Rank equalities for idempotent and involutory matrices

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Abstract

We establish several rank equalities for idempotent and involutory matrices. In particular, we obtain new formulas for the rank of the difference, the sum, the product and the commutator of idempotent or involutory matrices. Extensions to scalar-potent matrices are also included. Our matrices are complex and are not necessarily Hermitian. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction and preliminaries

A complex square matrix \(A\) is said to be idempotent, or a projector, whenever \(A^2 = A\); when \(A\) is Hermitian (real symmetric) and idempotent, it is often called an orthogonal projector, otherwise an oblique projector. As one of the fundamental building blocks in matrix theory, idempotent matrices are very useful in many contexts and have been extensively studied in the literature, see, e.g., [1–11,14–17].
In particular, many authors have studied the questions: if both $P$ and $Q$ are idempotent, then: Under what conditions are $P \pm Q$ and $PQ$ idempotent? Under what conditions are $P \pm Q$ nonsingular? Under what conditions do $P$ and $Q$ commute? In this paper we find several new and interesting rank equalities for the matrices $P \pm Q$, $PQ \pm QP$, $I - PQ$, and so on. Through these rank equalities we derive a variety of new properties for idempotent matrices, including some new solutions to the questions just mentioned.

We write $\mathbb{C}^{m \times n}$ for the set of all $m \times n$ matrices over the field of complex numbers. The symbols $A^*$, $A^-$, $r(A)$ and $R(A)$ denote, respectively, the conjugate transpose, a generalized inverse, the rank and the range space of a matrix $A$. The partitioned (block) matrix $M \in \mathbb{C}^{m \times (n+k)}$ with $A \in \mathbb{C}^{m \times n}$ placed next to $B \in \mathbb{C}^{m \times k}$ is denoted by $[A, B]$, and $N \in \mathbb{C}^{(m+l) \times n}$ with $A \in \mathbb{C}^{m \times n}$ placed above $C \in \mathbb{C}^{l \times n}$ by $[A, C]$.

Some well-known results on ranks of matrices used in the sequel are given in the following lemma due to Marsaglia and Styan [12].

**Lemma 1.1.** Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$ and $D \in \mathbb{C}^{l \times k}$ be given. Then they satisfy the following rank equalities:

\[
\begin{align*}
 r[A, B] &= r(A) + r(B - AA^B) = r(B) + r(A - BB^A), \\
 r\begin{bmatrix} A \\ C \end{bmatrix} &= r(A) + r(C - CA^A) = r(C) + r(A - AC^C), \\
 r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} &= r(B) + r(C) + r[(I_m - BB^B)A(I_n - C^C)].
\end{align*}
\]

where the generalized inverses $A^-$, $B^-$, $C^-$ are chosen arbitrarily.

2. **Rank equalities for idempotent matrices**

We begin this section with some rank equalities for the difference and for the sum of two idempotent matrices, and then consider various consequences. The ranks of $P \pm Q$ when both $P$ and $Q$ are idempotent were studied by Groß and Trenkler [9], where they also considered nonsingularity of the difference of two idempotent matrices; here we will present a more complete discussion of this topic.

**Theorem 2.1.** Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the difference $P - Q$ satisfies the following rank equalities:

\[
\begin{align*}
 r(P - Q) &= r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P, Q] - r(P) - r(Q) \\
 &= r(P - PQ) + r(PQ - Q) \\
 &= r(P - QP) + r(QP - Q).
\end{align*}
\]
Proof. It is easy to see by block Gaussian elimination that
\[
\begin{bmatrix}
-P & 0 & P \\
0 & Q & Q \\
P & Q & 0
\end{bmatrix}
= \begin{bmatrix}
-P & 0 & 0 \\
0 & Q & 0 \\
0 & 0 & P - Q
\end{bmatrix}
= r(P) + r(Q) + r(P - Q).
\]
On the other hand, since \(P^2 = P\) and \(Q^2 = Q\) it is also easy to find by block Gaussian elimination that
\[
\begin{bmatrix}
-P & 0 & P \\
0 & Q & Q \\
P & Q & 0
\end{bmatrix}
= \begin{bmatrix}
-P & 0 & P \\
0 & Q & 0 \\
P & Q & 0
\end{bmatrix}
= r\left[\begin{bmatrix} P \\ Q \end{bmatrix}\right] + r[P, Q].
\]
Combining the above two equalities yields (2.1). Then applying (1.1) and (1.2), respectively, to \([P, Q]\) in (2.1) yields
\[
\begin{align*}
r[P, Q] &= r(P) + r(Q - P Q), \\
r[P, Q] &= r(Q) + r(P - Q P), \\
r[P, Q] &= r(P) + r(Q - P Q), \\
r[P, Q] &= r(Q) + r(P - Q P).
\end{align*}
\]
Substitution in (2.1) yields (2.2) and (2.3). □

Some direct consequences of (2.1)–(2.3) are given in the following corollary. The result (d) is due to Hartwig and Styan [11] and (e), to Groß and Trenkler [9].

Corollary 2.2. Let \(P, Q \in \mathbb{C}^{m \times m}\) be any two idempotent matrices. Then:
(a) If \(P Q = 0\) or \(Q P = 0\), then \(r(P - Q) = r(P) + r(Q)\).
(b) If \(P Q = 0\), then \(r(P - Q P) + r(Q P - Q) = r(P) + r(Q)\).
(c) If \(Q P = 0\), then \(r(P - Q Q) + r(P Q - Q) = r(P) + r(Q)\).
(d) \(r(P - Q) = r(P) - r(Q) \iff P Q P = Q \iff R(Q) \subseteq R(P) \text{ and } R(Q^*) \subseteq R(P^*)\).
(e) The difference \(P - Q\) is nonsingular if and only if
\[
r[P, Q] = r(P, Q) = r(P) + r(Q) = m,
\]
or equivalently \(R(P) \oplus R(Q) = R(P^*) \oplus R(Q^*) = \mathbb{C}^m\).
We note that when $P$ is idempotent, then $I_m - P$ is also idempotent. Replacing $P$ in (2.1)–(2.3) by $I_m - P$ yields the following corollary.

**Corollary 2.3.** Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the rank of $I_m - P - Q$ satisfies the rank equalities

$$
\begin{align*}
 r(I_m - P - Q) &= r(PQ) + r(QP) - r(P) - r(Q) + m \\
 &= r(I_m - P - Q + PQ) + r(PQ) \\
 &= r(I_m - P - Q + QP) + r(QP).
\end{align*}
$$

Furthermore,

(a) $P + Q = I_m \iff PQ = QP = 0$ and $r(P + Q) = r(P) + r(Q) = m$.

(b) $I_m - P - Q$ is nonsingular if and only if $r(PQ) = r(QP) = r(P) = r(Q)$.

**Proof.** Replacing $P$ in (2.1) by $I_m - P$ yields

$$
r(I_m - P - Q) = r\begin{bmatrix} I_m - P \\ Q \end{bmatrix} + r[I_m - P, Q] - r(I_m - P) - r(Q). \tag{2.7}
$$

By (1.1) and (1.2), it follows that

$$
r[I_m - P, Q] = r(I_m - P) + r[Q - (I_m - P)Q] = m - r(P) + r(PQ)
$$

and

$$
r\begin{bmatrix} I_m - P \\ Q \end{bmatrix} = r(I_m - P) + r[Q - Q(I_m - P)] = m - r(P) + r(QP).
$$

Substitution in (2.7) yields (2.4). On the other hand, replacing $P$ in (2.2) by $I_m - P$ produces

$$
r[(I_m - P) - Q] = r[(I_m - P) - Q] + r((I_m - P)Q - Q)
$$

$$
= r(I_m - P - Q + PQ) + r(QP)
$$

establishing (2.5); the rank equality (2.6) follows similarly. The results in (a) and (b) are direct consequences of (2.4)–(2.6). □

**Theorem 2.4.** Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the sum $P + Q$ satisfies the rank equalities

$$
\begin{align*}
 r(P + Q) &= r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} - r(Q) = r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} - r(P) \\
 &= r(P - PQ + P Q) + r(Q) \\
 &= r(Q - PQ + QP) + r(P).
\end{align*}
$$

Furthermore,

(a) If $PQ = QP$, then $r(P + Q) = r(P - PQ) + r(Q) = r(Q - PQ) + r(P)$.

(b) If $PQ = 0$ or $QP = 0$, then $r(P + Q) = r(P) + r(Q)$. 

Our proof of Theorem 2.4 is essentially identical to that of Theorem 2.1 and so it is omitted.

We now use (2.8) to find some necessary and sufficient conditions for the sum \( P + Q \) to be nonsingular.

**Corollary 2.5.** Let \( P, Q \in \mathbb{C}^{m \times m} \) be any two idempotent matrices. Then the following five statements are equivalent:

(a) The sum \( P + Q \) is nonsingular.

(b) \( r \left[ \begin{array}{c} P \\ Q \\ 0 \end{array} \right] = m \) and \( R \left[ \begin{array}{c} P \\ Q \\ 0 \end{array} \right] \cap R \left[ \begin{array}{c} Q \\ 0 \end{array} \right] = \{0\}. \)

(c) \( r[P, Q] = m \) and \( R \left[ \begin{array}{c} P^* \\ Q^* \\ 0 \end{array} \right] \cap R \left[ \begin{array}{c} Q^* \\ 0 \end{array} \right] = \{0\}. \)

(d) \( r \left[ \begin{array}{c} Q \\ P \\ 0 \end{array} \right] = m \) and \( R \left[ \begin{array}{c} Q \\ P \\ 0 \end{array} \right] \cap R \left[ \begin{array}{c} P \\ 0 \end{array} \right] = \{0\}. \)

(e) \( r[Q, P] = m \) and \( R \left[ \begin{array}{c} Q^* \\ P^* \\ 0 \end{array} \right] \cap R \left[ \begin{array}{c} P^* \\ 0 \end{array} \right] = \{0\}. \)

**Proof.** From (2.8) we see that the sum \( P + Q \) is nonsingular if and only if

\[
r \left[ \begin{array}{c} P \\ Q \\ 0 \end{array} \right] = r(Q) + m \quad \text{or} \quad r \left[ \begin{array}{c} Q \\ P \\ 0 \end{array} \right] = r(P) + m.
\]

(2.11)

Combining (2.11) with the following inequalities:

\[
r \left[ \begin{array}{c} P \\ Q \\ 0 \end{array} \right] \leq r \left[ \begin{array}{c} P \\ Q \\ 0 \end{array} \right] + r \left[ \begin{array}{c} Q \\ 0 \end{array} \right] \leq m + r(Q),
\]

\[
r \left[ \begin{array}{c} P \\ Q \\ 0 \end{array} \right] \leq r[P, Q] + r[Q, 0] \leq m + r(Q),
\]

\[
r \left[ \begin{array}{c} Q \\ P \\ 0 \end{array} \right] \leq r \left[ \begin{array}{c} Q \\ P \\ 0 \end{array} \right] + r \left[ \begin{array}{c} P \\ 0 \end{array} \right] \leq m + r(P),
\]

\[
r \left[ \begin{array}{c} Q \\ P \\ 0 \end{array} \right] \leq r[Q, P] + r[P, 0] \leq m + r(P),
\]

yields (a)–(e) and our proof is complete. \( \square \)

Replacing \( P \) and \( Q \) in (2.8) by \( I_m - P \) and \( I_m - Q \), respectively, we obtain the following corollary.
Corollary 2.6. Let \( P, Q \in \mathbb{C}^{m \times m} \) be any two idempotent matrices. Then

(a) \( r(I_m + P - Q) = r(QPQ) - r(Q) + m \).

(b) \( r(2I_m - P - Q) = r(Q - QPQ) - r(Q) + m = r(P - PQP) - r(P) + m \).

(c) \( r(I_m + P - Q) = m \iff r(QPQ) = r(Q) \).

(d) \( r(2I_m - P - Q) = m \iff r(P - PQP) = r(P) \iff r(Q - QPQ) = r(Q) \).

We now present some rank equalities for the commutator \( PQ - QP \), where \( P \) and \( Q \) are idempotent; these rank equalities will lead us to some new necessary and sufficient conditions for the commutativity of \( P \) and \( Q \); see also [2].

Theorem 2.7. Let \( P, Q \in \mathbb{C}^{m \times m} \) be any two idempotent matrices. Then the rank of the commutator

\[
r(PQ - QP) = r(P - Q) + r(I_m - P - Q) - m
\]

\[
= r(P - Q) + r(PQ) + r(QP) - r(P) - r(Q)
\]

\[
= r(QP) + r(QP) + r(PQ) - 2r(P) - 2r(Q)
\]

\[
= r(P - PQ) + r(QP - Q) + r(QP) + r(PQ) - r(P) - r(Q)
\]

Furthermore, if both \( P \) and \( Q \) are Hermitian idempotent, then

\[
r(PQ - QP) = 2[r[P, Q] + r(PQ) - r(P) - r(Q)].
\]

Proof. Since \( PQ - QP = (P - Q)(P + Q - I_m) \), we have

\[
r(PQ - QP) = r\begin{bmatrix} I_m & P + Q - I_m \\ P - Q & 0 \end{bmatrix} - m.
\]

On the other hand, it is easy to verify by block Gaussian elimination that

\[
r\begin{bmatrix} I_m & P + Q - I_m \\ P - Q & 0 \end{bmatrix} = r(P - Q) + r(I_m - P - Q)
\]

and so the first equality is established. Substituting (2.4) yields the second equality, substituting (2.1) the third, and (2.2) and (2.3) the fourth and fifth equalities. \( \square \)

The first equality in Theorem 2.7 reveals an interesting relationship among the three matrices \( PQ - QP, P - Q \) and \( I_m - P - Q \). This leads at once to the following three corollaries.
Corollary 2.8. Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the following five statements are equivalent:

(a) $PQ = QP$.

(b) $r(P - Q) + r(I_m - P - Q) = m$.

(c) $r(P - Q) = r(P) + r(Q) - r(PQ)$.

(d) $r(P - PQ) = r(P) - r(PQ)$ and $r(Q - PQ) = r(Q) - r(PQ)$.

(e) $r(P - QP) = r(P) - r(QP)$ and $r(Q - QP) = r(Q) - r(QP)$.

Corollary 2.9. Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the following three statements are equivalent:

(a) $r(PQ - QP) = r(P - Q)$.

(b) $r(PQ) = r(QP) = r(P) = r(Q)$.

(c) $I_m - P - Q$ is nonsingular.

Proof. It is easy to see that (a) $\iff$ (b) follows from the second equality in Theorem 2.7, and (b) $\iff$ (c) from Corollary 2.3(b).

Corollary 2.10. Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the following three statements are equivalent:

(a) $PQ - QP$ is nonsingular.

(b) Both $P - Q$ and $I_m - P - Q$ are nonsingular.

(c) $R(P) \oplus R(Q) = R(P^*) \oplus R(Q^*) = \mathbb{C}^m$ and $r(PQ) = r(QP) = r(P) = r(Q)$.

Proof. It is easy to see that (a) $\iff$ (b) follows from the first equality in Theorem 2.7, and (b) $\iff$ (c) from Theorem 2.2(e) and Corollary 2.3(b).

Cochran’s Theorem (see, e.g., [1,12]) tells us that the sum $P + Q$ of two idempotent matrices $P$ and $Q$ is also idempotent if and only if $PQ = QP = 0$, and then rank is additive: $r(P + Q) = r(P) + r(Q)$. [To see that this holds without the matrices being Hermitian we note first that $(P + Q)^2 - (P + Q) = PQ + QP = 0$ by $P$ yields $PQ + QP = 0$ and so $QP = PQ$; post-multiplying $PQ + QP = 0$ by $P$ yields $PQP + QP = 0$, and so $PQP = QP = PQ$. Hence $QP = 0 = PQ$.]

The difference $P - Q$ of two idempotent matrices $P$ and $Q$ is idempotent if and only if $PQP = QP$ or equivalently $r(P - Q) = r(P) - r(Q)$, i.e., rank is subtractive and $P$ is said to be above $Q$ in the minus order, see, e.g., [5,10,11].

These results motivate us to consider the rank of $(P + Q)^2 - (P + Q)$ and the rank of $(P - Q)^2 - (P - Q)$ when $P$ and $Q$ are idempotent. Based on the following well-known rank formula for $A \in \mathbb{C}^{m \times m}$, see, e.g., [1]:

$$r(A^2 - A) = r(A) + r(A - I_m) - m \tag{2.12}$$

and the above several theorems and corollaries, we are now able to establish the following theorem, see also our Theorem 2.7 above.
Theorem 2.11. Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the sum $PQ + QP$ satisfies the four rank equalities

$$r(PQ + QP) = r(P + Q) + r(I_m - P - Q) - m$$
$$= r(P + Q) + r(PP) + r(QP) - r(P) - r(Q)$$
$$= r(P - PQ - QP + QP) + r(PP) + r(QP) - r(P)$$
$$= r(P - PQ - QP + QP) + r(PP) + r(QP) - r(Q).$$

Proof. Applying (2.12) to $(P + Q)^2 - (P + Q)$, we obtain the first equality. Then using (2.4), yields the second equality. The other equalities follow from Theorem 2.4. $\Box$

Combining the first rank equality for $r(PQ + QP)$ in Theorem 2.11 with the first rank equality for $r(PQ - QP)$ in Theorem 2.7 yields the following interesting rank formula:

$$r(P + Q) + r(PQ - QP) = r(P - Q) + r(PQ + QP). \quad (2.13)$$

Corollary 2.12. Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the following four statements are equivalent:

(a) $r(PQ + QP) = r(P + Q)$.
(b) $r(I_m - P - Q) = m$.
(c) $r(P) = r(Q)$.
(d) $r(PQ - QP) = r(P - Q)$. 

Proof. It is easy to see that (a) $\Leftrightarrow$ (b) follows from Theorem 2.11 and (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) from Corollary 2.9. $\Box$

Corollary 2.13. Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the following two statements are equivalent:

(a) $PQ + QP$ is nonsingular.
(b) $P + Q$ and $I_m - P - Q$ are nonsingular.

Proof. Corollary 2.13 follows directly from Theorem 2.11. $\Box$

Corollary 2.14. Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then

$$r((P - Q)^2 - (P - Q)) = r(I_m - P + Q) + r(P - Q) - m$$
$$= r(PP) - r(P) + r(P - Q).$$

Proof. The first equality follows at once from (2.12) and the second, from Corollary 2.6(a). $\Box$
These results easily lead to the following corollary, due to Hartwig and Styan [11].

**Corollary 2.15.** Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then the following four statements are equivalent:

(a) $P - Q$ is idempotent.
(b) $r(I_m - P + Q) = m - r(P - Q)$.
(c) $PQ P = Q$.
(d) $r(P - Q) = r(P) - r(Q)$.
(e) $R(Q) \subseteq R(P)$ and $R(Q^*) \subseteq R(P^*)$.

We now present rank equalities for $I_m - PQ$ and for $PQ - (PQ)^2$ and then use these to characterize the idempotency of $PQ$.

**Theorem 2.16.** Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then $I_m - PQ$ satisfies the rank equality

$$r(I_m - PQ) = r((2I_m - P - Q)).$$

**Proof.** Since rank is additive on the Schur complement, see, e.g., [12],

$$r \begin{bmatrix} I_m & I_m - PQ \\ Q & 0 \end{bmatrix} = m + r(Q - PQ);$$

(2.14)

using elementary row and column (block) operations we see that

$$r \begin{bmatrix} I_m & I_m - PQ \\ Q & 0 \end{bmatrix} = r \begin{bmatrix} I_m - PQ & I_m - PQ \\ Q & 0 \end{bmatrix} = r(Q) + r(I_m - PQ).$$

Thus

$$r(I_m - PQ) = r(Q - PQ) - r(Q) + m.$$  

(2.15)

Substituting (b) from Theorem 2.6 into (2.15) yields (2.16). □

Replacing $P$ and $Q$ in (2.14) by $I_m - P$ and $I_m - Q$ gives the following equality:

$$r(P + Q) = r(P + Q - PQ) = r(P + Q - PQ).$$

**Corollary 2.17.** Let $P, Q \in \mathbb{C}^{m \times m}$ be any two idempotent matrices. Then

$$r[PQ - (PQ)^2] = r(2I_m - P - Q) + r(PQ) - m.$$  

(2.16)

**Proof.** Applying (2.12) to $PQ - (PQ)^2$ gives

$$r[PQ - (PQ)^2] = r(I_m - PQ) + r(PQ) - m.$$  

(2.17)

Substituting (2.14) in (2.17) yields (2.16). □

Corollary 2.17 implies that the product $PQ$ of two idempotent matrices is idempotent if and only if $r(2I_m - P - Q) = m - r(PQ)$. Of course, there are several
other known equivalent conditions for the product \( PQ \) to be idempotent, see, e.g., [7,16].

We note that when the matrix \( P \) is idempotent then so is its conjugate transpose \( P^* \). Therefore:

**Corollary 2.18.** Let \( P \in \mathbb{C}^{m\times m} \) be an idempotent matrix. Then:

(a) \( r(P - P^*) = 2r[P, P^*] - 2r(P) \).

(b) \( r(I_m - P - P^*) = r(I_m + P - P^*) = m. \)

(c) \( r(P + P^*) = r(PP^* + P^*P) = r[P, P^*] \).

(d) \( R(P) \subseteq R(P + P^*) \) and \( R(P^*) \subseteq R(P + P^*) \).

(e) \( r(P^* - P^*P) = r(P - P^*) \).

(f) \( r(I_m - P^*P) = r(2I_m - P - P^*) \).

**Proof.** Part (a) follows from (2.1), (b) from (2.4) and Corollary 2.6(a), and (c) and (d) from (2.13); to prove (e) we use (a) from Corollary 2.9 and (b) here. Finally, (f) here follows from Theorem 2.16. \( \square \)

The results in the above theorems and corollaries can easily be extended to scalar-potent matrices, i.e., matrices with the property \( P^2 = \lambda P \) and \( Q^2 = \mu Q \), where \( \lambda \neq 0 \) and \( \mu \neq 0 \). In fact

\[
\left( \frac{1}{\lambda} P \right)^2 = \frac{1}{\lambda^2} P^2 = \frac{1}{\lambda} P \quad \text{and} \quad \left( \frac{1}{\mu} Q \right)^2 = \frac{1}{\mu^2} Q^2 = \frac{1}{\mu} Q
\]

and so \( P/\lambda \) and \( Q/\mu \) are idempotent. We obtain, e.g., the following rank equalities for scalar-potent matrices:

\[
r(\mu P - \lambda Q) = r \begin{bmatrix} P & Q \end{bmatrix} + r[P, Q] - r(P) - r(Q),
\]

\[
r(\mu P + \lambda Q) = r \begin{bmatrix} P & Q \end{bmatrix} - r(Q) = r \begin{bmatrix} Q & P \end{bmatrix} + r(P),
\]

\[
r(\lambda \mu I_m - \mu P - \lambda Q) = r(PQ) + r(QP) - r(P) - r(Q) + m,
\]

\[
r(PQ - QP) = r(\mu P - \lambda Q) + r(\lambda \mu I_m - \mu P - \lambda Q) - m,
\]

\[
r(PQ + QP) = r(\mu P + \lambda Q) + r(\lambda \mu I_m - \mu P - \lambda Q) - m,
\]

\[
r(\lambda \mu I_m - PQ) = r(2\lambda \mu I_m - \mu P - \lambda Q).
\]

We end this section with some rank equalities for the matrix difference \( PA - AQ \), with both \( P \) and \( Q \) idempotent. These equalities allow us to study the commutativity of an idempotent matrix \( P \) with an arbitrary matrix \( A \).
**Theorem 2.19.** Let \( A \in \mathbb{C}^{m \times n} \) be given and let \( P \in \mathbb{C}^{m \times m} \) and \( Q \in \mathbb{C}^{n \times n} \) be idempotent. Then the matrix difference \( PA - AQ \) satisfies the two rank equalities

\[
r(PA - AQ) = r \begin{bmatrix} PA \\ Q \end{bmatrix} + r[AQ, P] - r(P) - r(Q) \tag{2.18}
\]

\[
= r(PA - PAQ) + r(PAQ - AQ). \tag{2.19}
\]

Furthermore,

(a) If \( PAQ = 0 \), then \( r(PA - AQ) = r(PA) + r(AQ) \).

(b) \( PA = AQ \iff R(AQ) \subseteq R(P) \) and \( R[(PA)^*] \subseteq R(Q^*) \).

(c) If \( A \in \mathbb{C}^{m \times m} \), then \( PA = AP \iff R(AP) \subseteq R(P) \) and \( R[(PA)^*] \subseteq R(P^*) \).

**Proof.** It is easy to see by block Gaussian elimination that

\[
r \begin{bmatrix} -P & 0 & PA \\ 0 & Q & Q \\ P & AQ & 0 \end{bmatrix} = r(P) + r(Q) + r(PA - AQ) \tag{2.20}
\]

and

\[
r \begin{bmatrix} -P & 0 & PA \\ 0 & Q & Q \\ P & AQ & 0 \end{bmatrix} = r \begin{bmatrix} 0 & PAQ & PA \\ 0 & Q & Q \\ P & AQ & 0 \end{bmatrix} = r \begin{bmatrix} 0 & 0 & PA \\ 0 & 0 & Q \\ P & AQ & 0 \end{bmatrix} = r \begin{bmatrix} PA \\ Q \end{bmatrix} + r[AQ, P]. \tag{2.21}
\]

Combining (2.20) and (2.21) yields (2.18). Applying (1.1) and (1.2) to \([AQ, P]\) and \([PA]\) in (2.18), respectively, yields (2.19). The results in (b) and (c) are natural consequences of (2.18) and (2.19). \(\square\)

Applying Theorem 2.19 to \( PQ - QP \), with both \( P \) and \( Q \) idempotent, leads to some new formulas for the rank of \( PQ - QP \); we notice that \( P \) and \( Q \) are again interchangeable.

**Corollary 2.20.** Let \( P, Q \in \mathbb{C}^{m \times m} \) be two idempotent matrices. Then

\[
r(PQ - QP) = r \begin{bmatrix} PQ \\ P \end{bmatrix} + r[QP, P] - 2r(P)
\]

\[
= r \begin{bmatrix} QP \\ Q \end{bmatrix} + r[QP, Q] - 2r(Q)
\]

\[
= r(PQ - PQP) + r(PQP - QP)
\]

\[
= r(QP - QPQ) + r(QPQ - PQ) = r(QP - PQ).
\]
When both $P$ and $Q$ are Hermitian idempotent, then
\[ r(PQ - QP) = 2r(PQ - PQP) = 2r(QP - QPQ) \]
as established by Bérubé et al. [6]; Puntanen [14, p. 12], see also [15, Theorem 3.4.1, p. 34], showed that with both $P$ and $Q$ Hermitian idempotent
\[ r(PQ - PQP) = r(PQ) + r(Q - PQ) - r(Q) \]
\[ = r(QP) + r(P - PQ) - r(P). \] (2.22)
The rank equality (2.22) is of interest in the analysis of the Gauß–Markov linear statistical model \{y, X\beta, V\}; there $r(PQ - PQP)$ gives the number of unit canonical correlations between the ordinary least-squares fitted values and the residuals, with $P = XX^\dagger$ and $Q = VV^\dagger$; here the superscript $^\dagger$ denotes the Moore–Penrose inverse and so $P$ and $Q$ are the orthogonal projectors onto the range spaces $R(X)$ and $R(V)$, respectively. For further details see [14,15], as well as [4,6].

**Corollary 2.21.** Let $P, Q \in \mathbb{C}^{m \times m}$ be two idempotent matrices. Then
\[ r(P - Q) = r((P - PQ) + \alpha(QP - Q)) \] (2.23)
\[ = r([Q - PQ] + \alpha(QP - P)) \] (2.24)
holds for all $\alpha \in \mathbb{C}$ with $\alpha \neq 0$. In particular,
\[ r(P - Q) = r(P + Q - 2PQ) = r(P + Q - 2QP). \] (2.25)

**Proof.** We note that for all $\alpha \in \mathbb{C}$
\[ P - PQ + \alpha(PQ - Q) = P(P + \alpha Q) - (P + \alpha Q)Q. \]
Then, using (2.18), it follows that
\[ r(P - PQ + \alpha(QP - Q)) = r \left[ \begin{array}{c} P(P + \alpha Q) \\ Q \end{array} \right] \]
\[ + r \left[ (P + \alpha Q)Q, P \right] - r(P) - r(Q) \]
\[ = r \left[ \begin{array}{c} P \\ Q \end{array} \right] + r[\alpha Q, P] - r(P) - r(Q) \]
\[ = r \left[ \begin{array}{c} P \\ Q \end{array} \right] + r[P, Q] - r(P) - r(Q) \]
provided $\alpha \neq 0$. Combining this last equality with (2.1) yields (2.23); reversing $P$ and $Q$ yields (2.24). Putting $\alpha = -1$ in (2.23) and (2.24) yields (2.25). □
Replacing $P$ by $I_m - P$ in (2.23), we obtain the following corollary.

**Corollary 2.22.** Let $P, Q \in \mathbb{C}^{m \times m}$ be two idempotent matrices. Then

$$r(I_m - P - Q + \alpha PQ) = r(I_m - P - Q)$$

holds for all $\alpha \in \mathbb{C}$ with $\alpha \neq 1$.

### 3. Rank equalities for involutory matrices

A matrix $A \in \mathbb{C}^{m \times m}$ is said to be involutory if its square is the identity matrix, i.e., $A^2 = I_m$, see, e.g., [13, pp. 113, 325, 339, 485] and [19, Section 4.1]. In fact an involutory matrix is a nonsingular tripotent matrix. It is well known that involutory matrices and idempotent matrices are closely linked. Indeed, for any involutory matrix $A$, the two matrices $\frac{1}{2}(I_m + A)$ and $\frac{1}{2}(I_m - A)$ are idempotent. On the other hand, for any idempotent matrix $P$, the two matrices $\pm (I_m - 2P)$ are involutory. We may, therefore, extend all our results in Section 2 on idempotent matrices to involutory matrices. For example, we have the following theorem.

**Theorem 3.1.** Let $A, B \in \mathbb{C}^{m \times m}$ be two involutory matrices. Then the rank of $A + B$ satisfies the two equalities

$$r(A + B) = r\left[\begin{bmatrix} I_m + A \\ I_m - B \end{bmatrix}\right] + r[I_m + A, I_m - B]$$

$$- r(I_m + A) - r(I_m - B), \quad (3.1)$$

$$r(A + B) = r[(I_m + A)(I_m + B)] + r[(I_m - A)(I_m - B)]. \quad (3.2)$$

**Proof.** Since both $P = \frac{1}{2}(I_m + A)$ and $Q = \frac{1}{2}(I_m - B)$ are idempotent when $A$ and $B$ are involutory, we have

$$r(P - Q) = r[\frac{1}{2}(I_m + A) - \frac{1}{2}(I_m - B)] = r(A + B),$$

and

$$r\left[\begin{bmatrix} P \\ Q \end{bmatrix}\right] + r[P, Q] - r(P) - r(Q) = r\left[\begin{bmatrix} I_m + A \\ I_m - B \end{bmatrix}\right] + r[I_m + A, I_m - B]$$

$$- r(I_m + A) - r(I_m - B).$$

Substituting these equalities in (2.1) gives (3.1). Furthermore we have

$$r(P - PQ) = r[(I_m + A)(I_m - \frac{1}{2}(I_m - B))] = r[(I_m + A)(I_m + B)],$$

$$r(PQ - Q) = r[(\frac{1}{2}(I_m + A) - I_m)(I_m - B)] = r[(I_m - A)(I_m - B)].$$

Substituting these equalities in (3.1) yields (3.2). □
We note that when \( B \) is involutory, then \(-B\) is also involutory. Thus replacing \( B \) by \(-B\) in (3.1) and (3.2) yields two rank equalities for the difference \( A - B \) of two involutory matrices.

**Theorem 3.2.** Let \( A, B \in \mathbb{C}^{m \times m} \) be two involutory matrices. Then \( A + B \) and \( A - B \) satisfy the rank equalities

\[
\begin{align*}
r(A + B) &= r[(I_m + A)(I_m + B)] + r[(I_m + B)(I_m + A)] \\
&\quad - r(I_m + A) - r(I_m + B) + m, \quad (3.3) \\
r(A - B) &= r[(I_m + A)(I_m - B)] + r[(I_m - B)(I_m + A)] \\
&\quad - r(I_m + A) - r(I_m - B) + m. \quad (3.4)
\end{align*}
\]

**Proof.** Putting \( P = \frac{1}{2}(I_m + A) \) and \( Q = \frac{1}{2}(I_m + B) \) in (2.4) and simplifying yields (3.3). Replacing \( B \) by \(-B\) in (3.3) yields (3.4). \( \square \)

Combining (3.2) with (3.3) yields this interesting rank equality for involutory matrices \( A \) and \( B \):

\[
r[(I_m + B)(I_m + A)] = r(I_m + B) + r(I_m + A) - m \\
+ r[(I_m - A)(I_m - B)].
\]

**Theorem 3.3.** Let \( A, B \in \mathbb{C}^{m \times m} \) be two involutory matrices. Then

\[
r(AB - BA) = r(A + B) + r(A - B) - m, \quad (3.5)
\]

and so

\[
AB = BA \iff r(A + B) + r(A - B) = m. \quad (3.6)
\]

**Proof.** Putting \( P = \frac{1}{2}(I_m + A) \) and \( Q = \frac{1}{2}(I_m - B) \) in the first equality of Theorem 2.7 and simplifying yields (3.5). \( \square \)

Substituting (3.1)–(3.4) into (3.5) may yield some further rank equalities for \( AB - BA \). We plan to explore this in our future research.

**Theorem 3.4.** Let \( A \in \mathbb{C}^{m \times m} \) be an involutory matrix. Then

\[
r(A - A^*) = 2r[I_m + A, I_m + A^*] - 2r(I_m + A) \quad (3.7)
\]

\[
= 2r[I_m - A, I_m - A^*] - 2r(I_m - A), \quad (3.8)
\]

\[
r(A + A^*) = m, \quad (3.9)
\]

\[
r(AA^* - A^*A) = r(A - A^*). \quad (3.10)
\]
Proof. Putting $P = \frac{1}{2}(I_m \pm A)$ and $P^* = \frac{1}{2}(I_m \pm A^*)$ in Corollary 2.18 and simplifying yields (3.7)–(3.10). □

**Theorem 3.5.** Let $A \in \mathbb{C}^{m \times m}$ and $B \in \mathbb{C}^{n \times n}$ be two involutory matrices and $X \in \mathbb{C}^{m \times n}$. Then $AX - XB$ satisfies the rank equalities

$$
\begin{align*}
\text{r}(AX - XB) &= r \left[ \frac{(I_m + A)X}{I_n + B} \right] + r[X(I_n + B), I_m + A] \\
&\quad - r(I_m + A) - r(I_n + B), \quad (3.11) \\
&= r[(I_m + A)X(I_n - B)] + r[(I_m - A)X(I_n + B)]. \quad (3.12)
\end{align*}
$$

In particular,

$$
AX = XB \iff (I_m + A)X(I_n - B) = 0 \text{ and } (I_m - A)X(I_n + B) = 0.
$$

Proof. Putting $P = \frac{1}{2}(I_m + A)$ and $Q = \frac{1}{2}(I_n + B)$ in (2.18) and (2.19) yields (3.11) and (3.12). □

4. Conclusions and further extensions

In this paper we have established a variety of rank equalities for idempotent matrices. From these rank equalities we have obtained many new properties for sums, differences and products of two idempotent matrices. The product of two real symmetric idempotent matrices (orthogonal projectors) has been studied in multivariate statistical analysis since its eigenvalues are the squares of canonical correlations, see, e.g., [14,15].

We have applied our rank equalities to involutory matrices. There are also other applications in matrix theory. For example, from (2.1), we have for $A \in \mathbb{C}^{m \times m}$

$$
r(AA^\dagger - A^\dagger A) = 2r[A, A^*] - 2r(A), \quad (4.1)
$$

where $A^\dagger$ is the Moore–Penrose inverse. We may use (4.1) to characterize the commutativity of $A$ and $A^\dagger$.

Using (2.18), we see that

$$
r(A^k A^\dagger - A^\dagger A^k) = r \left[ \frac{A^k}{A^*} \right] + r[A^k, A^*] - 2r(A),
$$

which may be used to characterize the commutativity of $A^k$ and $A^\dagger$. Other rank equalities for a matrix and its Moore–Penrose inverse can also be derived in this way; these results appear in the first author’s paper [18].
Again from (2.1), we also find that
\[ \max_{A} r(AA - A^2 A) = \min_{A} \{2m - 2r(A), 2r(A)\} \]
and
\[ \min_{A} r(AA - A - A^2) = 2r(A) - 2r(A^2). \]
These rank equalities help characterize the nonsingularity and the rank invariance of the commutator \( AA - A^2 A \), as well as the commutativity of \( A \) and \( A^2 \). In addition, we can also determine from (2.1) and (2.18) the maximal and minimal ranks of \( A^k A - A^2 A^k \), \( AA - B - B \), and so on. We will present these results in a further paper.

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