Convergence of a monotonisation procedure for a non-monotone quasi-static model in poroplasticity

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Abstract

Existence theory to quasi-static initial-boundary value problem of poroplasticity is studied. The classical quasi-static Biot model for soil consolidation coupled with a nonlinear system of ordinary differential equations is considered. This article presents a convergence result for the coercive and monotone approximations to solution of the original non-coercive and non-monotone problem of poroplasticity such that the inelastic constitutive equation is satisfied in the sense of Young measures.

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1. Introduction

The concept of porous media is used in many areas of applied sciences and engineering for example: geomechanics, soil mechanics, rock mechanics. Porous materials are brittle, granular and they are often saturated by some liquids or gases. In this work we present mathematical analysis for quasi-static model in poroplasticity, which was introduced by W. Ehlers in [8]. These equations have a structure similar to the one from the theory of inelastic behavior of metals: linear partial differential equations (Biot model, see for more details [9]) are coupled with nonlinear ordinary differential equations describing inelastic deformation (constitutive equations).

The equations of the theory of poroplasticity can be written in the form

\[
\begin{align*}
\text{div}_x T(x,t) - \nabla_x p(x,t) &= -F(x,t), \\
c \Delta_x p(x,t) - \text{div}_x u_t(x,t) &= f(x,t), \\
T(x,t) &= D(\varepsilon(x,t) - \varepsilon^P(x,t)), \\
\varepsilon(x,t) &= \frac{1}{2} \left( \nabla_x u(x,t) + \nabla^T_x u(x,t) \right), \\
\varepsilon^P_t(x,t) &= F(Y(T(x,t))) \frac{\partial P}{\partial T} (T(x,t)).
\end{align*}
\] (1.1)

Here \( x \in \Omega \subset \mathbb{R}^3 \), where \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \) and \( t > 0 \) denotes time. The system (1.1) is built from the balance of momentum with a generalisation of the Hook law (the first and the third equation), the combination of the Darcy law with the fluid mass conservation (the second equation) and the so-called inelastic constitutive equation.
(the last equation). This inelastic constitutive relation was proposed by W. Ehlers in [8] and it is very often used in practice (the physical meaning of this equation can be also found in [8]). $\varepsilon(x,t)$ is the infinitesimal strain tensor.

In our problem, for all $T > 0$ we have to find the displacement field $u: \Omega \times [0, T] \rightarrow \mathbb{R}^3$, the pore pressure of the fluid $p: \Omega \times [0, T] \rightarrow \mathbb{R}$, the Cauchy stress tensor $T: \Omega \times [0, T] \rightarrow S^3$ and the plastic strain tensor $\varepsilon^p: \Omega \times [0, T] \rightarrow S^3$ ($S^3$ denotes the set of symmetric $3 \times 3$-matrices). $F: \Omega \times [0, T] \rightarrow \mathbb{R}^3$ and $f: \Omega \times [0, T] \rightarrow \mathbb{R}$ are given functions, which describe density of applied body forces and a forced fluid extraction or injection process, respectively. $D: S^3 \rightarrow S^3$ is the elasticity tensor which is assumed to be constant in time and space, symmetric and positive definite. $c > 0$ is a constant, which represents the permeability of the porous medium and the viscosity of the fluid.

The functions $Y: S^3 \rightarrow \mathbb{R}$ and $P: S^3 \rightarrow \mathbb{R}$ are two convex homogeneous polynomials of the same growth, which means that there exist constants $a, A > 0$ such that

$$aY(T) \leq P(T) \leq aY(T) \quad \text{for large } |T|. $$

$F: \mathbb{R} \rightarrow \mathbb{R}_+$ is a monotone and continuously differentiable scalar function with polynomial growth, which means that there exist $\alpha > 1$ and constants $m, M > 0$ such that

$$m|s|^\alpha \leq F(s) \leq M|s|^\alpha \quad \text{for large } |s|. $$

Moreover, we assume that $Y(0) = 0$ and $F(0) = 0$. The system (1.1) will be considered with Dirichlet boundary conditions

$$u(x, t) = g_0(x, t) \quad \text{for } x \in \partial \Omega \text{ and } t \geq 0, $$

$$p(x, t) = g_p(x, t) \quad \text{for } x \in \partial \Omega \text{ and } t \geq 0, $$

and with initial conditions

$$\text{div}_x u(x, 0) = \text{div} u^0(x), \quad \varepsilon^p(x, 0) = \varepsilon^{p, 0}(x) $$

(initial condition for the displacement means that we only know the divergence of $u(x, 0)$).

The free energy function associated with the system (1.1) is given by the formula

$$\rho \psi(\varepsilon, \varepsilon^p) = \frac{1}{2} D(\varepsilon - \varepsilon^p)(\varepsilon - \varepsilon^p), $$

where $\rho$ is the mass density which we assume to be constant. Note that the function $N(T) = F(Y(T)) \frac{\partial P}{\partial T}(T)$ satisfies the second law of thermodynamics: $-\rho \nabla_{\varepsilon^p} \psi(\varepsilon, \varepsilon^p) : N(T) \geq 0$ ($P$ is a homogeneous convex polynomial). This inequality is called the dissipative inequality (see [1] for the definition). The total energy is of the form:

$$\mathcal{E}(\varepsilon, \varepsilon^p)(t) = \int_{\Omega} \rho \psi(\varepsilon(x, t), \varepsilon^p(x, t)) \, dx. $$

We note, that the quadratic form $\psi$ in (1.4) is only semi-positive definite. Thus, our model is non-coercive (we say that the model from the inelastic deformation theory is coercive if the free energy function associated with this model is positive definite). Moreover, the constitutive function $N: S^3 \rightarrow S^3$ is not monotone. Therefore, our model is non-monotone (for the definitions see [1]). To the system (1.1) we will use a coercive approximation and monotisation procedure, which we will call a monotone approximation. In the literature there are not any mathematical results for this non-monotone model of poroplasticity. In [10] the existence theory for a model of monotone-gradient type in poroplasticity (the constitutive function can be written as a gradient of a convex function) was studied only. In the literature we can also find the article [11], where the authors consider the model describing the diffusion in poroplastic materials. Model from [11] is also of monotone type, because the nonlinear constitutive function is monotone. Moreover, most engineering models of inelastic behavior of metals are not coercive but these are sometimes monotone. In the article [6] the existence theory for non-coercive quasi-static models in inelastic deformation theory was studied. Strict monotonicity on the nonlinear constitutive equation is the main assumption in [6]. Under this assumption authors, using Young measures, could pass to the limit in the coercive approximation.

The considered constitutive function is non-monotone. Therefore, we define a new notion of a solution (for the definition see Section 3, Definition 3.1). This notion of a solution is weaker than the weak-type solution introduced in [2]. Definition from [2] was used for a model of monotone-gradient type in poroplasticity in [10]. In this paper we could only show that first two equations in (1.1) are satisfied in the “weak sense” (see Definition 3.1) and the inelastic constitutive equation is satisfied in the sense of Young measures.
2. Approximation procedure

In this section we approximate the problem (1.1) by a coercive and monotone problem. The idea of coercive approximation can be found in [4]. The idea of monotonisation is that we add to the constitutive function $N$ in (1.1) a monotone polynomial multiplied by a coefficient which will pass to zero. This polynomial will be a gradient of a convex function and it will have higher degree than the degree of $N$. Therefore, we will get a monotone structure.

Let $\eta > 0$ and $\beta > 1$, then the approximation is defined by

$$
\begin{align}
\text{div}_x T^\eta(x, t) - \nabla_x p^\eta(x, t) &= -F(x, t), \\
c\Delta_x p^\eta(x, t) - \text{div}_x u^\eta(x, t) &= f(x, t), \\
T^\eta(x, t) &= D(\epsilon^\eta(x, t) - \epsilon^{P,\eta}(x, t) + \eta\epsilon^\eta(x, t)), \\
\epsilon^\eta(x, t) &= \frac{1}{2}(\nabla_x u^\eta(x, t) + \nabla_x^T u^\eta(x, t)), \\
\epsilon_t^{P,\eta}(x, t) &= \eta\left|\frac{T^\eta(x, t)}{|T^\eta(x, t)|}\right|^\beta + \mathcal{F}(Y(T(x, t))) \frac{\partial P}{\partial T} \left(\tilde{T}^\eta(x, t)\right),
\end{align}
$$

(2.1)

where $\tilde{T}^\eta = D(\epsilon^\eta - \epsilon^{P,\eta}) = T^\eta - \eta \partial E\eta$. The free energy function associated with the system (2.1) is given by the formula

$$
\rho \psi^\eta(\epsilon^\eta, \epsilon^{P,\eta}) = \frac{1}{2} D(\epsilon^\eta - \epsilon^{P,\eta})(\epsilon^\eta - \epsilon^{P,\eta}) + \frac{1}{2} \eta \partial E\eta \epsilon^\eta.
$$

(2.2)

The problem (2.1) is considered with the Dirichlet boundary conditions

$$
\begin{align}
u^\eta(x, t) &= g_D(x, t), \quad x \in \partial\Omega, \quad t \geq 0, \\
p^\eta(x, t) &= g_P(x, t), \quad x \in \partial\Omega, \quad t \geq 0,
\end{align}
$$

(2.3)

and initial conditions

$$
\text{div}_x u^\eta(x, 0) = \text{div}_x u^0(x), \quad \epsilon^{P,0}(x, 0) = \epsilon^{P,0}(x).
$$

(2.4)

Assume that our data $F, f, g_D, g_P, \text{div} u^0, \epsilon^{P,0}$ have the following regularity

$$
\begin{align}
F &\in W^{1,\infty}(\{0, T\}; L^2(\Omega; \mathbb{R}^3)), \quad f \in W^{1,\infty}(\{0, T\}; L^2(\Omega; \mathbb{R})), \\
g_D &\in W^{2,\infty}(\{0, T\}; H^2(\partial\Omega; \mathbb{R}^3)), \quad g_P \in W^{1,\infty}(\{0, T\}; H^2(\partial\Omega; \mathbb{R})), \\
\text{div} u^0 &\in H^1(\Omega; \mathbb{R}), \quad \epsilon^{P,0} \in L^2(\Omega; \mathbb{R}^3), \quad \text{div} \epsilon^{P,0} \in L^2(\Omega; \mathbb{R}^3),
\end{align}
$$

(2.5)

and additionally $g_D(0)$ and $\text{div} u^0$ satisfy

$$
g_D(0) \in H^\frac{3}{2}(\partial\Omega; \mathbb{R}^3) \quad \text{and} \quad \int_{\partial\Omega} g_D(x, 0) n(x) \, ds = \int_{\Omega} \text{div} u^0(x) \, dx.
$$

(2.6)

Definition 2.1. We say that the nonlinear constitutive function $G: S^3 \rightarrow S^3$ belongs to the class $\mathcal{LM}$ if

$$
\exists L > 0 \quad (G(T_1) - G(T_2), T_1 - T_2) + L|T_1 - T_2|^2 \geq 0 \quad \text{for all} \ T_1, T_2 \in S^3.
$$

Note that the class $\mathcal{LM}$ is equal to the class of Lipschitz perturbations of monotone vector fields (see [5] for more details).

Lemma 2.2. Assume that $\beta > 1$ is not less than

$$
r' = (\alpha - 1) \deg(Y)(\deg(Y) - 1)^2.
$$

Then for all $\eta > 0$ and $T \in S^3$ the nonlinear constitutive function

$$
G^\eta(T) = \eta|T|^\beta \frac{T}{|T|} + \mathcal{F}(Y(T)) \frac{\partial P}{\partial T}(T)
$$

belongs to the class $\mathcal{LM}$.
Proof. Note that
\[ (G(T_1) - G(T_2), T_1 - T_2) = \nabla G(T^*) (T_1 - T_2)(T_1 - T_2), \]
where \( T^* \) belongs to the segment with ends \( T_1 \) and \( T_2 \). Therefore, it is sufficient to prove that there exists a constant \( L > 0 \) such that
\[ \nabla G(T^*) (T_1 - T_2)(T_1 - T_2) \geq -L|T_1 - T_2|^2. \] (2.9)

We easily calculate
\[ \nabla G(T^*) = \eta(\beta - 1)\frac{|T^*|^\beta - 2 T^* \otimes T^*}{|T^*|^\beta} + \eta|T^*|^{\beta - 1}(I_9 \otimes T^*) + F(Y(T^*)) \frac{\partial Y}{\partial T}(T^*) \otimes \frac{\partial P}{\partial T_2}(T^*), \]
where \( I_9 \) denotes the \( 9 \times 9 \) identity matrix.

Let \( C > 0 \) be a constant such that the inequality
\[ \left( \eta(\beta - 1)|T^*|^{\beta - 2} \frac{T^* \otimes T^*}{|T^*|^\beta} + \eta|T^*|^{\beta - 1}(I_9 \otimes T^*) + F(Y(T^*)) \frac{\partial Y}{\partial T}(T^*) \otimes \frac{\partial P}{\partial T_2}(T^*) \right) (T_1 - T_2)(T_1 - T_2) \geq 0 \]
holds for \( T^* \in \{|T| > C\} \). From the assumption, such a constant \( C \) exists because the matrix
\[ \frac{\partial^2}{\partial T^2} \left( \frac{\eta}{\beta + 1}|T|^\beta \right) \]
is positive definite for all \( T \in S^3 \). Therefore, for \( T^* \in \{|T| > C\} \) the inequality (2.9) is satisfied (\( P \) is a convex polynomial).

For \( T^* \in \{|T| \leq C\} \) we obtain that the coefficients of quadratic form on the left-hand side of (2.9) are bounded. The proof is complete. □

Before we formulate the existence theorem for each approximation step, let us consider the following problem
\[ \text{div}_x D(e^u(x, 0)) - \nabla_x p^u(x, 0) = -F(x, 0) + \text{div} D(e^p, 0), \]
\[ \text{div}_x u^0(x, 0) = \text{div} u^0(x) \] (2.10)
with boundary conditions
\[ u^0(x, 0) = g_D(x, 0), \quad x \in \partial \Omega, \]
\[ p^0(x, 0) = g_P(x, 0), \quad x \in \partial \Omega. \] (2.11)

Lemma 2.3. Assume (2.7) and (2.8), then there exist unique solutions \( u^0(0), p^0(0) \) such that \( u^0(0) \in H^2(\Omega; \mathbb{R}^3) \) and \( p^0(0) \in H^1(\Omega; \mathbb{R}) \). Moreover, we have
\[ \left\| u^0(0) \right\|_{H^2(\Omega; \mathbb{R}^3)} + \left\| p^0(0) \right\|_{H^1(\Omega; \mathbb{R})} \leq C(\Omega)(L(F(0))_{L^2(\Omega; \mathbb{R}^3)} + \left\| \text{div} D(e^p, 0) \right\|_{L^2(\Omega; S^3)} + \left\| g_D(0) \right\|_{H^\frac{1}{2}(\partial \Omega; \mathbb{R}^3)} + \left\| \text{div} u^0 \right\|_{H^1(\Omega; \mathbb{R})} + \left\| g_P(0) \right\|_{H^\frac{1}{2}(\partial \Omega; \mathbb{R})}). \]

Proof. Notice that the system (2.10) with (2.11) is the Stokes problem (see for instance [12]), so the proof will be omitted. □

From Lemma 2.2 we know that the constitutive function in (2.1) has monotone structure. Now, we are ready to formulate the existence of solutions for each approximation step.

Theorem 2.4. Let us suppose that for all \( T > 0 \) our data have regularity required in (2.5)–(2.8). Then for all \( \eta > 0 \) the system (2.1) with initial-boundary conditions (2.3)–(2.4) possesses unique solutions
\[ (u^0, e^{p^0}, p^0) \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; S^3) \times H^1(\Omega; \mathbb{R})) \]
and
\[ (u^0_t, e^{p^0}_t) \in L^2(0, T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; S^3)). \]

Proof. In Section 4 of the paper [10] the existence and uniqueness results to a model of gradient type in poroplasticity (constitutive function \( G \) can be written as the gradient of a convex function) was proved. Observe that \( G^0(T^0) = VM(T^0) + N(T^0) \), where \( M: S^3 \to \mathbb{R}_+ \) is a nonnegative, differentiable convex function such that \( M(T) = \frac{9}{2} T^\beta + 1 \). Moreover, the
degree of the polynomial \( N \) is less than the degree of \( \nabla M \). Using the same methods as in [10] we can easily show the existence and uniqueness for problems from the class \( L^M \) and the details are left to the reader. 

Let us calculate the time derivative of the energy

\[
\frac{d}{dt} (E(\eta, \eta^p, \eta^p)) (t) = \int_\Omega T^\eta_{\xi t} \ dx - \int_\Omega \tilde{T}^\eta_{\xi t} \ dx.
\] (2.12)

Moving the second term to the left and integrating (2.12) with respect to \( t \) we arrive at the equation

\[
E(\eta, \eta^p)(t) + \eta \int_0^t |\tilde{T}^\eta|^{\beta+1} \ dx d\tau + \eta \int_0^t \int_\Omega \mathcal{F}(\tilde{T}^\eta) \ \frac{\partial P}{\partial T}(\tilde{T}^\eta) \ dx d\tau
\]

\[
= E(\eta, \eta^p)(0) + \int_0^t \int_\Omega T^\eta_{\xi t} \ dx d\tau.
\] (2.13)

The third term on the left-hand side of (2.13) satisfies the dissipative inequality. We use Cauchy inequality to the second term of the right-hand side of (2.13). Next, from Lemma 2.3 and Theorem 2.4 we obtain that for all \( \eta > 0 \), \( T^\eta \in L^{\beta+1}(0, T; L^\beta(S^3)) \).

3. Main theorem

Before we present the main result of this paper, we define a new notion of the solutions of the initial-boundary value problem (1.1)–(1.3).

Definition 3.1. Suppose that the given data satisfy (2.5)–(2.8). Moreover, let \( F, F_t \in L^\infty(0, T; L^3(\Omega; \mathbb{R}^3)) \). We say that for \( \beta > 1 \) a vector \( u \in L^{1+\frac{1+\beta}{2}}(0, T; W^{1,1+\frac{1+\beta}{2}}(\Omega; \mathbb{R}^3)) \), the function \( p \in L^2(0, T; H^1(\Omega; \mathbb{R})) \), the inelastic deformation tensor \( \eta^p \in W^{1,1+\frac{1+\beta}{2}}(0, T; L^{1+\frac{1+\beta}{2}}(\Omega; S^3)) \) and Cauchy stress tensor \( T \in L^\infty(0, T; L^2(\Omega; S^3)) \) are solutions of the problem (1.1)–(1.3) if:

1. the functions \( u \) and \( p \) are in the form \( u(x, t) = v(x, t) + w(x, t), \ p(x, t) = \tilde{p}(x, t) + \tilde{w}(x, t), \) where \( w \in W^{2,\infty}(0, T; H^1(\Omega; \mathbb{R}^3)) \) and \( \tilde{w} \in W^{1,\infty}(0, T; H^2(\Omega; \mathbb{R})) \) are such functions that \( w_{\mid\Omega^2} = gD \) and \( \tilde{w}_{\mid\Omega^2} = gp \). Moreover functions \( v \) and \( \tilde{p} \) satisfy the following system of the equations:

\[
D(\epsilon(v) - \epsilon(\tilde{v})) \ dx - \int_\Omega p \ \text{div} \ \tilde{v} \ dx = \int_\Omega F \tilde{v} \ dx - \int_\Omega D(\epsilon(w)) \ dx, 
\]

\[
\int_{\Omega \times [0, T]} c \text{div} \tilde{\eta} \text{div} \phi \ dx dt - \int_{\Omega \times [0, T]} \text{div} u \phi_t \ dx dt + \int_{\Omega \times [0, T]} \text{div} u^0 \phi (0) \ dx dt - \int_{\Omega \times [0, T]} f \phi \ dx dt - \int_{\Omega \times [0, T]} \text{div} \tilde{\eta} \phi \ dx dt,
\]

where the first equation is satisfied for all \( \tilde{v} \in H^1_0(\Omega; \mathbb{R}^3) \) and for almost all \( t \in (0, T) \), the second equation is satisfied for all \( \phi \in C^\infty(\Omega \times [0, T]) \).

2. The fifth equation in (1.1) is satisfied in the sense of Young measures i.e.

\[
\epsilon^p_t (x, t) = \int_{S^3} \mathcal{F}(Y(S)) \ \frac{\partial P}{\partial T}(S) \ dx d\nu(x,t)(S),
\]

where \( \nu(x,t) \) is the Young measure generated by the sequence \( \{\tilde{T}^\eta\} \).

3. \( \epsilon^p(x, 0) = \epsilon^p(0, x) \).

To pass to the limit in the system (2.1) and trying to obtain the standard weak-type of the solutions (as in [10]) we have to get, from the energy estimate for the time derivatives of the approximate sequence that \( \text{div} u^\eta_t \) is bounded in \( L^2(L^2) \) independently of \( \eta \). The considered model is not monotone, so we are unable to show that the time derivative of the strain is bounded in \( L^2(L^2) \). Therefore, we change the definition of the weak-type solution. Observe, that to obtain the weak solution in the sense of Definition 3.1, we only have to prove that \( \epsilon(u^\eta) \) is bounded in \( L^{1+\frac{1}{\beta}}(L^{1+\frac{1}{\beta}}) \). The following theorem presents the main result of this paper:
Theorem 3.2. Assume that the given data have the regularity required in (2.5)–(2.8). Moreover, let \( F, F_t \in L^\infty(0, T; L^3(\Omega; \mathbb{R}^3)) \). Then for all \( T > 0 \) the sequence \( \{(T^\eta, u^\eta, p^\eta, v^\eta, \eta^\eta)\}_{\eta > 0} \) converges as \( \eta \to 0^+ \) to the solution (in the sense of Definition 3.1) \((T, u, p, v^P)\) of the problem (1.1). Additionally this solution has the regularity

\[
(T, p, u) \in L^\infty(0, T; L^2(\Omega; S^3)) \times L^2(0, T; H^1(\Omega; \mathbb{R})) \times L^{1+\frac{2}{\beta}}(0, T; W^{1,1+\frac{2}{\beta}}(\Omega; S^3)),
\]

\[
e^p \in W^{1,1+\frac{2}{\beta}}(0, T; L^{1+\frac{2}{\beta}}(\Omega; S^3))
\]

for every \( \beta > r' > r = \alpha \deg(Y)(\deg(Y) - 1) > 1 \).

The idea of the proof of the theorem above is: first we will prove that the approximation functions are bounded independently of \( \eta \), finally we will pass to the limit \( \eta \to 0^+ \).

4. Energy estimate

This section is the main part of the proof of Theorem 3.2. We are going to prove some bounds of the approximate solutions and their derivatives. First, note that from the assumptions on the given boundary data, we have that there exists a function \( p^* \in W^{1,\infty}(0, T; H^2(\Omega; \mathbb{R})) \) such that \( p_t^* = \beta \). Therefore, the system (2.1) can be written in the form:

\[
\begin{align*}
\text{div}_x T^\eta(x, t) - \nabla_x (p^\eta(x, t) - p^*(x, t)) &= -F(x, t) + \nabla_x p^*(x, t), \\
\beta \Delta_x (p^\eta(x, t) - p^*(x, t)) - \text{div}_x u^\eta_t(x, t) &= f(x, t) - \beta \Delta x p^*(x, t).
\end{align*}
\]

Now we are ready to prove some estimates for the approximation sequence.

Theorem 4.1 (Energy estimate). Suppose that our data have regularity required in (2.5)–(2.8). Additionally assume that \( F, F_t \in L^\infty(0, T; L^3(\Omega; \mathbb{R}^3)) \). Moreover, let \( \beta > r' > r = \alpha \deg(Y)(\deg(Y) - 1) > 1 \). Then, there exists a positive constant \( C(T) \) (not depending on \( \eta \)) such that the inequality

\[
\mathcal{E}^\eta(t) \leq C(T)
\]

holds.

Proof. Again, we calculate the time derivative of the energy

\[
\frac{d}{dt}(\mathcal{E}^\eta(t)) = \int_\Omega T^\eta \text{div}_x \partial^\eta_x dx - \int_\Omega \hat{T}^\eta \text{div}^\eta_x dx.
\]

Integrating three times by parts in the first integral, using first two equations in (4.1), and integrating the whole inequality with respect to time we obtain

\[
\begin{align*}
\mathcal{E}^\eta(t) &= \mathcal{E}^\eta(0) - \int_0^t \int_\Omega (p^\eta - p^*) \partial^\eta_x dx d\tau - \int_0^t \int_\Omega \nabla p^\eta \nabla u^\eta dx d\tau - \int_0^t \int_\Omega \nabla (p^\eta - p^*) \nabla u^\eta dx d\tau + \int_0^t \int_\Omega |\nabla (p^\eta - p^*)|^2 dx d\tau \\
&= \int_0^t \int_\Omega (p^\eta - p^*) \partial^\eta_x dx d\tau - \int_0^t \int_\Omega \nabla p^\eta \nabla u^\eta dx d\tau - \int_0^t \int_\Omega \nabla (p^\eta - p^*) \nabla u^\eta dx d\tau + \int_0^t \int_\Omega |\nabla (p^\eta - p^*)|^2 dx d\tau.
\end{align*}
\]

From Lemma 2.3 the first term on the right-hand side of (4.3) is bounded independently of \( \eta \). The second term on the right-hand side of (4.3) is estimated as follows:

\[
\int_0^T \int_\Omega F_t^\eta dx d\tau = -\int_0^T \int_\Omega F_t u^\eta dx d\tau + \int_\Omega F^\eta dx - \int_\Omega F(0) u^\eta(0) dx.
\]
Using regularity of $F$, $u^\eta(0)$ we see that the last integral in (4.4) is bounded. Using Hölder inequality to the second integral in (4.4) we get
\[
\int_0^t \int_\Omega F_t u^\eta \, dx \, d\tau \leq \int_0^t \|F_t\|_{L^3(\Omega; \mathbb{R}^3)} \|u^\eta\|_{L^2(\Omega; \mathbb{R}^3)} \, d\tau.
\] (4.5)

Let $LD(\Omega; \mathbb{R}^3) = \{u \in L^1(\Omega; \mathbb{R}^3); \ v(u) \in L^1(\Omega; \mathbb{S}^3)\}$. It is known that $LD(\Omega; \mathbb{R}^3)$ is continuously embedded in $L^2(\Omega; \mathbb{R}^3)$ (see for example [13]). Therefore,
\[
\int_0^t \int_\Omega F_t u^\eta \, dx \, d\tau \leq C(\Omega, \beta) \left( 1 + \int_0^t \|F_t\|_{L^3(\Omega; \mathbb{R}^3)} \|e^\eta\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)} \, d\tau \right)
\]
\[
\leq C(\Omega, \beta) \left( 1 + \int_0^t \|F_t\|_{L^3(\Omega; \mathbb{R}^3)} \|e^\eta - e^{P,\eta}\|_{L^2(\Omega; \mathbb{S}^3)} + \|e^{P,\eta}\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)} \right) \, d\tau.
\] (4.6)

Using Cauchy and Young inequalities on the right-hand side of (4.6) we obtain
\[
\int_0^t \int_\Omega F_t u^\eta \, dx \, d\tau \leq C(\Omega, \beta, v) + C(\Omega, \beta) \left( \frac{1}{2} \int_0^t \|\hat{\eta}\|_{L^2(\hat{\Omega}; \mathbb{S}^3)}^2 \, d\tau + v \int_0^t \|e^{P,\eta}\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)} \, d\tau \right),
\] (4.7)

where $v > 0$ is any positive number and $C(\Omega, \beta, v) > 0$ does not depend on $\eta$. Estimating the third integral in (4.4), similarly to the second integral in (4.4) and using the following inequality
\[
\|e^{P,\eta}\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)} \leq \|e^{P,0}\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)}^\beta + c \int_0^t \|e^{P,\eta}\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)} \, d\tau,
\] (4.8)

we obtain
\[
\int_\Omega F u^\eta \, dx \leq C(\Omega, \beta, v) + C(\Omega, \beta) \left( v \|\hat{\eta}\|_{L^2(\hat{\Omega}; \mathbb{S}^3)}^2 + v \int_0^t \|e^{P,\eta}\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)} \, d\tau \right).
\] (4.9)

From the inelastic constitutive equation we have (note, that $1 + \frac{1}{\beta} < 1 + \frac{1}{r}$)
\[
\int_0^t \|e^{P,\eta}\|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S}^3)} \, d\tau = \int_0^t \|e^{P,\eta}\|_{L^{1+\frac{1}{r}}(\Omega; \mathbb{S}^3)} \, d\tau
\]
\[
\leq 2^\frac{1}{2} \left( \int_0^t \|\hat{\eta}\|_{L^2(\hat{\Omega}; \mathbb{S}^3)}^\beta \, d\tau + \int_0^t \|\nabla \mathcal{F}(\hat{\eta}) \|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S})} \, d\tau \right)^{\frac{1}{2}}
\]
\[
\leq \eta \|\hat{\eta}\|_{L^2(\hat{\Omega}; \mathbb{S}^3)}^\beta + \eta \|\hat{\eta}\|_{L^2(\hat{\Omega}; \mathbb{S}^3)} \|\nabla \mathcal{F}(\hat{\eta}) \|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S})}
\]
\[
\leq C(\beta, \eta) \int_0^t \|\hat{\eta}\|_{L^{1+\frac{1}{r}}(\Omega; \mathbb{S}^3)} \, d\tau + C(\beta, r) \int_0^t \|\mathcal{F}(\hat{\eta}) \|_{L^{1+\frac{3}{r}}(\Omega; \mathbb{S})} \, d\tau.
\] (4.10)

The third term on the right-hand side of (4.3) is estimated just like the second term. Moreover, the fourth term on the right-hand side of (4.3) is moved to the left and other terms on the right-hand side of (4.3) are bounded by the standard methods: continuity of the trace operator and the weighted Cauchy inequality (see for instance [10]). Therefore, if using the inequalities (4.7)–(4.10) then we obtain that the inequality
exists a constant side of (4.15) is less than theorem (see [7, Theorem 4.21, p. 274]) we finish the proof.

From Theorem 4.1, the integral on the right-hand side of (4.15) is bounded. We can choose holds for all \( \hat{\eta} \) is satisfied for \( \hat{\eta} \) is bounded in \( \hat{\eta} \). Therefore, from (4.14) and (4.15) we have that the inequality

\[
L \leq C(\Omega, T, \beta) + C(\Omega, \beta) \int_{0}^{t} \left\| \hat{\eta} \right\|_{L^2(\Omega; S^3)}^2 \, d\tau
\]

(4.14)

is satisfied for \( \hat{\eta} \in \|T\| > D \). Observe, that for \( \hat{\eta} \in \|T\| \leq D \) the last integral on the right-hand side of (4.11) is bounded independently of \( \eta > 0 \). So, in the inequality (4.11) we can choose \( \nu > 0 \) so small that the inequality

\[
L \leq C(\Omega, T, \beta) + C(\Omega, \beta) \int_{0}^{t} \left\| \hat{\eta} \right\|_{L^2(\Omega; S^3)}^2 \, d\tau
\]

(4.15)

holds for \( \hat{\eta} \in \|T\| \leq D \). Therefore, from (4.14) and (4.15) we have that the inequality

\[
L \leq C(\Omega, T, \beta) + C(\Omega, \beta) \int_{0}^{t} \left\| \hat{\eta} \right\|_{L^2(\Omega; S^3)}^2 \, d\tau
\]

(4.16)

holds for all \( \hat{\eta} \in S^3 \). Using the Gronwall inequality in (4.16) we complete the proof. \( \square \)

**Theorem 4.2.** Let us suppose that all hypotheses of Theorem 4.1 hold. Then, the sequence \( \{e^{p, \eta}_t\} \) is sequentially weakly precompact in \( L^1(0; T; L^2(\Omega; S^3)) \).

**Proof.** Notice that, from Theorem 4.1 we obtain that \( e^{p, \eta}_t \) is bounded in \( L^{1+\frac{1}{p}}(0, T; L^{1+\frac{1}{p}}(\Omega; \mathbb{R})) \). Let us fix \( \epsilon > 0 \) and let \( E \subseteq \Omega \times (0, T) \) be a measurable set such that \( |E| < \mu \). Then

\[
\int_{E} \left| e^{p, \eta}_t \right| \, dx \, dt \leq |E|^\frac{d}{p + d} \left( \int_{\Omega \times (0, T)} \left| e^{p, \eta}_t \right|^{\beta + 1} \, dx \, dt \right)^{\frac{1}{\beta + 1}}. \tag{4.17}
\]

From Theorem 4.1, the integral on the right-hand side of (4.15) is bounded. We can choose \( \mu \) so small that the left-hand side of (4.15) is less than \( \epsilon > 0 \). Therefore, we proved the equi-integrability of the sequence \( \{e^{p, \eta}_t\} \). From the Dunford–Pettis theorem (see [7, Theorem 4.21, p. 274]) we finish the proof. \( \square \)
5. Passing to the limit in the sense of Young measures

Before we start proving Theorem 3.2, we would like to present the main theorem from [3].

Theorem 5.1 (The fundamental theorem for Young measures). (See [3].) Let \( \Omega \subset \mathbb{R}^n \) be Lebesgue measurable, let \( K \subset \mathbb{R}^m \) be closed and let \( z^j : \Omega \to \mathbb{R}^m, j = 1, 2, \ldots, \) be a sequence of Lebesgue measurable functions such that for any given open neighbourhood \( U \) of \( K \) in \( \mathbb{R}^m, \lim_{j \to \infty} ||x \in \Omega : z^j(x) \notin U|| = 0. \)

Then, there exists a subsequence \( z^{j_k} \) of \( z^j \) and family \( (\nu_k)_k \) of positive measures on \( \mathbb{R}^m, \) depending measurably on \( x. \) Suppose further that \( z^{j_k} \) satisfies the boundedness condition

\[
\lim_{k \to \infty} \sup_{\mu \in \mathcal{M}} \left| \left| z^{j_k}(x) \right| \right| = 0.
\]

for every \( R > 0, \) where \( B_R = \mathbb{B}(0, R). \) Then, \( \|\nu_k\|_{\mathcal{M}} = 1 \) for a.e. \( x \in \Omega \) and given any measurable subset \( A \) of \( \Omega \)

\[
f(z^{j_k}) \rightharpoonup (\mu_x, f) = \int f(\lambda) d\nu_x(\lambda) \quad \text{in} \quad L^1(A)
\]

for any continuous function \( f : \mathbb{R}^m \to \mathbb{R} \) such that \( \{f(z^{j_k})\} \) is sequentially weakly precompact in \( L^1(A). \)

Section 4 gives us the following results:

The sequence \( \{\hat{T}^\eta, p^\eta\}_{\eta > 0} \) is bounded in \( L^\infty(0, T; L^2(\Omega; S^3)) \times L^2(0, T; H^1(\Omega; \mathbb{R}^3)), \) \( \{\varepsilon_{T}^\eta, \varepsilon_0^\eta\} \) is bounded in \( L^1 \left( (0, T; L^1(\Omega; S^3)) \right) \times L^1 \left( (0, T; \sqrt{\eta} H^1(\Omega; \mathbb{R}^3)) \right), \) \( \{\varepsilon_{p}^\eta, \varepsilon_0^\eta\} \) is bounded in \( L^1 \left( (0, T; L^1(\Omega; S^3)) \right) \) and \( \{\varepsilon_{\varepsilon_0^\eta}^\eta\}_{\eta > 0} \) is weakly precompact in \( L^1(0, T; L^1(\Omega; S^3)). \)

Now we are ready to prove the main theorem.

Proof of Theorem 3.2. From Theorem 2.4 we have that \( u^0(x, t) = v^0(x, t) + w(x, t), \) \( p^0(x, t) = \tilde{p}^0(x, t) + \tilde{w}(x, t), \) where \( w \) and \( \tilde{w} \) are such that trace of \( w \) on \( \partial \Omega \) is equal \( g_D \) and trace \( \tilde{w} \) on \( \partial \Omega \) is equal \( g_p. \) Moreover functions \( v^0 \) and \( \tilde{p}^0 \) satisfy the following system of the equations:

\[
\int_{\Omega} D((1 + \eta) \varepsilon(v^0, \eta) - p^0 \phi) v^0 dx - \int_{\Omega} p^0 \phi \nabla v^0 dx = \int_{\Omega} \nabla \bar{v} dx - \int_{\Omega} D((1 + \eta) \varepsilon(w, \eta)) v^0 dx,
\]

\[
\int_{\Omega \times [0, T]} c \nabla \tilde{p}^0 \nabla \phi dx dt - \int_{\Omega \times [0, T]} \nabla u^0_\phi \phi dx dt = - \int_{\Omega \times [0, T]} f \phi dx dt - \int_{\Omega \times [0, T]} \nabla \tilde{w} \phi dx dt,
\]

where the first equation is satisfied for all \( \bar{v} \in H^1_0(\Omega; \mathbb{R}^3) \) and almost all \( t \in (0, T), \) the second equation is satisfied for all \( \phi \in C^\infty(\Omega \times [0, T]) \) and third equation is satisfied for almost all \( (x, t) \in \Omega \times (0, T). \) Integrating by parts in the second term on the left-hand side of the second equation above and passing to the weak limits \( \eta \to 0^+ \) (passing to subsequence if necessary) we obtain the limit functions satisfy the following system of equations

\[
\int_{\Omega} D(\varepsilon(v) - p^0 \phi) v dx - \int_{\Omega} p \phi v dx = \int_{\Omega} \nabla \bar{v} dx - \int_{\Omega} D(\varepsilon(w)) v^0 dx,
\]

\[
\int_{\Omega \times [0, T]} c \nabla \tilde{p} \nabla \phi dx dt - \int_{\Omega \times [0, T]} \nabla u_\phi \phi dx dt + \int_{\Omega \times [0, T]} \nabla u_0^0 \phi(0) dx = - \int_{\Omega \times [0, T]} f \phi dx dt - \int_{\Omega \times [0, T]} \nabla \tilde{w} \phi dx dt,
\]

where \( \varepsilon^0_\phi = w - \lim_{\eta \to 0^+} \varepsilon^\eta_\phi = \tilde{x}. \)

Moreover, note that from Theorem 4.1 the term \( \eta|\hat{T}^\eta|^\beta |\hat{T}^\eta| \) goes to zero in \( L^1(0, T; L^1(\Omega; S^3)) \) as \( \eta \to 0^+. \) Therefore, from Theorem 4.2 the sequence

\[
\left\{ \mathcal{F} \left( Y(\hat{T}^\eta) \right) \frac{\partial p}{\partial \varepsilon_\eta} \right\}_{\eta > 0}
\]

is weakly precompact in \( L^1(0, T; L^1(\Omega; S^3)). \) From the fundamental theorem for Young measures, there exists the family of Young measures \( \nu_{\varepsilon(\eta)} \) generated by the sequence \( \{\varepsilon(v^0, \eta)\}_{\eta > 0} \) (\( \hat{T}^\eta \) is bounded in \( L^\infty(0, T; L^2(\Omega; S^3)) \) and the boundedness
condition from Theorem 5.1 is satisfied). By Theorem 5.1 we conclude that the weak limit $\hat{\chi}$ is in the form

$$\hat{\chi}(x, t) = \int_{\mathcal{S}^3} \mathcal{F}(Y(S)) \frac{\partial P}{\partial T}(S) d\nu(x, t)(S).$$

The proof is complete. □

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