(t)z'(t)\big| + \big|z(t_0)z'(t_0)\big| + \int_{t_0}^{t} \big|z'(s)\big|^2\,ds = O(1) \quad \text{as } t \to \infty. \tag{36}$$

By Lemma 3 and Hölder's inequality we have

$$\left| \int_{t_0}^{t} z'(s)z''(s)\,ds \right| \leqslant \left(\int_{t_0}^{t} \big|z'(s)\big|^2\,ds \right)^{1/2} \left(\int_{t_0}^{t} \big|z''(s)\big|^2\,ds \right)^{1/2} = O(1) \quad \text{as } t \to \infty. \tag{37}$$

By Lemma 2, integration by parts, and arguing as in (37), we obtain

$$\left| \int_{t_0}^{t} z^p(s)z''(s)\,ds \right| \leqslant \big|z^p(t)z'(t)\big| + \big|z^p(t_0)z'(t_0)\big| + \left| \int_{t_0}^{t} p z^{p-1}(s) z'(s)^2\,ds \right| = O(1)$$
$$\text{as } t \to \infty. \tag{38}$$

Using again Lemma 2 we conclude that

$$\left|\int_{t_0}^{t} z'''(s)z''(s)\,ds\right| \leqslant \frac{1}{2}\left|z''(t)\right|^2 + \frac{1}{2}\left|z''(t_0)\right|^2 = O(1) \quad \text{as } t \to \infty. \tag{39}$$

Finally, after integration by parts we infer that

$$\int_{t_0}^{t} \left|z'''(s)\right|^2 ds = z'''(t)z''(t) - z'''(t_0)z''(t_0) - \int_{t_0}^{t} z^{(4)}(s)z''(s)\,ds = O(1) \quad \text{as } t \to \infty$$

in view of Lemmas 2, 3, and (35)–(39). This completes the proof of the lemma. □

Lemma 5. *We have*

$$\int_{t_0}^{\infty} \left|z^{(4)}(s)\right|^2 ds < \infty.$$

Proof. We multiply Eq. (13) by $z^{(4)}$ and integrate over (t_0, t) to obtain

$$\int_{t_0}^{t} \left|z^{(4)}(s)\right|^2 ds = \int_{t_0}^{t} \big(z^p(s) - K_0 z(s) + K_1 z'(s) - K_2 z''(s) + K_3 z'''(s)\big) z^{(4)}(s)\,ds. \tag{40}$$

Arguing as in the proof of Lemma 4 one can easily prove that the right-hand side of (40) remains bounded as $t \to \infty$. This completes the proof of the lemma. □

Lemma 6. *We have*

$$\int_{t_0}^{\infty} z^2(s)\left|z^{p-1}(s) - K_0\right|^2 ds < \infty.$$

Proof. Using the differential equation (13) we obtain

$$\left[z^{(4)}(s) - K_3 z'''(s) + K_2 z''(s) - K_1 z'(s)\right]^2 = z^2(s)\left|z^{p-1}(s) - K_0\right|^2.$$

The proof of the lemma follows immediately from Lemmas 3–5. □

Proof of Theorem 3. Let $W = (w_1, w_2, w_3, w_4)$ be the solution of the dynamical system (17) corresponding to a radial singular solution u of (1), and let P and O be the stationary points introduced in (19). In view of Lemmas 3–6 we infer that at least one of the two alternatives holds true

$$\exists \{\sigma_k\} \quad \text{s.t. } \sigma_{k+1} < \sigma_k,\ \lim_{k\to\infty} \sigma_k = -\infty,\ \lim_{k\to\infty} W(\sigma_k) = P; \tag{41}$$

or

$$\exists\{\sigma_k\} \quad \text{s.t. } \sigma_{k+1} < \sigma_k, \ \lim_{k\to\infty} \sigma_k = -\infty, \ \lim_{k\to\infty} W(\sigma_k) = O. \tag{42}$$

Arguing as in Proposition 7 in [9] we conclude that

$$\lim_{t\to-\infty} W(t) = P$$

or

$$\lim_{t\to-\infty} W(t) = O$$

respectively in the cases (41) and (42). In view of Theorem 6 we may exclude the second case since it would imply that u is a regular solution. Therefore, only the first case may occur and hence, Theorem 6 implies that u is a weakly singular solution. □

The energy functional defined above in (33) will also help to specify the behaviour of (weakly) singular solutions of (1) near $r = 0$. To this end we may assume that

$$\lim_{t\to\infty} z(t) = K_0^{1/(p-1)}. \tag{43}$$

Lemma 7. *Let u_s be a weakly singular solution of* (1) *with parameter λ_s and*

$$z(t) : \left(-\frac{1}{4}\log\lambda_s, \infty\right) \to (0, \infty)$$

the corresponding solution of (13). *Then, it cannot happen that $z'(t_0) = 0$ for some t_0.*

Proof. Assume for contradiction that $z'(t_0) = 0$. Then, by (16), we have that $z'(-\frac{1}{4}\log\lambda_s) \neq 0$ and hence, z is not a constant. For any $t > t_0$ we obtain by arguing as in Lemma 3

$$E(t) - E(t_0) = z'(t)z'''(t) - K_3 z'(t)z''(t) + \int_{t_0}^{t} \left(K_3 z''(s)^2 - K_1 z'(s)^2\right) ds.$$

Letting $t \to \infty$ and observing Proposition 1 yields

$$\begin{aligned} E(\infty) - E(t_0) &= \int_{t_0}^{\infty} \left(K_3 z''(s)^2 - K_1 z'(s)^2\right) ds > 0 \\ \Rightarrow \quad -\frac{p-1}{2(p+1)} K_0^{(p+1)/(p-1)} &> \frac{1}{p+1} z(t_0)^{p+1} - \frac{K_0}{2} z(t_0)^2 + \frac{1}{2}\left|z''(t_0)\right|^2 \\ &\geqslant \min_{\zeta\geqslant 0}\left(\frac{\zeta^{p+1}}{p+1} - \frac{K_0}{2}\zeta^2\right) = -\frac{p-1}{2(p+1)} K_0^{(p+1)/(p-1)}, \end{aligned}$$

a contradiction. □

Proof of Theorem 5. We know from (16) that

$$v'\left(\frac{1}{4}\log\lambda_s\right)=\frac{4}{p-1}v\left(\frac{1}{4}\log\lambda_s\right)>0$$

so that

$$z'\left(-\frac{1}{4}\log\lambda_s\right)<0.$$

The previous lemma then shows that for all $t\geqslant -\frac{1}{4}\log\lambda_s$

$$z'(t)<0 \quad\Rightarrow\quad z(t)>K_0^{1/(p-1)}$$

and

$$U(x)>K_0^{1/(p-1)}|x|^{-4/(p-1)}$$

so that

$$u_s(x)>\left(\frac{K_0}{\lambda_s}\right)^{1/(p-1)}|x|^{-4/(p-1)}-1.$$

Moreover, we have in particular $0=u_s(1)$ so that

$$\lambda_s>K_0. \qquad \square$$

Remark. With a completely analogous proof, one can show for any (weakly) singular radial solution u_s of the Dirichlet problem for the biharmonic equation $\Delta^2 u_s=\lambda_s\exp(u_s)$ that $\lambda_s>8(n-2)(n-4)$ and that

$$u_s(x)>-4\log|x|+\log\frac{8(n-2)(n-4)}{\lambda_s}.$$

This complements [2, Theorem 4].

7. Stability of the minimal regular solution

In this section we shall give the proof of Theorem 2.

Let $\lambda\in(0,\lambda^*)$, and let u_λ be the corresponding minimal solution. By Theorem 7 we know that u_λ is a regular solution. Consider the following weighted η-eigenvalue problem

$$\begin{cases}\Delta^2\psi=\eta\lambda p(1+u_\lambda)^{p-1}\psi & \text{in } B,\\ \psi=|\nabla\psi|=0 & \text{on } \partial B\end{cases} \tag{44}$$

and let

$$\eta_1(\lambda)=\inf_{\psi\in H_0^2(B)\setminus\{0\}}\frac{\int_B|\Delta\psi|^2\,dx}{\lambda\int_B p(1+u_\lambda)^{p-1}\psi^2\,dx} \tag{45}$$

be the corresponding first eigenvalue. Since $u_\lambda \in L^\infty(B)$, by compactness of the embedding $H_0^2(B) \subset L^2(B)$ we infer that the minimum in (45) is achieved. Note that by the Lagrange multiplier method any minimizer ψ_1 of $\eta_1(\lambda)$ solves (44) with $\eta = \eta_1(\lambda)$.

Since u_λ is a regular solution of (1), by L^q-estimates for fourth order elliptic equations and Schauder estimates we infer that both u_λ and ψ_1 are classical solutions of (1) and (44), respectively. In the next lemma we show that ψ_1 does not change sign in B.

Lemma 8. *Let ψ_1 be a minimizer for $\eta_1(\lambda)$. Then, $\psi_1 > 0$ in B up to a constant multiple.*

Proof. Assume for contradiction that ψ_1 is a sign changing minimizer for $\eta_1(\lambda)$ and consider the problem

$$\begin{cases} \Delta^2 w = \eta_1(\lambda)\lambda p(1+u_\lambda)^{p-1}|\psi_1| & \text{in } B,\\ w = |\nabla w| = 0 & \text{on } \partial B. \end{cases} \tag{46}$$

By Boggio's maximum principle [4], we deduce that $w > |\psi_1|$ in B and hence

$$\int_B |\Delta w|^2\,dx = \eta_1(\lambda)\int_B \lambda p(1+u_\lambda)^{p-1}|\psi_1| w\,dx < \eta_1(\lambda)\int_B \lambda p(1+u_\lambda)^{p-1} w^2\,dx$$

so that

$$\frac{\int_B |\Delta w|^2\,dx}{\lambda\int_B p(1+u_\lambda)^{p-1}w^2\,dx} < \eta_1(\lambda).$$

This contradicts the definition of $\eta_1(\lambda)$. Therefore, ψ_1 is a function of constant sign and hence, up to a constant multiple, we may assume that $\psi_1 \geqslant 0$ in B. The strict positivity of ψ_1 follows from (44) and Boggio's maximum principle [4]. □

Lemma 9. *Let $\eta_1(\lambda)$ be the first eigenvalue of* (44). *Then $\eta_1(\lambda) > 1$.*

Proof. Fix $\bar\lambda \in (\lambda, \lambda^*)$ and consider the corresponding minimal solution $u_{\bar\lambda}$ of (1). Since $u_\lambda, u_{\bar\lambda}$ are minimal solutions for the respective problems we have that $u_\lambda \leqslant u_{\bar\lambda}$ in B. Boggio's maximum principle yields $u_\lambda < u_{\bar\lambda}$ in B. By Lemma 8 we may fix a positive minimizer ψ_1 of (45). Convexity of $s \mapsto (1+s)^p$ yields

$$\begin{aligned} \eta_1(\lambda)\int_B (u_{\bar\lambda}-u_\lambda)\lambda p(1+u_\lambda)^{p-1}\psi_1\,dx &= \int_B (u_{\bar\lambda}-u_\lambda)\Delta^2\psi_1\,dx\\ &= \int_B \big[\bar\lambda(1+u_{\bar\lambda})^p - \lambda(1+u_\lambda)^p\big]\psi_1\,dx\\ &> \lambda\int_B \big[(1+u_{\bar\lambda})^p - (1+u_\lambda)^p\big]\psi_1\,dx\\ &\geqslant \lambda\int_B p(1+u_\lambda)^{p-1}(u_{\bar\lambda}-u_\lambda)\psi_1\,dx. \end{aligned}$$

This proves that $\eta_1(\lambda) > 1$. □

Proof of Theorem 2. Consider now the first eigenvalue $\mu_1(\lambda)$ for the linearized operator $\Delta^2 - \lambda p(1+u_\lambda)^{p-1}$. We have

$$\mu_1(\lambda) = \inf_{w \in H_0^2(B)\setminus\{0\}} \frac{\int_B |\Delta w|^2\,dx - \lambda \int_B p(1+u_\lambda)^{p-1} w^2\,dx}{\int_B w^2\,dx}.$$

For any $w \in H_0^2(B)$ we have

$$\begin{aligned}\int_B |\Delta w|^2\,dx - \lambda \int_B p(1+u_\lambda)^{p-1} w^2\,dx &\geqslant \left(1 - \frac{1}{\eta_1(\lambda)}\right) \int_B |\Delta w|^2\,dx \\ &\geqslant \lambda_1 \left(1 - \frac{1}{\eta_1(\lambda)}\right) \int_B w^2\,dx,\end{aligned}$$

where λ_1 denotes the first eigenvalue of (2). Lemma 9 now yields

$$\mu_1(\lambda) \geqslant \lambda_1 \left(1 - \frac{1}{\eta_1(\lambda)}\right) > 0.$$

This completes the proof of the theorem. □

For $1 < p \leqslant \frac{n+4}{n-4}$, we may define the action functional J_λ associated with the Euler–Lagrange equation (1)

$$J_\lambda(u) = \frac{1}{2} \int_B |\Delta u|^2\,dx - \frac{\lambda}{p+1} \int_B |1+u|^{p+1}\,dx \quad \forall u \in H_0^2(B).$$

Since for $u \in H_0^2(B)$

$$J_\lambda''(u) = \Delta^2 - \lambda p|1+u|^{p-1} \quad \text{in } \mathcal{L}\big(H_0^2(B); H^{-2}(B)\big),$$

by Theorem 2 we immediately obtain

Corollary 1. *Let $1 < p \leqslant \frac{n+4}{n-4}$ and $\lambda \in (0, \lambda^*)$. Then, the corresponding minimal solution u_λ is a local minimum for the functional J_λ.*

By Theorem 2 and [3, Proposition 3.6] we immediately obtain that Theorem 1 holds true for any superlinear exponent $p > 1$. Moreover, for subcritical and critical $1 < p \leqslant \frac{n+4}{n-4}$, according to the related result [3, Theorem 2.2], we expect the existence of two distinct solutions for $\lambda \in (0, \lambda^*)$.

Appendix A. Proof of Proposition 3

In order to avoid confusion with respect to the time direction we switch to the solution z of (13):

$$z^{(4)}(t) - K_3 z'''(t) + K_2 z''(t) - K_1 z'(t) + K_0 z(t) = z^p(t) \quad \left(t > -\frac{1}{4}\log\lambda\right).$$

For contradiction, assume first that L is finite and $L \notin \{0, K_0^{1/(p-1)}\}$. Then, $z^p(t) - K_0 z(t) \to \alpha := L^p - K_0 L \neq 0$ and for all $\varepsilon > 0$ there exists $T > 0$ such that

$$\alpha - \varepsilon \leqslant z^{(4)}(t) - K_3 z'''(t) + K_2 z''(t) - K_1 z'(t) \leqslant \alpha + \varepsilon \quad \forall t \geqslant T. \tag{A.1}$$

Take $\varepsilon < |\alpha|$ so that $\alpha - \varepsilon$ and $\alpha + \varepsilon$ have the same sign and let

$$\delta := \sup_{t \geqslant T} \bigl| z(t) - z(T) \bigr| < \infty.$$

Integrating (A.1) over $[T, t]$ for any $t \geqslant T$ yields

$$(\alpha - \varepsilon)(t - T) + C - |K_1|\delta \leqslant z'''(t) - K_3 z''(t) + K_2 z'(t) \leqslant (\alpha + \varepsilon)(t - T) + C + |K_1|\delta \quad \forall t \geqslant T,$$

where $C = C(T)$ is a constant containing all the terms $z(T)$, $z'(T)$, $z''(T)$ and $z'''(T)$. Repeating twice more this procedure gives

$$\frac{\alpha - \varepsilon}{6}(t - T)^3 + O\bigl(t^2\bigr) \leqslant z'(t) \leqslant \frac{\alpha + \varepsilon}{6}(t - T)^3 + O\bigl(t^2\bigr) \quad \text{as } t \to \infty.$$

This contradicts the assumption that z admits a finite limit as $t \to +\infty$.

Next, we exclude the case $L = +\infty$. For contradiction, assume that

$$\lim_{t \to +\infty} z(t) = +\infty. \tag{A.2}$$

Then, there exists $T \in \mathbb{R}$ such that

$$z^{(4)}(t) - K_3 z'''(t) + K_2 z''(t) - K_1 z'(t) \geqslant \frac{z^p(t)}{2} \quad \forall t \geqslant T.$$

Moreover, by integrating this inequality over $[T, t]$ (for $t \geqslant T$), we get

$$z'''(t) - K_3 z''(t) + K_2 z'(t) - K_1 z(t) \geqslant \frac{1}{2} \int_T^t z^p(s)\, ds + C \quad \forall t \geqslant T, \tag{A.3}$$

where $C = C(T)$ is a constant containing all the terms $z(T)$, $z'(T)$, $z''(T)$ and $z'''(T)$. From (A.2) and (A.3) we deduce that there exists $T' \geqslant T$ such that $\alpha := z'''(T') - K_3 z''(T') + K_2 z'(T') - K_1 z(T') > 0$. Since (10) is autonomous, we may assume that $T' = 0$. Therefore, we have

$$z^{(4)}(t) - K_3 z'''(t) + K_2 z''(t) - K_1 z'(t) \geqslant \frac{z^p(t)}{2} \quad \forall t \geqslant 0, \tag{A.4}$$

$$z'''(0) - K_3 z''(0) + K_2 z'(0) - K_1 z(0) = \alpha > 0. \tag{A.5}$$

We may now apply the test function method developed by Mitidieri–Pohožaev [16]. More precisely, fix $T_1 > T > 0$ and a nonnegative function $\phi \in C_c^4[0, \infty)$ such that

$$\phi(t) = \begin{cases} 1 & \text{for } t \in [0, T], \\ 0 & \text{for } t \geqslant T_1. \end{cases}$$

In particular, these properties imply that $\phi(T_1) = \phi'(T_1) = \phi''(T_1) = \phi'''(T_1) = 0$. Hence, multiplying inequality (A.4) by $\phi(t)$, integrating by parts, and recalling (A.5) yields

$$\int_0^{T_1} \left[\phi^{(4)}(t) + K_3\phi'''(t) + K_2\phi''(t) + K_1\phi'(t)\right] z(t)\,dt \geqslant \frac{1}{2}\int_0^{T_1} z^p(t)\phi(t)\,dt + \alpha. \tag{A.6}$$

We now apply Young's inequality in the following form: for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$z\phi^{(i)} = z\phi^{1/p}\frac{\phi^{(i)}}{\phi^{1/p}} \leqslant \varepsilon z^p\phi + C(\varepsilon)\frac{|\phi^{(i)}|^{p/(p-1)}}{\phi^{1/(p-1)}}, \quad \phi^{(i)} = \frac{d^i\phi}{dt^i}\ (i = 1, 2, 3, 4).$$

Then, provided ε is chosen sufficiently small, (A.6) becomes

$$C\sum_{i=1}^{4}\int_0^{T_1}\frac{|\phi^{(i)}(t)|^{p/(p-1)}}{\phi^{1/(p-1)}(t)}dt \geqslant \frac{1}{4}\int_0^{T} z^p(t)\,dt + \alpha \tag{A.7}$$

where $C = C(\varepsilon, K_i) > 0$. We now choose $\phi(t) = \phi_0(\frac{t}{T})$, where $\phi_0 \in C_c^4[0, \infty)$, $\phi_0 \geqslant 0$ and

$$\phi_0(\tau) = \begin{cases} 1 & \text{for } \tau \in [0, 1], \\ 0 & \text{for } \tau \geqslant \tau_1 > 1. \end{cases}$$

As noticed in [16], there exists a function ϕ_0 in such class satisfying moreover

$$\int_0^{\tau_1}\frac{|\phi_0^{(i)}(\tau)|^{p/(p-1)}}{\phi_0^{1/(p-1)}(\tau)}\,d\tau =: A_i < \infty \quad (i = 1, 2, 3, 4).$$

Then, thanks to a change of variables in the integrals, (A.7) becomes

$$C\sum_{i=1}^{4} A_i T^{1-ip/(p-1)} \geqslant \frac{1}{4}\int_0^{T} z^p(t)\,dt + \alpha \quad \forall T > 0.$$

Letting $T \to \infty$, the previous inequality contradicts (A.2). $\square$

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math. 12 (1959) 623–727.

[2] G. Arioli, F. Gazzola, H.-Ch. Grunau, E. Mitidieri, A semilinear fourth order elliptic problem with exponential nonlinearity, SIAM J. Math. Anal. 36 (2005) 1226–1258.

[3] E. Berchio, F. Gazzola, Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities, Electron. J. Differential Equations 2005 (34) (2005) 1–20.

[4] T. Boggio, Sulle funzioni di Green d'ordine m, Rend. Circ. Mat. Palermo 20 (1905) 97–135.
[5] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, Adv. Differential Equations 1 (1996) 73–90.
[6] H. Brezis, J.L. Vazquez, Blow up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid 10 (1997) 443–469.
[7] R. Dalmasso, Uniqueness theorems for some fourth-order elliptic equations, Proc. Amer. Math. Soc. 123 (1995) 1177–1183.
[8] J.I. Diaz, M. Lazzo, P.G. Schmidt, Large solutions of a polyharmonic equation with superlinear growth, preprint.
[9] F. Gazzola, H.-Ch. Grunau, Radial entire solutions for supercritical biharmonic equations, Math. Ann. 334 (2006) 905–936.
[10] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981) 525–598.
[11] H.-Ch. Grunau, G. Sweers, Positivity properties of elliptic boundary value problems of higher order, in: Proc. 2nd World Congress of Nonlinear Analysts, Nonlinear Anal. 30 (1997) 5251–5258.
[12] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.
[13] D.D. Joseph, T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Ration. Mech. Anal. 49 (1973) 241–269.
[14] P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982) 441–467.
[15] F. Mignot, J.P. Puel, Sur une classe de problèmes nonlinéaires avec nonlinéarité positive, croissante, convexe, Comm. Partial Differential Equations 5 (1980) 791–836.
[16] E. Mitidieri, S. Pohožaev, Apriori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, Proc. Steklov Inst. Math. 234 (2001) 1–362 (translated from Russian).
[17] P. Oswald, On a priori estimates for positive solutions of a semilinear biharmonic equation in a ball, Comment. Math. Univ. Carolin. 26 (1985) 565–577.
[18] S.I. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965) 1408–1411.
[19] S.I. Pohožaev, Solvability of an elliptic problem in $\mathbb{R}^n$ with a supercritical index of nonlinearity, Dokl. Akad. Nauk SSSR 313 (1990) 1356–1360, English translation in Soviet Math. Dokl. 42 (1991) 215–219.
[20] W. Reichel, Uniqueness results for semilinear polyharmonic boundary value problems on conformally contractible domains, I, J. Math. Anal. Appl. 287 (2003) 61–74;
W. Reichel, Uniqueness results for semilinear polyharmonic boundary value problems on conformally contractible domains, II, J. Math. Anal. Appl. 287 (2003) 75–89.
[21] R. Soranzo, A priori estimates and existence of positive solutions of a superlinear polyharmonic equation, Dyn. Systems Appl. 3 (1994) 465–487.
[22] X. Wang, On the Cauchy problem for reaction–diffusion equations, Trans. Amer. Math. Soc. 337 (1993) 549–590.