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Algebraic Nijenhuis operators and Kronecker Poisson pencils

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Abstract

We give a criterion of (micro-)kroneckerity of the linear Poisson pencil on $\mathfrak{g}^*$ related to an algebraic Nijenhuis operator $N:\mathfrak{g}\to\mathfrak{g}$ on a finite-dimensional Lie algebra $\mathfrak{g}$. As an application we get a series of examples of completely integrable systems on semisimple Lie algebras related to Borel subalgebras and a new proof of the complete integrability of the free rigid body system on $\mathfrak{gl}_n$.

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0. Introduction

This paper is devoted to a method of constructing completely integrable systems based on the micro-local theory of bihamiltonian structures [2,7–10,18,23]. The main tool are the so-called micro-Kronecker bihamiltonian structures [23], which will be called Kronecker in this paper for short (in [10] the term Kronecker was used for the micro-Kronecker structures with some additional condition of "flatness" which will not be essential in this paper).

A Kronecker bihamiltonian structure on a manifold M is a Poisson pencil $\{s_1\theta_1+s_2\theta_2\}_{(s_1,s_2)\in\mathbb{K}^2}$, i.e. a two-dimensional linear space over a base field $\mathbb{K}$ in the set of all Poisson structures on M, satisfying an additional condition of the constancy of rank: $\operatorname{rank}_{\mathbb{C}}\theta^s=\text{const}$, $s:=(s_1,s_2)\in\mathbb{C}^2\setminus(0,0)$, $\theta^s:=s_1\theta_1+s_2\theta_2$ (in the real case we should pass to the complexification of the pencil). The kroneckerity condition is important due to the fact that it automatically implies the existence (at least locally) on M of the complete involutive with respect to any bivector θ^s set of functions. This set is functionally generated by the Casimir functions of the bivectors θ^s (see Proposition 2.4). Geometrically this set corresponds to the intersection over $s\in\mathbb{K}^2\setminus\{(0,0)\}$ of all symplectic leaves of maximal dimension of the Poisson structures θ^s and the completeness of this set reflects the fact that this intersection is lagrangian in any fixed symplectic leaf (see [8,18]).

The main result of this paper (Theorem 2.5) gives a criterion of kroneckerity for the Poisson pencils related to diagonalizable algebraic Nijenhuis operators. An algebraic Nijenhuis operator N on a Lie algebra $\mathfrak{g}$ (see [4,12], for

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example) is a linear operator $N : \mathfrak{g} \to \mathfrak{g}$ with the condition of the vanishing of the so-called Nijenhuis torsion (see Definition 1.1). Given a linear operator $N : (\mathfrak{g}, [,]) \to (\mathfrak{g}, [,])$, the condition of vanishing of its Nijenhuis torsion guarantees that the infinithesimal part $[,]_N$ of the trivial deformation $(\mathrm{Id} + \lambda N)^{-1}[(\mathrm{Id} + \lambda N)\cdot, (\mathrm{Id} + \lambda N)\cdot]$ of the Lie bracket $[,]$ is again a Lie bracket. This new Lie bracket $[,]_N$ is automatically compatible with $[,]$, thus any Nijenhuis operator N "produces" the pencil of Lie brackets $[,]^s := s_1[,] + s_2[,]_N$ and, consequently, the corresponding pencil $\{\theta_N^s\}_{s\in\mathbb{K}^2}$ of the Lie–Poisson structures on $\mathfrak{g}^*$.

Let us look more closely at the problem of the kroneckerity of the Poisson pencil $\{\theta_N^s\}$. It can be shown (Proposition 1.2) that if N is Nijenhuis, $(N - \lambda\,\mathrm{Id})^{-1}[(N - \lambda\,\mathrm{Id})\cdot, (N - \lambda\,\mathrm{Id})\cdot] = [\cdot,\cdot]_N - \lambda[\cdot,\cdot]$. In particular, all the Lie brackets $[,]^s$ are isomorphic to $[,]$ except those corresponding to $s = (s_1, s_2)$ with $\lambda = -s_1/s_2$ belonging to $\mathrm{Sp}\,N$, the spectrum of N. Thus the problem of kroneckerity of $\{\theta_N^s\}$ (modulo some not very restrictive assumption on the codimension of the set of singular coadjoint orbits of $\mathfrak{g}$, see (2.5.1)) reduces to the problem of calculating the dimension of the coadjoint orbits of the exceptional brackets $[,]_N - \lambda_i[,]$, $i = 1, \dots, n$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of N. In fact, due to the semicontinuity of the function $\mathrm{rank}\,\theta^s$, in order to prove the kroneckerity it is sufficient to find for any i a particular coadjoint orbit O_i of a Lie bracket $[,]_N - \lambda_i[,]$ such that $\dim O_i = \dim O$, where O is the generic coadjoint orbit of $[,]$.

One possibility of finding the orbits O_i is the following. If N is a Nijenhuis operator, then $N : (\mathfrak{g}, [,]_N) \to (\mathfrak{g}, [,])$ is a homomorphism of Lie algebras [12]. Hence $\mathrm{im}\,N$ is a subalgebra of $(\mathfrak{g}, [,])$ and we have the Poisson inclusion ${}^tN : ((\mathrm{im}\,N)^*, \theta_{\mathrm{st}}) \hookrightarrow (\mathfrak{g}^*, \theta_N)$, where θ_{st} is the standard Lie–Poisson structure on $(\mathrm{im}\,N)^*$ and θ_N corresponds to $[,]_N$. In particular, one can take O_i to be a symplectic leaf in $([\mathrm{im}(N - \lambda_i\,\mathrm{Id})]^*, \theta_{\mathrm{st}}) \subset (\mathfrak{g}^*, \theta_{(N-\lambda_i\,\mathrm{Id})})$ (the operator $N - \lambda_i\,\mathrm{Id}$ is also Nijenhuis). Choosing O_i to be a generic coadjoint orbit and passing to codimensions we get the following sufficient condition of kroneckerity: if $\mathrm{ind}\,\mathrm{im}(N - \lambda_i\,\mathrm{Id}) + \mathrm{codim}\,\mathrm{im}(N - \lambda_i\,\mathrm{Id}) = \mathrm{ind}\,\mathfrak{g}$ for any i, where ind stands for the index of a Lie algebra, i.e. the codimension of a generic coadjoint orbit, then the Poisson pencil $\{\theta_N^s\}$ is Kronecker (cf. Corollary 2.6).

In general, however, this condition is not necessary because it may happen that the generic coadjoint orbits in $(\mathrm{im}\,N)^*$ are not generic in $(\mathfrak{g}^*, \theta_N)$. For example, take $\mathfrak{g} = \mathfrak{sl}(2) = \mathfrak{n}_- \oplus \mathfrak{b}_+$, where $\mathfrak{b}_+$ is the upper Borel subalgebra and $\mathfrak{n}_-$ is the lower nilpotent subalgebra. Let $N = P_{\mathfrak{n}_-}$ be the projector to the first summand along the second one. Then coadjoint orbits of $\mathrm{im}\,N$ are points, whereas the algebra $(\mathfrak{g}, [,]_N)$ is nonabelian and has also coadjoint orbits of dimension 2.

So our main theorem generalizes the above mentioned sufficient condition and gives necessary and sufficient conditions of the kroneckerity of the pencil $\{\theta_N^s\}$ (for a diagonalizable N). The method of proof of this result consists in showing that the above mentioned exceptional brackets are in fact semi-direct products and using the Raïs type formulas for their indices.

We illustrate our method by two examples. First of them, a generalization of the example above, relates a complete involutive set of functions on a semisimple split Lie algebra with the Nijenhuis operator P that is a projector onto the lower nilpotent subalgebra along the upper Borel subalgebra (in fact we use operators of the form $N = s_1 P + s_2\,\mathrm{Id}$, all such operators generate the same Poisson pencil, see Section 4).

Note that for $s_1 = 2, s_2 = -1$ such N is a classical R-matrix in the sense of Semenov-Tian-Shansky [22]. However, our method is essentially different, since it (1) exploits another modified bracket $[\cdot,\cdot]_N = [N\cdot,\cdot] + [\cdot, N\cdot] - N[\cdot,\cdot]$ (in the R-matrix approach $[\cdot,\cdot]_R = [R\cdot,\cdot] + [\cdot, R\cdot]$); (2) uses the whole pencil of brackets, generated by $[\cdot,\cdot]$ and $[\cdot,\cdot]_N$ (the R-matrix approach uses only $[\cdot,\cdot]_R$); (3) is applicable to the generic coadjoint orbits (while R-matrix approach generates involutive sets of functions on the orbits of dimension $2r = 2\,\mathrm{ind}\,\mathfrak{g}$). The reader is also referred to the reference [12] for another application of algebraic Nijenhuis operators which are projectors. This application, related with the so-called Kostant–Symes theorem, is also different from our since it produces involutive sets of functions on the dual spaces to subalgebras of $\mathfrak{g}$.

Our second example (see Section 5), which in fact inspired this paper, uses the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ of $n \times n$-matrices and the operator $N = L_A$ of the left multiplication by a diagonal matrix $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{g}$ with $\lambda_i \neq \lambda_j$, $i \neq j$. The corresponding modified commutator is of the form $[x, y]_N = xAy - yAx$, $x, y \in \mathfrak{g}$, and was considered earlier in [2] and [16] in the context of the free rigid body system on $\mathfrak{so}_n$. In these papers it was proved that the related Euler vector field is hamiltonian with respect to the corresponding Lie–Poisson structure θ_N. It is also known that this vector field is hamiltonian with respect to the standard Lie–Poisson structure θ_{st} on $\mathfrak{so}_n$ with the hamiltonian function $\mathrm{Tr}(Ax^2)$. As a consequence of our method we get an alternative proof of the complete integrability of the analogue of the n-dimensional free rigid body system on $\mathfrak{gl}_n$. The traditional proof, which goes back to the papers of Manakov

[14] and Mishchenko–Fomenko [15] (see also [2]), uses the so-called method of the argument translation, i.e. the pencil of the affine Poisson structures generated by the linear Poisson structure θ_{st} and the constant Poisson structure $\theta_{\mathrm{st}}|_A$. The complete involutive family of functions (which includes the function $\mathrm{Tr}(Ax^2)$) built by our method in fact coincides with the family obtained by the method of the argument translation (see Proposition 5.3), however the two families of functions are obtained differently and their equality is not seen at first glance. The proof of the equality uses recurrence relations between two families.

In order to use our method for the proof of the integrability of "true" free rigid body (i.e. on $\mathfrak{so}_n$) one should extend the method. Note that the pencil of brackets $[x,y]_{(N-\lambda\,\mathrm{Id})} = x(A-\lambda\,\mathrm{Id})y - y(A-\lambda\,\mathrm{Id})x$ is correctly defined on the subalgebra $\mathfrak{so}_n \subset \mathfrak{gl}_n$, although the Nijenhuis operator N does not preserve this subalgebra: $N\mathfrak{so}_n \not\subset \mathfrak{so}_n$ (cf. Example 1.6 and Remark 1.7). One can consider the following generalization of this situation: let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{k} \subset \mathfrak{g}$ its subalgebra, $N : \mathfrak{g} \to \mathfrak{g}$ a Nijenhuis operator such that $(N-\lambda\,\mathrm{Id})\mathfrak{k}$ again is a Lie subalgebra for any λ (we do not require $N\mathfrak{k} \subset \mathfrak{k}$). Then $(N-\lambda\,\mathrm{Id})^{-1}[(N-\lambda\,\mathrm{Id})\cdot, (N-\lambda\,\mathrm{Id})\cdot]$ is a Lie bracket on $\mathfrak{k}$ which is equal to $[\cdot,\cdot]_N|_{\mathfrak{k}} - \lambda[\cdot,\cdot]|_{\mathfrak{k}} =: [\cdot,\cdot]^\lambda$.

So one of the possible extensions of our method is the study of the kroneckerity of the pencils of the Lie–Poisson structures related to the pencil of the brackets $[\cdot,\cdot]^\lambda$. Another one is the consideration of the so-called weak Nijenhuis operators [5], i.e. operators whose Nijenhuis torsion is a cocycle with the coefficients in the adjoint module. Such operators also generate pencils of Lie–Poisson structures and the question of their kroneckerity seems reasonable and can provide with new examples of completely integrable systems or new proofs of their complete integrability.

1. Algebraic Nijenhuis operators and pencils of Lie algebras

The following definition, which is basic for this paper, is taken from [12].

Definition 1.1. Let $(\mathfrak{g}, [,])$ be a finite-dimensional Lie algebra over a field $\mathbb{K}$, where $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. A linear operator $N : \mathfrak{g} \to \mathfrak{g}$ is called Nijenhuis if

$$[Nx, Ny] - N\big([Nx, y] + [x, Ny]\big) + N^2[x, y] = 0 \tag{1.1.1}$$

for all $x, y \in \mathfrak{g}$.

The word "algebraic" in the titles of this section and the paper is used to distinguish the algebraic situation from the geometric one, where Nijenhuis operators are the endomorphisms of the tangent bundle to a manifold. Since these last will not be used in the paper we shall omit the term algebraic.

Given a Nijenhuis operator, it can be showed (see [12]) that the operation

$$[,]_N : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad [x, y]_N := [Nx, y] + [x, Ny] - N[x, y], \quad x, y \in \mathfrak{g}, \tag{1.1.2}$$

is again a Lie algebra bracket and moreover so is any linear combination $[x, y]^\lambda := [x, y]_N - \lambda[x, y]$, $\lambda \in \mathbb{K} \cup \infty$. This fact also follows from the proposition below, which will be used for calculating the Casimir functions of the Poisson pencil corresponding to the pencil of Lie algebras $(\mathfrak{g}, [,]^\lambda)$ and for other purposes.

Proposition 1.2. *Let N be a Nijenhuis operator. Then the following equality holds for any $x, y \in \mathfrak{g}$ and any $\lambda \notin \operatorname{Sp} N$, where $\operatorname{Sp} N$ stands for the spectrum of N:*

$$(N-\lambda\,\mathrm{Id})^{-1}\big[(N-\lambda\,\mathrm{Id})x, (N-\lambda\,\mathrm{Id})y\big] = [x, y]_N - \lambda[x, y] = [x, y]^\lambda. \tag{1.2.1}$$

Proof. It is straightforward:

$$\begin{aligned}
&(N-\lambda\,\mathrm{Id})^{-1}\big[(N-\lambda\,\mathrm{Id})x, (N-\lambda\,\mathrm{Id})y\big] \\
&\quad = (N-\lambda\,\mathrm{Id})^{-1}\big([Nx, Ny] - \lambda\big([Nx, y] + [x, Ny]\big) + \lambda^2[x, y]\big) \\
&\quad = (N-\lambda\,\mathrm{Id})^{-1}\big(N\big([Nx, y] + [x, Ny]\big) - N^2[x, y] - \lambda\big([Nx, y] + [x, Ny]\big) + \lambda^2[x, y]\big) \\
&\quad = (N-\lambda\,\mathrm{Id})^{-1}\big((N-\lambda\,\mathrm{Id})\big([Nx, y] + [x, Ny] - N[x, y]\big) - \lambda\big(N[x, y] - \lambda[x, y]\big)\big) \\
&\quad = (N-\lambda\,\mathrm{Id})^{-1}\big((N-\lambda\,\mathrm{Id})\big([x, y]_N - \lambda[x, y]\big)\big) = [x, y]^\lambda. \qquad \square
\end{aligned}$$

Now, the LHS of the proved equality is a Lie bracket for almost all λ, hence by continuity $[,]^\lambda$ is a Lie bracket for all λ.

The next lemma together with results of [12] allow to give a description of Nijenhuis operators, which is complete in the diagonalizable case.

Lemma 1.3. *Let N be a Nijenhuis operator. Then $N - \lambda\,\mathrm{Id}$ is a Nijenhuis operator for any $\lambda \in \mathbb{K} \cup \infty$.*

Proof. Let us note that by definition an invertible operator A is Nijenhuis if and only if $A^{-1}[Ax, Ay] = [x, y]_A$ for any $x, y \in \mathfrak{g}$. Now, by previous proposition for $A := N - \lambda\,\mathrm{Id}$ and for almost all λ we have

$$A^{-1}[Ax, Ay] = [Nx, y] + [x, Ny] - N[x, y] - \lambda[x, y] = [x, y]_A.$$

By continuity we conclude that $N - \lambda\,\mathrm{Id}$ is Nijenhuis for any λ. □

Proposition 1.4. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, let N be a Nijenhuis operator, and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be its decomposition to root spaces. Then this decomposition has the following property: the subspace of the form $\mathfrak{g}_{i_1} \oplus \cdots \oplus \mathfrak{g}_{i_k}$, $i_1 < \cdots < i_k$, is a Lie subalgebra for any $k < n$ (equivalently, $g_i + g_j$ is a subalgebra for any i, j).*

Conversely, any direct decomposition of $\mathfrak{g}$ to subspaces with the property above determines a diagonalizable Nijenhuis operator uniquely up to a choice of the eigenvalues $\lambda_1, \ldots, \lambda_n$ corresponding to the subspaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$. In particular, any decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ to two subalgebras determines a Nijenhuis operator.

Proof. Let N be Nijenhuis and let $\mathrm{Sp}\,N = \{\lambda_1, \ldots, \lambda_n\}$. By [12, Section 2.1] and by Lemma 1.2 the subspaces $\mathfrak{g}_i = \ker A_i^{r_i}$, $\check{\mathfrak{g}}_i := \bigoplus_{j \neq i} \mathfrak{g}_j = \mathrm{im}\,A_i^{r_i}$ are Lie subalgebras, where we put $A_i := N - \lambda_i\,\mathrm{Id}$ and r_i is the Riesz index of A_i, i.e. the smallest integer with the property that $\mathrm{im}\,A_i^{r_i} = \mathrm{im}\,A_i^{r_i+1} = \cdots$, while $\mathrm{im}\,A_i^{r_i-1} \neq \mathrm{im}\,A_i^{r_i}$. Obviously, the restriction of a Nijenhuis operator to a subalgebra is again a Nijenhuis operator. So we can pass to $N|_{\check{\mathfrak{g}}_i}$ and repeat the considerations above. By induction we get the desired property.

Now, let the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be such that $\mathfrak{g}_i + \mathfrak{g}_j$ is a subalgebra for any i, j. Define N by $N|_{\mathfrak{g}_i} := \lambda_i\,\mathrm{Id}_{\mathfrak{g}_i}$, $i = 1, \ldots, n$. By the bilinearity it is enough to prove equality (1.1.1) for $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, $1 \leqslant i, j \leqslant n$:

$$\begin{aligned}
&[Nx, Ny] - N\big([Nx, y] + [x, Ny]\big) + N^2[x, y] \\
&\quad = \lambda_i\lambda_j\big([x, y]_i + [x, y]_j\big) - N\big(\lambda_i\big([x, y]_i + [x, y]_j\big) + \lambda_j\big([x, y]_i + [x, y]_j\big)\big) + N\big(\lambda_i[x, y]_i + \lambda_j[x, y]_j\big) \\
&\quad = \lambda_i\lambda_j\big([x, y]_i + [x, y]_j\big) - \big(\lambda_i^2[x, y]_i + \lambda_i\lambda_j[x, y]_j + \lambda_i\lambda_j[x, y]_i + \lambda_j^2[x, y]_j\big) \\
&\qquad + \big(\lambda_i^2[x, y]_i + \lambda_j^2[x, y]_j\big) = 0
\end{aligned}$$

(here we denote by $[x, y]_i$ the ith component of the element $[x, y]$ with respect to the decomposition above). □

Some examples of Nijenhuis operators can be found in [12], another can be built using the second part of the proposition above.

The fundamental example for this paper is as follows.

Example 1.5. Let $\mathfrak{g}$ be an associative algebra and the Lie bracket be the commutator: $[x, y] := xy - yx$. Then the operator L_a of left (associative) multiplication by an element $a \in \mathfrak{g}$ is a Nijenhuis operator: $[L_a x, L_a y] - L_a([L_a x, y] + [x, L_a y]) + L_a^2[x, y] = axay - ayax - a(axy - yax + xay - ayx) + a^2(xy - yx) = 0$. In particular, if $\mathfrak{g} = \mathfrak{gl}_n$ is the algebra of $n \times n$-matrices we get important examples of: (a) nilpotent Nijenhuis operator if $a \in \mathfrak{gl}_n$ is nilpotent; (b) diagonalizable Nijenhuis operator if $a \in \mathfrak{gl}_n$ is diagonalizable. If $a = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathfrak{gl}_n$ is diagonal with $\lambda_i \neq \lambda_j$ while $i \neq j$, then the corresponding eigenspaces $\mathfrak{g}_i$ are equal to the Lie subalgebras of matrices whose all rows except the ith one are the zero vectors.

It is easy to see that $[x, y]_{L_a} = xay - yax$ and the corresponding pencil of Lie brackets is of the form $[x, y]^\lambda = x(a - \lambda)y - y(a - \lambda)x$.

Now we want to remark that the last formula admits a generalization for a wider class of Lie algebras. Below we shall give a construction of pencils of Lie brackets on a class of subalgebras of $\mathfrak{gl}_n$, which also come from a Nijenhuis operator but this operator in a sense is an “outer” one.

Example 1.6. Fix a matrix $I \in \mathfrak{gl}_n$ and put $\mathfrak{g}_I := \{B \in \mathfrak{gl}_n \mid BI + IB^* = 0\}$, $\mathfrak{h}_I := \{A \in \mathfrak{gl}_n \mid AI - IA^* = 0\}$, where $*$ denotes some involution on $\mathfrak{gl}_n$ such that $(AB)^* = B^*A^*$ for any $A, B \in \mathfrak{gl}_n$. Then it is easy to see that $\mathfrak{g}_I$ is a Lie subalgebra in $\mathfrak{gl}_n$ and that so is $L_A\mathfrak{g}_I$ for any $A \in \mathfrak{h}_I$: $[B,C]I = BCI - CBI = -BIC^* + CIB^* = IB^*C^* - IC^*B^* = -I([B,C])^*$, $[AB, AC] = A(BAC - CAB)$, $(BAC - CAB)I = -BAIC^* + CAIB^* = -BIA^*C^* + CIA^*B^* = IB^*A^*C^* - IC^*A^*B^* = -I(BAC - CAB)^*$, $B, C \in \mathfrak{g}_I$. This shows that the formula $[B,C]_A := BAC - CAB$ defines a new Lie bracket on $\mathfrak{g}_I$. Since for any $\lambda \in \mathbb{K}$ we have $A - \lambda I_n \in \mathfrak{h}_I$, where I_n is the unity matrix, the brackets $[,]_A$ and $[,]$ generate the pencil of Lie brackets $[,]^\lambda := [,]_{(A-\lambda I_n)} = [,]_A - \lambda[,]$. In general, this pencil is not generated by an "inner" Nijenhuis operator because in general $L_A\mathfrak{g}_I \neq \mathfrak{g}_I$. However, the formula of Proposition 1.2 is still valid: $[B,C]^\lambda = (A - \lambda I_n)^{-1}[(A - \lambda I_n)B, (A - \lambda I_n)C]$, $B, C \in \mathfrak{g}_I$.

Remark 1.7. One can generalize this construction to the following one. Let N be a Nijenhuis operator on $\mathfrak{g}$ and let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra such that $(N - \lambda\,\mathrm{Id})\mathfrak{k}$ again is a Lie subalgebra for any λ. Then $(N - \lambda\,\mathrm{Id})^{-1}[(N - \lambda\,\mathrm{Id})\cdot, (N - \lambda\,\mathrm{Id})\cdot]$ is a Lie bracket on $\mathfrak{k}$ which is equal to $[\cdot,\cdot]_N|_{\mathfrak{k}} - \lambda[\cdot,\cdot]|_{\mathfrak{k}} =: [\cdot,\cdot]^\lambda$ by Proposition 1.2. In particular, $[\cdot,\cdot]^\lambda$ is a correctly defined pencil of Lie brackets on $\mathfrak{k}$.

2. Preliminaries on Poisson pencils and formulation of main results

All definitions below admit the real (C^∞) and the complex (holomorphic) versions. However, for the purposes of this paper we shall mainly need the last one. So all objects in the next two sections are complex analytic, M stands for a connected manifold. We refer the reader to the book [6] for the preliminaries on Poisson structures.

Definition 2.1. A pair (θ_1, θ_2) of linearly independent bivector fields (bivectors for short) on a manifold M is called Poisson if $\theta^s := s_1\theta_1 + s_2\theta_2$ is a Poisson bivector for any $s = (s_1, s_2) \in \mathbb{C}^2$; the whole family of Poisson bivectors $\{\theta^s\}_{s\in\mathbb{C}^2}$ is called a Poisson pencil or a bi-Poisson structure (or bihamiltonian structure).

A bi-Poisson structure $\{\theta^s\}$ (we shall often skip the parameter space in the notations) can be viewed as a two-dimensional vector space of Poisson bivectors, the Poisson pair (θ_1, θ_2) as a basis in this space. Of course, the basis can be changed.

Example 2.2. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ with a Nijenhuis operator N. Denote by θ_1, θ_2 the canonical linear Poisson bivectors (the so-called Lie–Poisson bivectors) on the dual space $\mathfrak{g}^*$ related to the Lie brackets $[,]$ and $[,]_N$, respectively. Then, since these brackets generate a pencil of Lie brackets, the pair θ_1, θ_2 is Poisson. The corresponding Poisson pencil will be denoted by $\{\theta_N^s\}$. In the real case we complexify $\mathfrak{g}$ and N and then build the holomorphic Poisson pencil on $(\mathfrak{g}^{\mathbb{C}})^*$ as above.

The following definition is due to I. Zakharevich [23]

Definition 2.3. Let $\{\theta^s\}$ be a Poisson pencil on M. It is called Kronecker at a point $x \in M$ if $\operatorname{rank}_{\mathbb{C}} \theta_x^s$ is constant with respect to $s \in \mathbb{C}^2 \setminus \{0\}$. We say that $\{\theta^s\}$ is micro-Kronecker (Kronecker for short) if it is Kronecker at any point of some open dense set in M.

The next proposition shows the importance of Kronecker Poisson pencils, which serve as a convenient formalism allowing to construct and investigate completely integrable systems. For the proof see [2,18].

Proposition 2.4. *Let $\{\theta^s\}$ be a Kronecker Poisson pencil on M. Assume that an open set $U \subset M$ is such that the set $Z^{\theta^s}(U)$ of Casimir functions for θ^s over U is complete* (*i.e. the common level sets of functions from $Z^{\theta^s}(U)$ coincide with the regular part of the symplectic foliation of θ^s on U*) *for any $s \neq 0$. Then the set*

$$Z^{\{\theta^s\}}(U) := \sum_{s\neq 0} Z^{\theta^s}(U)$$

is a complete involutive set of functions for any $\theta^s \neq 0$, that is, the common level sets of functions from $Z^{\{\theta^s\}}(U)$ form a lagrangian foliation in any regular symplectic leaf of θ^s on U. (*Here we understand the sum as the linear span of an infinite family of linear subspaces of functions, which in fact is generated by some finite subfamily.*)

Now we are ready to formulate the main result of this paper which gives necessary and sufficient conditions of the kroneckerity of the Poisson pencil $\{\theta_N^s\}$ built by means of a Nijenhuis operator N (see Example 2.2).

Given a homogeneous space G/H, where $G \supset H$ are Lie groups, the Lie group H is naturally acting on it and since the point eH, where $e \in G$ is the neutral element, is stabilized by this action one can extend it to the linear action in the cotangent space $T^*_{eH}(G/H)$. This representation of H is denoted by ρ and is called the coisotropy representation. Identifying $T_{eH}(G/H)$ with $\mathfrak{g}/\mathfrak{h}$ we obtain the following formula: $\rho : H \to \operatorname{Aut}((\mathfrak{g}/\mathfrak{h})^*)$, $h \overset{\rho}{\mapsto} {}^t(g + \mathfrak{h} \mapsto \operatorname{Ad}_h(g) + \mathfrak{h})$, where $g \in \mathfrak{g}, h \in H$ and ${}^t(\cdot)$ stands for the transposed operator. For any element $a \in (\mathfrak{g}/\mathfrak{h})^*$ we introduce two numbers: $\operatorname{ind} a$, which is the index of the Lie algebra $\mathfrak{h}^a$ of the stabilizer H^a of a with respect to the coisotropy action, and $\operatorname{codim} a$, which is the codimension of the orbit $H \cdot a$ of a with respect to the coisotropy action (recall that the index of a Lie algebra is by definition the codimension of the generic coadjoint orbit).

Theorem 2.5. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let N be a diagonalizable Nijenhuis operator with the spectrum $\operatorname{Sp} N = \{\lambda_1, \dots, \lambda_n\}$. Assume the following condition is satisfied*:

$$\text{the complement to the set } \bigcup_{\lambda \in \mathbb{C} \setminus \operatorname{Sp} N} (N - \lambda \operatorname{Id})(\operatorname{Sing} \mathfrak{g}^*) \text{ in } \mathfrak{g}^* \text{ contains an open dense set.} \tag{2.5.1}$$

(*Here we have denoted by* $\operatorname{Sing} \mathfrak{g}^*$ *the union of all coadjoint orbits of nonmaximal dimension. In particular, if $\mathfrak{g}$ is reductive,* $\operatorname{codim} \operatorname{Sing} \mathfrak{g}^* \geqslant 3$ *and the assumption above is satisfied.*)

Put $\check{\mathfrak{g}}_i := \operatorname{im}(N - \lambda_i \operatorname{Id})$, $i = 1, \dots, n$ ($\check{\mathfrak{g}}_i$ are Lie algebras by Lemma 1.3 and by equality (1.1.1), see also Proposition 1.4).

Then the corresponding Poisson pencil $\{\theta_N^s\}$ is Kronecker if and only if one can find elements $c_1, \dots, c_n$, $c_i \in (\mathfrak{g}/\check{\mathfrak{g}}_i)^*$, $i = 1, \dots, n$, such that for any i, $1 \leqslant i \leqslant n$,

$$\operatorname{ind} c_i + \operatorname{codim} c_i = \operatorname{ind} \mathfrak{g}, \tag{2.5.2}$$

where $\operatorname{ind} c_i, \operatorname{codim} c_i$ *are the corresponding numbers related to the coisotropy representation $\rho_i : \check{G}_i \to (\mathfrak{g}/\check{\mathfrak{g}}_i)^*$, $\check{G}_i$ being the Lie subgroup of G with the Lie algebra $\check{\mathfrak{g}}_i$.*

The proof of this result is postponed to Section 3. Taking $c_i = 0$, $i = 1, \dots, n$, we get the following corollary.

Corollary 2.6. *Under the assumptions of Theorem 2.5, if for any i, $1 \leqslant i \leqslant n$,*

$$\operatorname{ind} \check{\mathfrak{g}}_i + \operatorname{codim} \check{\mathfrak{g}}_i = \operatorname{ind} \mathfrak{g},$$

then the Poisson pencil $\{\theta_N^s\}$ is Kronecker.

The last part of this section is devoted to some definitions which are based on Ref. [11] and which will be used in the proof of Theorem 2.5.

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a Lie algebra which is a direct sum of its subalgebras $\mathfrak{g}_1, \mathfrak{g}_2$. Then the Lie bracket on $\mathfrak{g}$ can be decomposed as follows:

$$[x, y] = [x_1, y_1]_1 + \big([x_1, y_2]_1 + [x_2, y_1]_1\big) + \big([x_1, y_2]_2 + [x_2, y_1]_2\big) + [x_2, y_2]_2, \tag{2.6.1}$$

where the indices refer to the corresponding projections onto $\mathfrak{g}_1$ or $\mathfrak{g}_2$. It turns out that the maps $A_1 : x_1 \mapsto [x_1, \cdot]_2 : \mathfrak{g}_1 \to \operatorname{End}(\mathfrak{g}_2)$ and $A_2 : x_2 \mapsto [x_2, \cdot]_1 : \mathfrak{g}_2 \to \operatorname{End}(\mathfrak{g}_1)$ are Lie algebra homomorphisms, where $\operatorname{End}(\mathfrak{g}_i)$ is the Lie algebra of the endomorphisms of the vector space $\mathfrak{g}_i$. The representations A_1, A_2 also satisfy some additional conditions making them cocycles, which will be inessential for us. The formula above rewritten in terms of the pairs of elements (x_1, x_2), $x_i \in \mathfrak{g}_i$, is the following:

$$\big[(x_1, x_2), (y_1, y_2)\big] = \big([x_1, y_1] + A_2(x_2)y_1 - A_2(y_2)x_1, [x_2, y_2] + A_1(x_1)y_2 - A_1(y_1)x_2\big). \tag{2.6.2}$$

In particular, if one put here $A_2 \equiv 0$, one gets the formulas

$$\big[(x_1, x_2), (y_1, y_2)\big] = \big([x_1, y_1], A_1(x_1)y_2 - A_1(y_1)x_2\big), \tag{2.6.3}$$

or

$$[x, y] = [x_1, y_1]_1 + \big([x_1, y_2]_2 + [x_2, y_1]_2\big), \tag{2.6.4}$$

in which one recognizes the multiplication in the semidirect product $\mathfrak{g}_1 \times_{A_1} \mathfrak{g}_2$, where $\mathfrak{g}_2$ is regarded as a vector space (abelian Lie algebra).

Definition 2.7. We shall refer to the direct sum of Lie algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the bracket (2.6.1) (or (2.6.2)) as to a twilled Lie algebra. The same vector space with the multiplication (2.6.3) (or (2.6.4)) will be called the truncation of the twilled Lie algebra.

3. Proof of the main result

Proof of Theorem 2.5. The idea of the proof is as follows. As we have already mentioned in the Introduction formula (1.2.1) shows that almost all bivectors of the Poisson pencil $\{\theta_N^s\}$ built from the algebraic Nijenhuis operator N are isomorphic, in particular have the same corank equal to the index of the algebra $\mathfrak{g}$. The exception are the bivectors $\theta^{(-\lambda_i,1)}$, $i = 1, \ldots, n$, corresponding to the eigenvalues λ_i of N and all that we need to control the corank of the whole pencil is to control the rank of these bivectors which will be called exceptional. To this end we shall show that the Lie brackets corresponding to the exceptional bivectors are semidirect products and we shall use the Raïs type formula for the index of such algebras.

Let us fix i, $1 \leqslant i \leqslant n$, and consider the Nijenhuis operator $M := N - \lambda_i \operatorname{Id}$. Then the following formula describes the deformed bracket corresponding to M (see [1,4,21]):

$$[x, y]_M = M|_{E_1}^{-1}[Mx, My]_1 + [Mx, y]_2 + [x, My]_2 - M[x, y]_2,$$

where we put for a moment $E_1 := \check{\mathfrak{g}}_i = \operatorname{im} M$, $E_2 := \mathfrak{g}_i = \ker M$ and the subscripts refer to the projections onto E_1 or E_2. Note that the last term is zero. We claim that the bracket $[,]_M$ is the truncated bracket (see Definition 2.7) corresponding to the twilled Lie algebra structure on $\mathfrak{g} = E_1 \times E_2$ given by the bracket $[,]_L$ defined below.

Define a new Nijenhuis operator L on $\mathfrak{g}$ by the formula $L = M \circ P_1 + P_2$, where P_i stands for the projector onto E_i, $i = 1, 2$. In other words L acts as M on E_1 and identically on E_2. Now let $[,]_L$ stand for the deformed bracket corresponding to L by formula (1.1.2). Then by (1.2.1) we have

$$\begin{aligned}[x, y]_L &= L^{-1}[Lx, Ly] = L|_{E_1}^{-1} P_1[Lx, Ly] + P_2[Lx, Ly] \\ &= M|_{E_1}^{-1}\big([Mx_1, My_1]_1 + [Mx_1, y_2]_1 + [x_2, My_1]_1\big) + [Mx_1, y_2]_2 + [x_2, My_1]_2 + [x_2, y_2]_2.\end{aligned}$$

Now, the truncated bracket equals $M|_{E_1}^{-1}[Mx_1, My_1]_1 + [Mx_1, y_2]_2 + [x_2, My_1]_2$ which coincides with $[x, y]_M$ and the claim is proved.

Note that the twilled Lie algebras $(\check{\mathfrak{g}}_i \times \mathfrak{g}_i, [,])$ and $(\check{\mathfrak{g}}_i \times \mathfrak{g}_i, [,]_L)$ are isomorphic. Indeed, the isomorphism is given by the operator $L : (\check{\mathfrak{g}}_i \times \mathfrak{g}_i, [,]_L) \to (\check{\mathfrak{g}}_i \times \mathfrak{g}_i, [,])$. This isomorphism is compatible with the truncations, i.e. the corresponding truncated algebras also are isomorphic. In particular, in the considerations below concerning the codimensions of the coadjoint orbits we can regard simply the semidirect products which are the truncations of the twilled Lie algebras $(\check{\mathfrak{g}}_i \times \mathfrak{g}_i, [,])$, $i = 1, \ldots, n$.

Now we can use the standard facts about semidirect products, which we recall below. Given a semidirect product $\mathfrak{g} \times_\rho V$ of a Lie algebra $\mathfrak{g}$ with a vector space V by means of a representation ρ, one can show (see [20], for example) that: (1) any covector $a \in V^*$ is contained in a set $V_a \subset (\mathfrak{g} \times_\rho V)^*$ which is a Poisson submanifold in $(\mathfrak{g} \times_\rho V)^*$ isomorphic to T^*G/G^a (here G, G^a are the Lie groups corresponding to the Lie algebras $\mathfrak{g}, \mathfrak{g}^a$, where $\mathfrak{g}^a$ is the stabilizer of a); (2) the coadjoint orbits contained in V_a are isomorphic to the symplectic leaves of T^*G/G^a, in particular, the generic (in V_a) orbits have codimension in V_a equal to $\operatorname{ind} \mathfrak{g}^a$; (3) the submanifold V_a is of the form $\mathfrak{g}^* \times O_a$, where $O_a \subset V^*$ is the orbit of a in V^*, in particular, $\operatorname{codim}_{(\mathfrak{g} \times_\rho V)^*} V_a = \operatorname{codim}_{V^*} O_a$. Summarizing all this, we can say that, given an element $a \in V^*$, one can associate with it a coadjoint orbit S_a of $(\mathfrak{g} \times_\rho V)^*$ such that $\operatorname{codim}_{(\mathfrak{g} \times_\rho V)^*} S_a = \operatorname{codim}_{V^*} O_a + \operatorname{ind} \mathfrak{g}^a$. Taking a generic a we get the so-called Raïs formula [19]: $\operatorname{ind}(\mathfrak{g} \times_\rho V)^* = \operatorname{codim}_{V^*} O_a + \operatorname{ind} \mathfrak{g}^a$.

Now let us complete the proof of Theorem 2.5. We claim that the kroneckerity of the Poisson pencil $\{\theta_N^s\}$ is equivalent to existing for any i, $1 \leqslant i \leqslant n$, of a symplectic leaf S_i of the bivector $\theta_N^{(-\lambda_i,1)}$ such that $\operatorname{codim} S_i =$

$\operatorname{ind}\mathfrak{g}$. Indeed, by formula (1.2.1) we have $(N-\lambda\,\mathrm{Id})_*\theta_1=\theta_N^{(-\lambda,1)}$ for any $\lambda\notin\operatorname{Sp}N$. Since $\operatorname{Sing}\mathfrak{g}^*$ is the union of symplectic leaves of θ_1 of nonmaximal dimension, the assumption (2.5.1) implies that on the open dense set mentioned in it the corank of the bivectors $\theta_N^{(-\lambda,1)}$, $\lambda\notin\operatorname{Sp}N$, is equal to $\operatorname{ind}\mathfrak{g}$. But these bivectors up to rescaling exhaust all nonexceptional bivectors. Now it is clear that the kroneckerity is equivalent to the condition $\operatorname{corank}\theta_N^{(-\lambda_i,1)}=\operatorname{ind}\mathfrak{g}$, $i=1,\dots,n$. On the other hand, in general $\operatorname{ind}\mathfrak{g}\leqslant\operatorname{corank}\theta_N^{(-\lambda_i,1)}\leqslant\operatorname{codim}S_i$ and we have proved the claim.

Now let us pick out a point $c_i\in(\mathfrak{g}/\check{\mathfrak{g}}_i)^*\simeq(\mathfrak{g}_i)^*$ for any $i=1,\dots,n$. By the considerations above this is equivalent to picking out a symplectic leaf S_i of the exceptional bivector $\theta^{(-\lambda_i,1)}$ such that $\operatorname{codim}_{\mathfrak{g}^*}S_i=\operatorname{codim}_{\mathfrak{g}_i^*}O_{c_i}+\operatorname{ind}\check{\mathfrak{g}}_i^{c_i}$, where O_{c_i} is the $\check{G}_i$-orbit of c_i in $\mathfrak{g}_i^*$ and $\check{\mathfrak{g}}_i^{c_i}$ is its stabilizer. $\square$

4. First example: Kronecker Poisson pencils related to Borel subalgebras

Let $\mathfrak{g}$ be a semisimple real or complex Lie algebra of rank r. If $\mathfrak{g}$ is real, assume that it is split. Consider a Cartan subalgebra $\mathfrak{h}$ and the corresponding direct decompositions $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{b}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, where $\mathfrak{b},\mathfrak{n}_-,\mathfrak{n}_+$ are the Borel and the maximal nilpotent subalgebras respectively. Define a linear operator $N:\mathfrak{g}\to\mathfrak{g}$ by $N|_{\mathfrak{n}_-}=\lambda_1\,\mathrm{Id}$, $N|_{\mathfrak{b}}=\lambda_2\,\mathrm{Id}$. Then by Proposition 1.4 N is a Nijenhuis operator.

Theorem 4.1. *The corresponding Poisson pencil $\{\theta_N^s\}$ is Kronecker.*

Proof. We need to check that $(\mathfrak{g},N)$ satisfies the criterion of the kroneckerity, Theorem 2.5. Note that since $\mathfrak{g}$ is semisimple, $\operatorname{codim}\operatorname{Sing}\mathfrak{g}\geqslant3$ and condition (2.5.1) is satisfied and also $\operatorname{ind}\mathfrak{g}=\operatorname{rank}\mathfrak{g}$.

Now we shall consider the corresponding coisotropy representations. Denote by N_-,B the corresponding Lie groups. Using the Killing form we obtain the following natural identifications: $(\mathfrak{g}/\mathfrak{b})^*\simeq\mathfrak{n}_+$, $(\mathfrak{g}/\mathfrak{n}_-)^*\simeq\mathfrak{b}_-:=\mathfrak{n}_-\oplus\mathfrak{h}$, which are B- and N_--equivariant respectively. We need to find two elements $c_1\in\mathfrak{n}_+$ and $c_2\in\mathfrak{b}_-$ satisfying condition (2.5.2).

First take $c_1:=e$, a principal nilpotent element [3,13]. Its stabilizer $\mathfrak{g}^e$ with respect to the adjoint action of $\mathfrak{g}$ is an abelian subalgebra of dimension r. Moreover, $\mathfrak{g}^e\subset\mathfrak{n}_+$, hence $\mathfrak{g}^e=\mathfrak{b}^e$ and $\operatorname{ind}c_1=r$. The dimension of the adjoint B-orbit of c_1 equals $\dim\mathfrak{n}_+$, i.e. $\operatorname{codim}c_1=0$ and (2.5.2) is satisfied for c_1.

Now let $c_2:=f\in\mathfrak{h}$ be a regular semisimple element. Then $\mathfrak{g}^f=\mathfrak{h}$ and $\mathfrak{n}_-^f=\mathfrak{g}^f\cap\mathfrak{n}_-=0$. Thus $\operatorname{ind}c_2=0$ and the dimension of the N_--orbit of c_2 in $\mathfrak{b}_-$ is $\dim\mathfrak{n}_-$, i.e. $\operatorname{codim}c_2=\dim\mathfrak{b}_--\dim\mathfrak{n}_-=r$. $\square$

Remark 4.2. Let us exhibit what functions in involution we get on $\mathfrak{g}^*$ with the help of N (for general pair $(\mathfrak{g},N)$). Recall that they are generated by the Casimir functions of all bivectors of the pencil $\{\theta_N^s\}$ (see Proposition 2.4). By formula (1.2.1) the Casimirs of θ_N^s, $s=(s_1,s_2)$, are functionally generated by $C_j((N-\lambda\,\mathrm{Id})^{-1}x)$, $j=1,\dots,r$, where $\lambda=-s_1/s_2$ and $C_1,\dots,C_r$ are the independent Casimirs of $\theta_N^{(0,1)}$, i.e. the invariants of the coadjoint action of $\mathfrak{g}$. In fact it is enough to choose a finite number of bivectors $\theta_N^{s_i}$, $s_i=(s_i^1,s_i^2)$, $i=1,\dots,p$, where p is sufficiently large, and one can take $-s_i^1/s_i^2$ to be not equal to the eigenvalues of N. Thus our family of functions in involution is functionally generated by $C_j((s_i^2N+s_i^1\,\mathrm{Id})^{-1}x)$, $j=1,\dots,r$, $i=1,\dots,p$.

If $C_1,\dots,C_r$ are polynomials as in the case of semisimple $\mathfrak{g}$, another way to obtain this family of functions is to consider the coefficients of the expansion of $C_j((N-\lambda\,\mathrm{Id})^{-1}x)$ in the powers of λ (or $1/\lambda$).

Taking $\lambda_1=1$, $\lambda_2=-1$ for the Nijenhuis operator N built above we obtain the following simple formula for the resolvent: $(N-\lambda\,\mathrm{Id})^{-1}=(1-\lambda^2)^{-1}(N+\lambda\,\mathrm{Id})$. In particular, for $\mathfrak{g}=\mathfrak{sl}_n$ we can take the coefficients of the expansion in λ of the following functions $\operatorname{Tr}((N+\lambda\,\mathrm{Id})x)^k$, $k=2,\dots,n$ (we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form). These functions form a complete involutive set on any adjoint orbit of maximal dimension. It is easy to see that the following quadratic hamiltonians are in this family: $\sum_{i=1}^n x_{ii}^2$, $\sum_{i<j}x_{ij}x_{ji}$.

5. Second example: n-dimensional free rigid body and the method of argument translation

Theorem 5.1. *Let $\mathfrak{g}=\mathfrak{gl}_n$ be the Lie algebra of $n\times n$-matrices and let $N=L_A$ be a Nijenhuis operator of left multiplication by the diagonal matrix $A=\operatorname{diag}(\lambda_1,\dots,\lambda_n)\in\mathfrak{g}$ with $\lambda_i\neq\lambda_j$, $i\neq j$.*

Then the corresponding Poisson pencil $\{\theta_N^s\}$ is Kronecker.

Proof. Obviously, the subalgebra $\check{\mathfrak{g}}_i$ for the Nijenhuis operator L_A equals the set of matrices with the zero ith row, hence $\operatorname{codim}\check{\mathfrak{g}}_i = n$. The proof follows from Corollary 2.6 and from the following lemma. □

Lemma 5.2. *Let $\check{\mathfrak{g}}_i \subset \mathfrak{g} = \mathfrak{gl}_n$ be the Lie subalgebra of matrices with the zero ith row. Then* $\operatorname{ind}\check{\mathfrak{g}}_i = 0$ (*i.e.* $\check{\mathfrak{g}}_i$ *is Frobenius*).

Proof. We refer the reader to the reference [17], where the following fact is proved: the Lie algebra of the endomorphisms of a finite-dimensional vector space with the images in a fixed subspace of codimension 1 is Frobenius. □

Let us look at the functions in involution obtained from this example. By Remark 4.2 they can be functionally generated by the coefficients of the expansion of the functions $\operatorname{Tr}((N-\lambda\,\mathrm{Id})^{-1}x)^k$, $k = 1,\dots,n$, in λ (we have identified $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the "trace" form). Let us rewrite the resolvent $(N-\lambda\,\mathrm{Id})^{-1}$ in the form $-\lambda^{-1}(\mathrm{Id}-\frac{1}{\lambda}N)^{-1} = -\lambda^{-1}\sum_{j=0}^{\infty}\frac{1}{\lambda^j}N^j$. Write f_{kl} for the coefficient of $1/\lambda^l$ in $\operatorname{Tr}(\sum_{j=0}^{\infty}\frac{1}{\lambda^j}N^jx)^k$. Recall also that the so-called Manakov integrals h_{kl} are the coefficients of λ^l in $h_k^\lambda(x) := \frac{1}{k}\operatorname{Tr}(x+\lambda A)^k$.

Proposition 5.3. *The functions h_{kl}, $k = 1,\dots,n$, $l = 0,\dots,k-1$, and f_{kl}, $k = 1,\dots,n$, $l = 0,\dots,k-1$, generate the same families of functions in involution.*

Proof. We shall use the recursion relations satisfied by both the families.

Lemma 5.4. *Let θ_1,θ_2 be the Lie–Poisson structures corresponding to the Lie brackets* $[,]$, $[,]_N$ *on* $\mathfrak{g}$ *respectively. Then* (1) $\theta_1(h_{k+1,l+1}) = \theta_2(h_{kl})$; (2) $\theta_1(f_{k,l+1}) = \theta_2(f_{kl})$.

Proof. To prove the first relation we adapt the proof of the analogous relation for $\mathfrak{g} = \mathfrak{so}_n$ in [16]. We notice that $d_xh_k^\lambda(x) = (x+\lambda A)^{k-1}$ (here the matrix in the RHS is a functional via the "trace" form) and $\theta_1|_x x' = [x,x']$, $\theta_2|_x x' = xx'A - Ax'x$ (here θ_i is a map $T^*\mathfrak{g}^* \to T\mathfrak{g}^* \simeq \mathfrak{g}^* \overset{\mathrm{Tr}}{\simeq} \mathfrak{g}$, $x' \in T_x^*\mathfrak{g}^* \simeq \mathfrak{g}$). Now it is straightforward to show that $\theta_1\,dh_{k+1}^\lambda = \lambda\theta_2\,dh_k^\lambda$, which proves (1).

To show the second relation we use the fact that $\operatorname{Tr}((N-\lambda\,\mathrm{Id})^{-1}x)^k$ is a Casimir for $\theta_2 - \lambda\theta_1$, i.e. $(\theta_2 - \lambda\theta_1)(\sum_j \frac{1}{\lambda^j}f_{kj}) = 0$. Comparing the coefficients we get (2). □

Now we are ready to prove Proposition 5.3. We have the following two hierarchies of functions:

$$
\begin{array}{llll}
h_{10} & & & \\
h_{20} & h_{21} & & \\
h_{30} & h_{31} & h_{32} & \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\qquad\qquad
\begin{array}{llll}
f_{10} & & & \\
f_{20} & f_{11} & & \\
f_{30} & f_{21} & f_{12} & \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

We shall use the induction on the number l of the column of the second table. It is easy to see that $h_{k0} = \frac{1}{k}\operatorname{Tr}x^k = \frac{1}{k}f_{k0}$. Now fix l and suppose the function f_{ml}, $1 \leqslant m \leqslant n$, can be expressed as a function of h_{kj}, $j = 0,\dots,l$, $k = 1,\dots,n$. We should prove that $f_{m,l+1}$, $1 \leqslant m \leqslant n$, also can be expressed as a function of h_{kj}, $j = 0,\dots,l+1$, $k = 1,\dots,n$. Indeed, since by the lemma above $\theta_1(f_{m,l+1}) = \theta_2(f_{ml})$, this vector field can be expressed as a linear combination of vector fields $\theta_2(h_{kj})$, $j \leqslant l$, which in turn are equal to $\theta_1(h_{k+1,j+1})$. This shows that the function $f_{m,l+1}$ can be functionally expressed by the functions h_{kj}, $j = 0,\dots,l+1$, $k = 1,\dots,n$ (the Casimirs of θ_1 are included in this family). □

References

[1] N. Bedjaoui-Tebbal, Contractions d'algèbres de Lie et torsion de Nijenhuis, Bull. Belg. Math. Soc. 7 (2000) 303–310.

[2] A. Bolsinov, Compatible Poisson brackets on Lie algebras and completeness of families of functions in involution, Math. USSR Izvestiya 38 (1992) 69–90. Translation from the Russian.

[3] N. Bourbaki, Groups et algèbres de Lie, IX, Masson, 1982.

[4] J. Carinẽna, J. Grabowski, G. Marmo, Contractions: Nijenhuis and Saletan tensors for general algebraic structures, J. Phys. A 34 (2001) 3769–3789.
[5] J. Carinẽna, J. Grabowski, G. Marmo, Courant algebroid and Lie bialgebroid contractions, J. Phys. A 37 (2004) 5189–5202.
[6] A. Canas da Silva, A. Weinstein, Geometric Models for Noncommutative Algebras, AMS, Providence, RI, 1999.
[7] I. Gelfand, I. Zakharevich, Spectral theory for a pair of skew-symmetrical operators on S^1, Functional Anal. Appl. 23 (2) (1989) 85–93. Translation from the Russian.
[8] I. Gelfand, I. Zakharevich, Webs, Veronese curves, and bihamiltonian systems, J. Funkt. Anal. 99 (1991) 150–178.
[9] I. Gelfand, I. Zakharevich, On the local geometry of a bihamiltonian structure, in: The Gelfand Mathematical Seminars 1990–1992, Birkhäuser, 1993, pp. 51–112.
[10] I. Gelfand, I. Zakharevich, Webs, Lenard schemes, and the local geometry of bihamiltonian Toda and Lax structures, Selecta-Math. (N.S.) 6 (2000) 131–183.
[11] Y. Kosmann-Schwarzbach, F. Magri, Poisson–Lie groups and complete integrability I. Drinfeld bigebras, dual extensions and their canonical representations, Ann. Inst. Henri Poincaré 49 (1988) 433–460.
[12] Y. Kosmann-Schwarzbach, F. Magri, Poisson–Nijenhuis structures, Ann. Inst. Henri Poincaré 53 (1990) 35–81.
[13] B. Kostant, The principal thee-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. 81 (1959) 973–1032.
[14] S.V. Manakov, A remark on the integration of the Euler equation of the dynamics of an n-dimensional rigid body, Funktsional. Anal. Appl. 10 (1976) 328–329.
[15] A.S. Mishchenko, A.T. Fomenko, Euler equations on finite dimensional Lie groups, Math. USSR Izvestiya 12 (1978) 371–389. Translation from the Russian.
[16] C. Morosi, L. Pizzocchero, On the Euler equation: Bi-Hamiltonian structure and integrals in involution, Lett. Math. Phys. 37 (1996) 117–135.
[17] A. Ooms, On Lie algebras having a primitive universal enveloping algebra, J. Algebra 32 (1974) 488–500.
[18] A. Panasyuk, Veronese webs for bihamiltonian structures of higher corank, Banach Center Publications 51 (2000) 251–261.
[19] M. Raïs, L'indice des produits semi-directs $E \times_\rho \mathfrak{g}$, C. R. Acad. Sci. Paris, Ser. A 287 (1978) 195–197.
[20] A.G. Reyman, M.A. Semenov-Tian-Shansky, Group-theoretical methods in the theory of integrable systems, in: Encyclopaedia of Math. Sciences (Dynamical Systems VII), vol. 16, Springer, 1994, pp. 116–225.
[21] E. Saletan, Contraction of Lie groups, J. Math. Phys. 2 (1961) 1–21.
[22] M.A. Semenov-Tian-Shansky, Lectures on r-matrices, Poisson–Lie groups and integrable systems, in: Y. Kosmann-Schwarzbach, O. Babelon, P. Cartier (Eds.), Lectures on Integrable Systems, In memory of J.-L. Verdier, Proc. of the CIMPA School on Integrable Systems, Nice 1991, World Scientific, 1994, pp. 269–317.
[23] I. Zakharevich, Kronecker webs, bihamiltonian structures, and the method of argument translation, Transformation Groups 6 (2001) 267–300.